BIFURCATIONS IN CONTINUOUS SYSTEMS

FRANCESCA AICARDI

In this lesson we will see the *bifurcations*, i.e., the typical changes of the behaviour of a continuous dynamical system depending on a parameter.

1. O.D.E.

A continuous dynamical system in $\mathbb{R}^n$ is defined by a system of n ordinary first order differential equations (O.D.E.):

$$\begin{aligned} &\dot{x}_1 = f_1(x_1, x_2, \ldots, x_n) \\ &\dot{x}_2 = f_2(x_1, x_2, \ldots, x_n) \\ &\ldots \\ &\dot{x}_n = f_n(x_1, x_2, \ldots, x_n) \end{aligned}$$

where $\dot{x}$ indicate the derivative of the variable x with respect to the time.

We will denote by $\mathbf{x}$ the vector in $U \subset \mathbb{R}^n$ with components $x_1, \ldots x_n$ and by $\dot{\mathbf{x}}$ the velocity vector with components $\dot{x}_1, \ldots, \dot{x}_n$. The O.D.E. will be written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$

A solution of this O.D.E. in the interval

$$I \subset \mathbb{R}, \qquad I \text{ is open}$$

is a map

$$\varphi : I \to U \qquad (\mathbb{R} \to \mathbb{R}^n)$$

such that

$$\frac{d}{dt}\varphi(t)|_{t=\tau} = \mathbf{f}(\varphi(\tau)),$$

for all $\tau \in I$.

Example. Let $n = 1$, and the phase-space be the entire line: $U = \mathbb{R}$. Let the equation be

$$\dot{\mathbf{x}} = 3\mathbf{x}.$$

The solutions are defined for all times in $(-\infty, +\infty)$, so that $I = \mathbb{R}$. For every initial condition $x_0 \in \mathbb{R}$

$$\varphi(t) = \mathsf{x}_0 e^{3t},$$

is a solution of the equation, indeed:

$$\frac{d}{dt}e^{3t}|_{t=\tau} = 3\mathsf{x}_0 e^{3\tau}.$$

Definition. The image in $\mathbb{R}^n$ of the map φ is called *trajectory.*

1.1. **Existence and uniqueness of the solutions.** *Theorem.* If $\mathbf{f}$ is a C^1-function defined in an open set $U \subset \mathbb{R}^n$, then the solution $\mathbf{x}(\mathsf{t})$ of the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

satisfying $\mathbf{x}(0) = \mathbf{x}_0$, for any $\mathbf{x}_0 \in U$, **exists** and is **unique** in a suitable interval of time $(-s, s)$.

This theorem is the basis of the theory of continuous dynamical systems. From a geometrical point of view, it says the following. A solution of the equation with initial condition $\mathbf{x}_0$ defines, for every $\tau > 0$, $\tau \in I$, a curve contained in U, with an extreme in $\mathbf{x}_0$ and the other extreme in $\varphi(\tau)$. Such a curve is called trajectory starting at $\mathbf{x}_0$. The equation defines in $U \subset \mathbb{R}^n$ a vector field, i.e. in each point of U is well defined a vector, which will be interpreted as the velocity of a trajectory passing through that point. If the vector field changes with continuity in space, then for each point of U there is one and only one trajectory passing through that point.

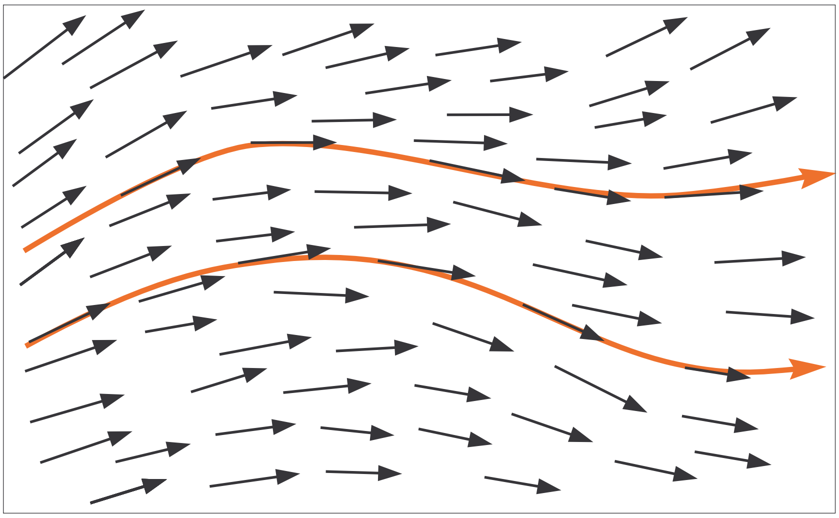

Figure 1. A vector field and two trajectories

1.2. **Fixed points.** A **fixed point** or *equilibrium point* of the flow φ is a point $\mathbf{p}$ satisfying

$$\varphi^{\mathsf{t}}(\mathbf{p}) = \mathbf{p} \quad \forall \mathsf{t} \in \mathbb{R}$$

Theorem. The point $\mathbf{p}$ is a fixed point for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ if and only if $\mathbf{f}(\mathbf{p}) = 0$.

Evidently, if the vector field vanishes at $\mathbf{p}$, the velocity of a trajectory lying at $\mathbf{p}$ is zero so it cannot move out from $\mathbf{p}$.

This means that the point $\mathbf{p}$ is itself a trajectory, and, by the uniqueness theorem, no other trajectories can pass trough this point.

Therefore equilibrium points are, from a mathematical point of view, isolated solutions.

However, there are particular fixed points that are *attracting*, or *stable equilibrium points*. I.e., if $\mathbf{p}$ is a stable equilibrium point, there is a neighbourhood V of it such that all trajectories starting in V remains in V and as time grows they become closer and closer to $\mathbf{p}$.

EXAMPLE. Consider the equation for the real pendulum (i.e., with friction):

$$\ddot{\theta} = -a \sin(\theta) - b\dot{\theta},$$

where a and b are positive constant.

This is a non linear equation of second order, but we transform it in a two dimensional dynamical system: Let $\theta := x_1$, and $\dot{\theta} = x_2$, we get the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin(x_1) - bx_2 \end{aligned} \tag{1}$$

Observe that x_1, being the angle θ, is defined on $\mathbb{R}$ modulo 2π.

The phase space is constituted by the coordinate x_1, the angle, and x_2, the angular velocity.

The point $(0, 0)$ is a fixed point (prove it!). This implies that if the pendulum is initially in the equilibrium position, then it remains in that position forever. But this point is also *stable*. This means that, starting far from the equilibrium position, after a finite time the pendulum will be very close to the equilibrium.

If we put the pendulum in the position $x_1(0) > 0$, with zero initial speed $(x_2(0) = 0)$, the trajectory is a spiral which approaches the equilibrium. We see, in fact, that while the pendulum is oscillating, the amplitude of oscillations decreases, till a moment when it 'ceases to oscillate'. This is uncorrect from the mathematical view point: the pendulum continues to oscillate for every finite time, but the amplitude becomes so little that the oscillations are invisible. In practice, however, when we say that the trajectories tends to the equilibrium, we have in mind that after a finite time they belong to a neighbourhood of the equilibrium as small as we like.

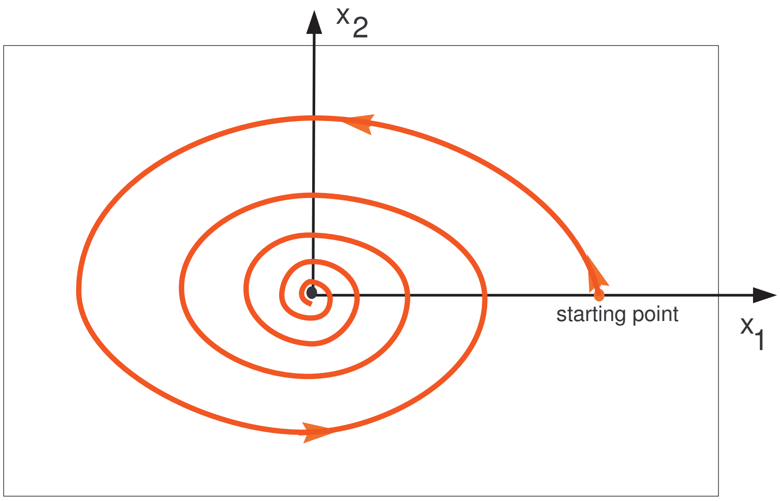

Figure 2. One trajectory in the phase space of the real pendulum.

1.3. **Asymptotic behaviour.** In most cases what we need to know is the solution of the equation for big time.

If the system has an attracting set, a point for instance, after a finite time all trajectories become so close to the fixed point, that we say that the system has reached the equilibrium position. (Remember what we have observed: mathematically the equilibrium is reached only at infinite time, but a small neighbourhood of it, which is what we are able to measure in physical systems, is reached in a finite, and even short, time).

For this reason the most important object of a dynamical system is its "attracting set", which allows us to know with certainty (and with a chosen precision) the behaviour of the system after a finite interval of time.

1.4. **Hyperbolic systems.** The existence and uniqueness theorem is extremely important, since in most cases, even if the system of O.D.E is given by very simple functions but non linear, precise analytical solutions do not exist. Therefore to study the behaviour of the system one needs to find approximate numerical solutions.

The main problem arising in the approximation of the solution is to know whether the difference between the exact solution and the numerical solution remains small or even decreases. Otherwise, the numerical computation does not help to find the behaviour of the system.

The word 'hyperbolic' is used in many different fields of mathematics and indicates different properties. In the theory of dynamical systems, this adjective indicate a "good" behaviour, i.e. a behavior which is robust with respect to small perturbations and hence can be investigated with approximating techniques.

The words 'hyperbolic' was invented as opposite of the words 'elliptic'. Consider the system (1) with $b = 0$. It is exactly the equation of the ideal pendulum, i.e., without friction.

If x_1 and x_2 are both zero, the field is still zero, i.e. the point $(0, 0)$ is an equilibrium point.

But if we start at a point $x_1 > 0$, with zero velocity ($x_2 = 0$), the point representing the solution moves indefinitely along a closed curve, close to an *ellipse*, i.e. the pendulum oscillates indefinitely in a perpetual motion. We obtain a closed trajectory for every initial condition different from the origin and with initial speed equal to zero. The equilibrium point is in this case non hyperbolic.

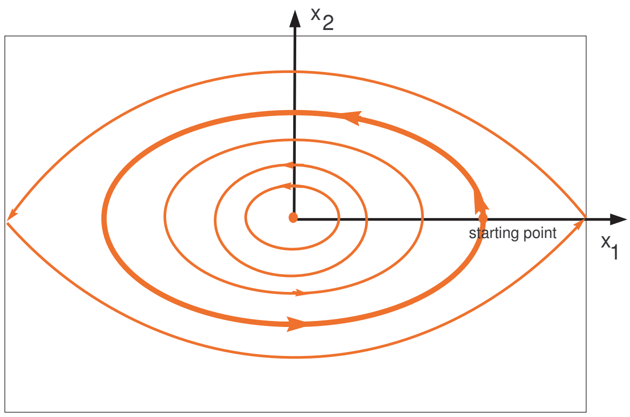

Figure 3. The trajectories in the phase space of the ideal pendulum

The entire system is said non hyperbolic. It is quite evident that such a system is not 'robust' with respect to perturbations. In the hyperbolic case, if we vary the coefficient b, i.e. we augment or diminish the friction, the amplitude of the oscillations will decreases more or less rapidly, but the asymptotic behaviour will be the same (eventually the pendulum will reach the equilibrium). In the elliptic case, it suffices to introduce a non zero coefficient b very small, and the closed trajectories will be transformed either into spirals going towards the origin or into spiral coming out from the equilibrium, so that the asymptotic behaviour will be absolutely different from the initial 'elliptic' cases.

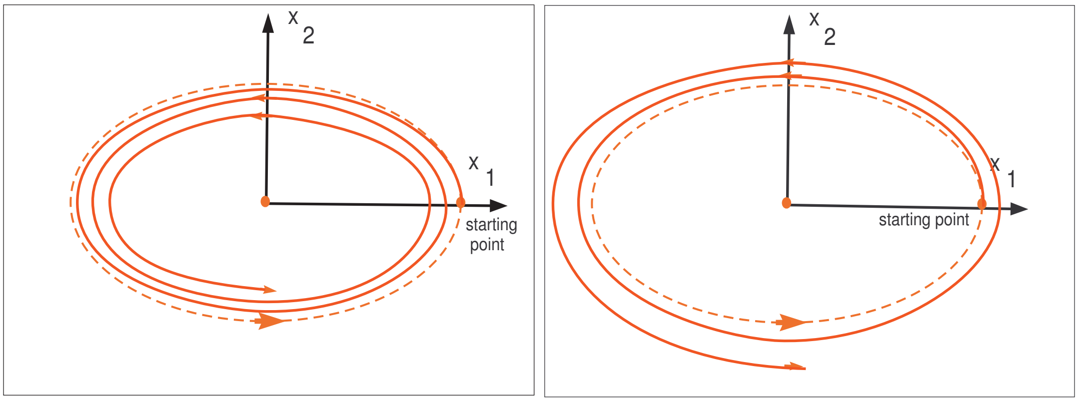

Figure 4. Results of perturbations in the phase space of the ideal pendulum

2. Stability of fixed points

2.1. **One-dimensional systems.** One dimensional systems are defined by a unique equation:

$$\dot{x} = f(x) \tag{2}$$

where $x \in U \subset \mathbb{R}$, and f is a C^1 function. A point $x \in U$ is a fixed points if and only if $f(x) = 0$. The following theorem gives a criterion to establish if a fixed point is hyperbolic, is a stable equilibrium or is an unstable equilibrium.

Theorem. The point p satisfying $f(p) = 0$ is hyperbolic if $\frac{df}{dx}|_{x=p} \neq 0$. Moreover, p is a stable equilibrium (attracting fixed point) if $\frac{df}{dx}|_{x=p} < 0$ and an unstable equilibrium (repelling fixed point) if $\frac{df}{dx}|_{x=p} > 0$

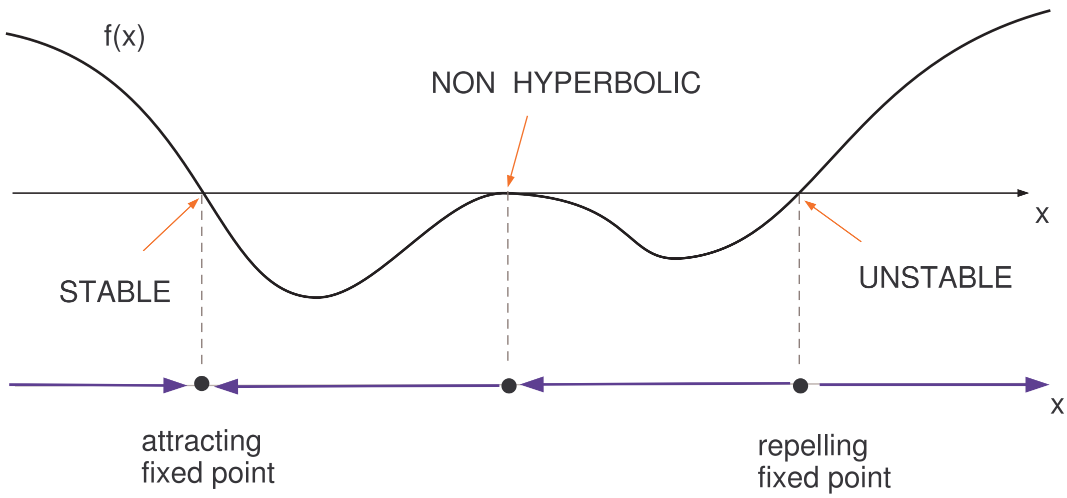

Figure 5. Trajectories (bottom) of the dynamical system defined by the function $f(x)$ (top) with 3 fixed points.

EXERCISE. Let $U = \mathbb{R}$ and $f(x)$ be a polynomial of degree $2n+1$. Which is the minimal number of fixed points? Which is the maximal number of non hyperbolic points? Which is the minimal number (and the maximal number) of stable and unstable points?

2.2. **Bifurcations of equilibrium points.** As we have said previously, a non hyperbolic situation does not persist under perturbation. On the other hand, if we have a system depending on a parameter, then for some special (isolate) value of the parameter a non hyperbolic point may appear. In this case we may have a bifurcation: the behaviour of the system changes.

2.2.1. *Saddle-node bifurcation.* Consider for instance the case where the one-parameter family of monodimensional systems is given by

$$\dot{x} = f(x) + a,$$

where f is the function depicted in Figure 5, and a is a parameter.

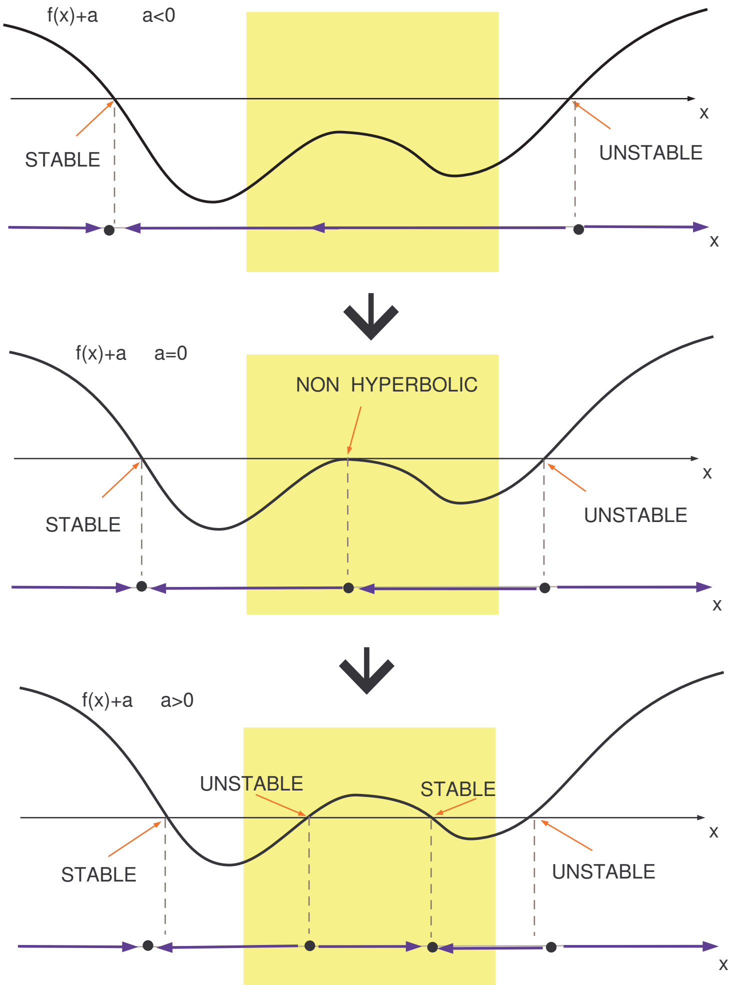

Figure 6. Saddle-node bifurcation in a monodimensional dynamical system

The birth of a pair of hyperbolic fixed points, one attracting and the other repelling, in a segment after the appearance of a point p such that $f(p) = 0$, $f'(p) = 0$ and $f'' \neq 0$ (f' denotes the derivative with respect to x), is called *saddle-node bifurcation*. See Fig.7.

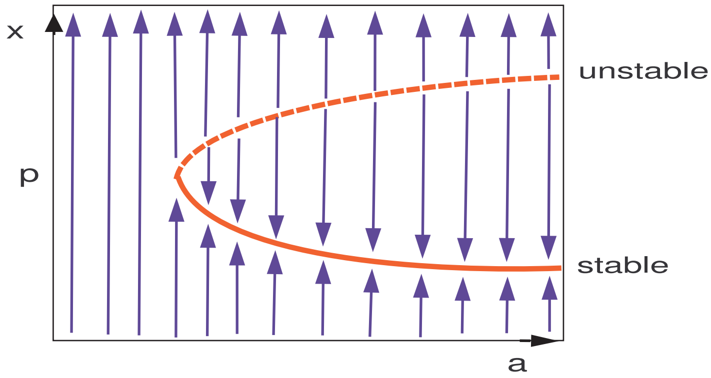

Figure 7. The vertical lines are the corresponding trajectories.

2.2.2. *Pitch-fork bifurcation.* For an isolate value of the parameter the function f_a has a non hyperbolic fixed point p satisfying $f_a(p) = 0$, $f'_a(p) = 0$, $f''_a(p) = 0$ and $f'''_a(p) \neq 0$. See figure.

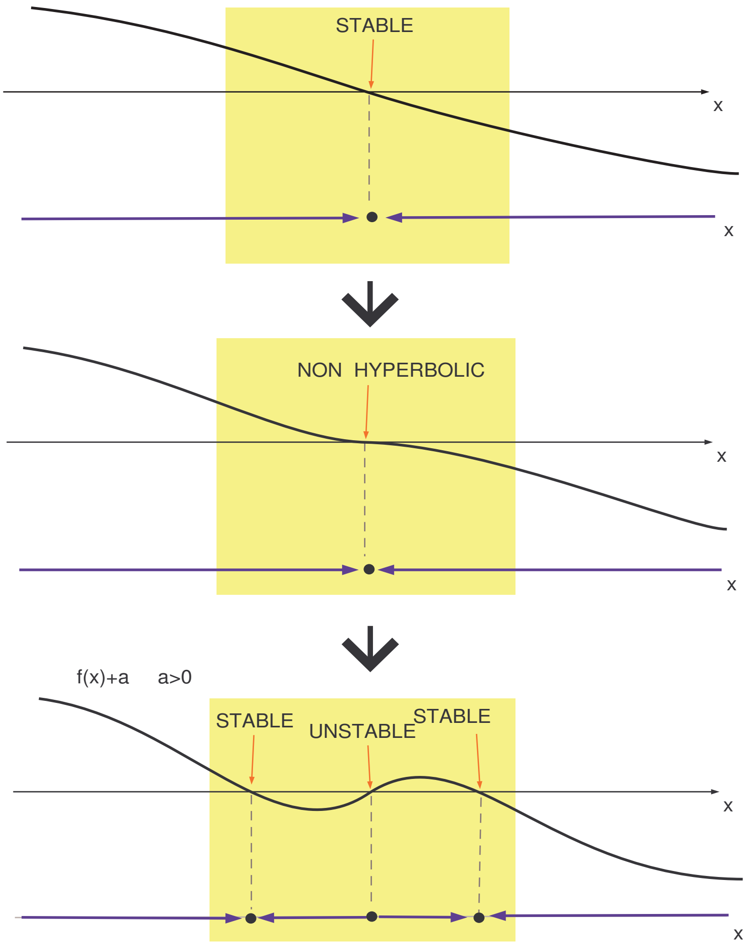

Figure 8. Pitch-fork bifurcation in a monodimensional dynamical system

Observe that in Figure 8 $f'''_a(p) > 0$: the fixed point p loses stability and the new born fixed points are stable. In the opposite case ($f'''_a(p) < 0$): the unstable fixed point p becomes stable and the new born fixed points are unstable.

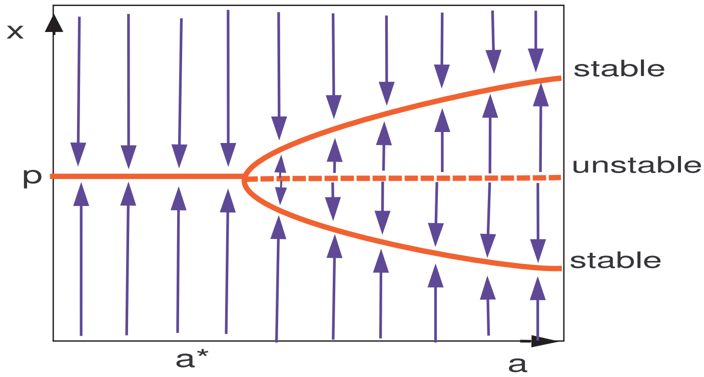

Figure 9. Diagram of a saddle-node bifurcation in a monodimensional dynamical system

EXERCISE. Consider the family of systems

$$\dot{x} = x^3 - ax \tag{3}$$

Say what happen when a changes from negative to positive values.

2.3. **Two-dimensional systems.** In the two dimensional case we may expect new types of behaviour in continuous dynamical systems. In fact, in the asymptotic behaviour there are stable fixed points and limit cycles.

2.3.1. *Hyperbolic fixed points.* A fixed point $\mathbf{p}$ of the equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{p})$ (where $\mathbf{f}$ is $\mathbb{C}^1$) satisfies $\mathbf{f}(\mathbf{p}) = 0$. As in the monodimensional case, we want to know the stability properties of $\mathbf{p}$ (i.e., the behaviour of the system in the vicinity of $\mathbf{p}$) by the properties of $\mathbf{f}$ at $\mathbf{p}$. The derivative, in dimension 2, is replaced by the Jacobian matrix: if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{f} = (f_1, f_2)$ then $J(p) = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{pmatrix}$.

Observe that J is a real matrix and its eigenvalues are roots of a quadratic equation: hence they are either real or complex conjugate.

Theorem. I) The fixed point $\mathbf{p}$ is hyperbolic if both the eigenvalues of $J(p)$ are real and non zero or are complex with non zero real part. II) The fixed point $\mathbf{p}$ is stable (attracting) if both the eigenvalues of $J(p)$ are real and negative or are complex with negative real part. III) The fixed point $\mathbf{p}$ is unstable (repelling) if both the eigenvalues of $J(p)$ are real and positive or are complex with positive real part.

The most important results is that the hyperbolicity (i.e. the 'robustness') of the fixed points is determined only by the Jacobian, so that non linear systems, near to fixed points, are similar and behave in the same way as the linear systems (for which we are able to find the exact solutions).

In the tables at the end of paper we show the vector field and some trajectories around the hyperbolic fixed points. The main observations we have to do are the following. Note that the set of trajectories defined by a vector field is called *phase portrait*.

1) The trajectories going towards an attracting fixed points (or going away from a repelling fixed point) have an asymptotic direction (in the case of real eigenvalues: the point is called *node*) or have not an asymptotic direction, i.e., they approach the point along spirals (in the case of complex conjugate eigenvalues: the point is called *focus*).

2) There are hyperbolic fixed points, which are attracting along one eigen-direction (corresponding to the negative eigenvalue) and repelling along the other direction (corresponding to the positive eigenvalues). They are called *saddle*.

2.3.2. *Non hyperbolic fixed points and bifurcations.* As in the monodimensional case, if the system depends on a parameter, i.e. we are considering a family of dynamical systems $\dot{\mathbf{x}} = \mathbf{f}_a(\mathbf{x})$, for some isolate values of the parameter a non hyperbolic fixed point may appear, so giving rise to a bifurcation.

The simplest case of a hyperbolic point is when one eigenvalue is zero. Consider the example

$$\begin{aligned} \dot{x}_1 &= x_1^2 - a \\ \dot{x}_2 &= -x_2 \end{aligned} \tag{4}$$

If $a < 0$, there are no fixed points. The trajectories look like in Figure 10, left.

If $a = 0$ the origin is a fixed point, but $J((0,0)) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, therefore the origin is non hyperbolic.

For $a > 0$ there are two fixed points $\mathbf{p}_1 = (-\sqrt{a}, 0)$ and $\mathbf{p}_2 = (\sqrt{a}, 0)$. Moreover $J(\mathbf{p}_1) = \begin{pmatrix} -2\sqrt{a} & 0 \\ 0 & -1 \end{pmatrix}$, has two negative eigenvalues, hence $\mathbf{p}_1$ is a node, whereas $J(\mathbf{p}_2) = \begin{pmatrix} 2\sqrt{a} & 0 \\ 0 & -1 \end{pmatrix}$, has one positive and one negative eigenvalue, hence it is a saddle (see Figure 10, right). This type of bifurcation gives the name 'saddle-node' to all bifurcation where a pair of fixed points (stable-unstable) appear.

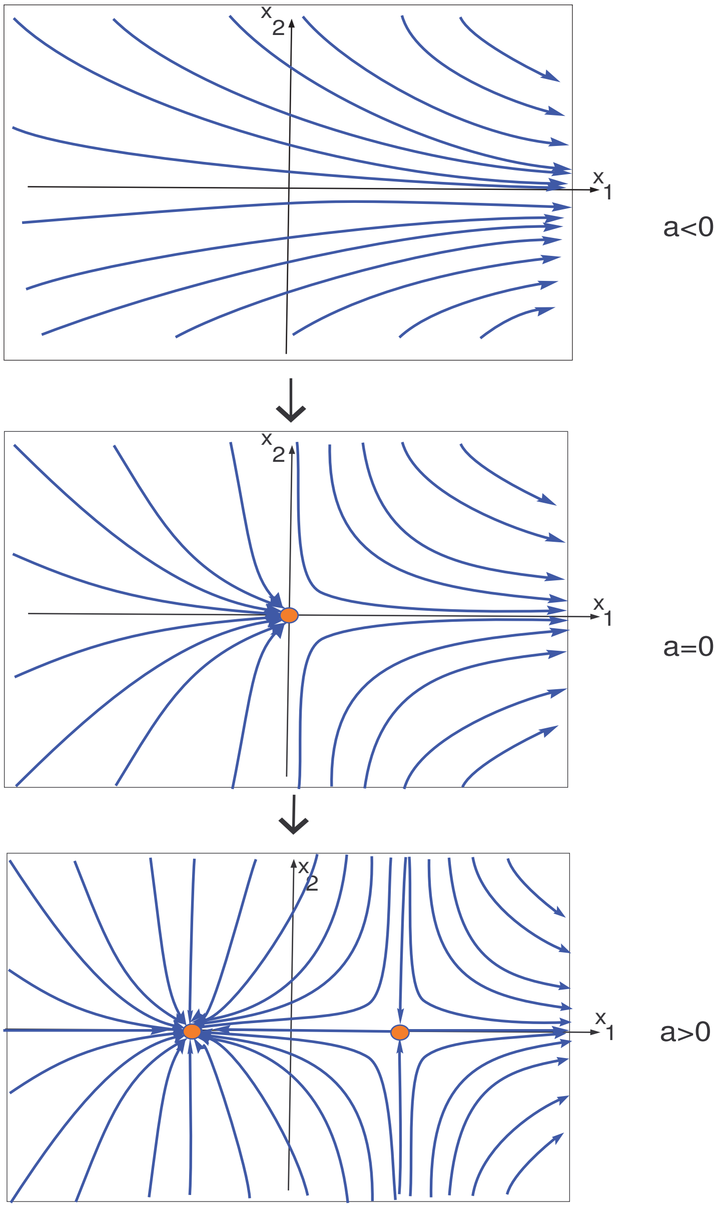

Figure 10. Saddle-node bifurcation in a two-dimensional system

EXERCISES: 1) Consider the family of systems (4) where the second equation is replaced by $\dot{x}_2 = x_2$. How the phase portraits of Figure 10 are changing?

2) Consider the family of systems

$$\begin{aligned} \dot{x}_1 &= x_1^3 - a x_1 \\ \dot{x}_2 &= -x_2 \end{aligned} \tag{5}$$

Say what happen when a changes from negative to positive values.

2.3.3. *Hopf bifurcation.* Another type of bifurcation arising in dynamical systems at least two-dimensional is the so-called Hopf[1] bifurcation.

It may happen when the real part of a pair of complex conjugate eigenvalues of the Jacobian matrix at a fixed point vanishes for a certain value of the parameter. Let us

[1]Eberhard Frederich Ferdinand Hopf (1902 Salzburg, Austria 1983 Bloomington, USA.

suppose that for $a < 0$ the real part is negative and for $a > 0$ the real part is positive. Observe that for linear system when $a = 0$ the fixed point is a 'centre', i.e. is neither attracting nor repelling, and the trajectories around it are closed curves, like ellipses. For $a \neq 0$ these closed curves are transformed into spirals (see also Figure 4), and the fixed point becomes a focus, attracting for $a < 0$ and repelling for $a > 0$.

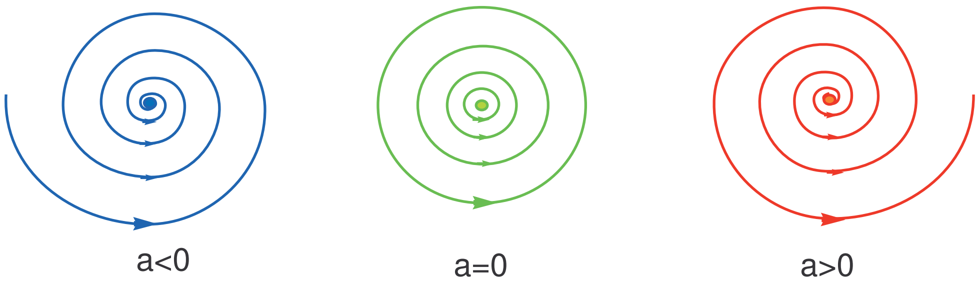

Figure 11. A stable focus becomes unstable

For a family of non linear systems the intermediate situation of 'centre' may be replaced by the birth or the death of a *limit cycle*.

In the *supercritical Hopf bifurcation*, the fixed points loses stability, and passes its stability to an attracting limit cycle that arises for $a = 0$ and grows around it for $a > 0$.

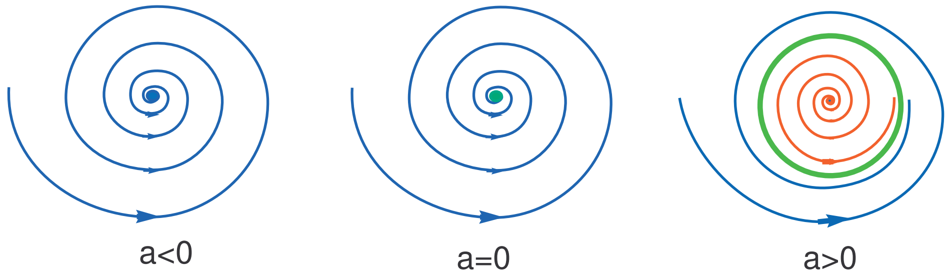

Figure 12. Supercritical Hopf bifurcation

In the *subcritical Hopf bifurcation*, the fixed point loses stability since receives instability from a repelling limit cycle that diminishes around it for $a < 0$ and disappears for $a = 0$.

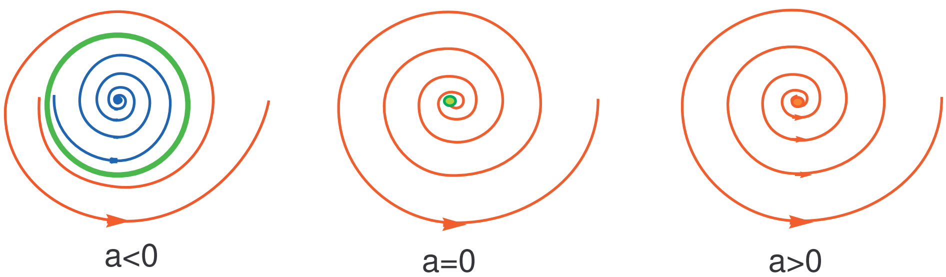

Figure 13. Subcritical Hopf bifurcation

Remark. The Andronov-Hopf theorem gives the necessary condition for the occurrence of the Hopf-bifurcations.

2.4. **Limit cycles in two-dimensional systems.** In the preceding section we have seen the appearance of the limit cycle: it is a closed trajectory, composed of periodic points. If L is a limit cycle for the flow φ^t of a dynamical system at least two-dimensional, then there is a finite time T such that, for every point $\mathbf{x} \in L$, $\phi^T(\mathbf{x}) = \mathbf{x}$. The limit cycles, as the fixed points, are invariant sets for the flow and constitute particular trajectories.

In hyperbolic two-dimensional systems, limit cycles are either attracting or repelling (like in Figures 12 and 13, respectively). But how characterize their stability properties?

2.5. **Poincaré map.** Suppose that the circle of radius 1 centered at the origin in the (xy)-plane is an attracting cycle for a flow φ^t defined by a planar dynamical system. We shall define a function on the semiaxis as follows: for every $x > 0$ we define $f(x)$ as the value x' such that $\varphi^t(x, 0) = (x', 0)$ for the minimal value of t. In other words, we follow the trajectory starting at $(x, 0)$ till the moment when it encounters for the first time the semiaxis $x > 0$. The x-coordinate of this point is $f(x)$. This map is well defined and one can prove that it is smooth if the original flow φ^t is smooth. Then we consider the following discrete one-dimensional system on $x > 0$:

$$x(n+1) = f(x(n)), \tag{6}$$

where n counts the number of intersections of the trajectory with the semiaxis. Evidently, $f(1) = 1$, since the point $(1, 0)$ belongs to the cycle by hypothesis. Therefore 1 is a fixed point for the dynamical system (6), and we can apply the theorem holding for the monodimensional maps: if the fixed point is attracting (repelling) the limit cycle will be attracting (repelling), too. The constructed map is called Poincar'e map.

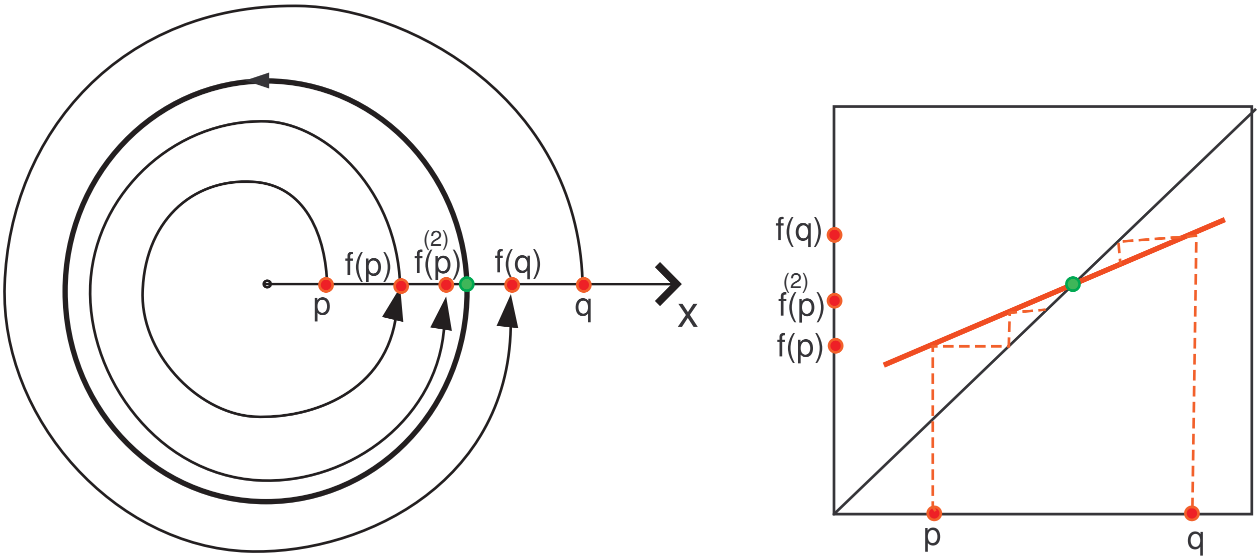

Figure 14. Poincaré map of an attracting cycle in the plane.

The limit cycle is stable if $|df/dx| < 0$ at the fixed point, unstable if $|df/dx| > 0$.

EXERCISE. What happens to the cycle of a system depending on a parameter, if its Poincaré map has a pitch-fork bifurcation?

2.6. **Higher dimensional systems.** A fixed point for a dynamical system in $\mathbb{R}^n$ is a point $\mathbf{p}$ where the vector field vanishes: $\mathbf{f}(\mathbf{p}) = 0$. The hyperbolicity of a fixed point in $\mathbb{R}^n$ is defined as in $\mathbb{R}^2$: a fixed point $\mathbf{p}$ is hyperbolic if the eigenvalues of the Jacobian $J(\mathbf{p})$ are either real and different from zero or complex conjugate with real part different from zero.

A fixed point $\mathbf{p}$ in $\mathbb{R}^n$ is attracting (repelling) if all eigenvalues of $J(\mathbf{p})$ are real and negative (resp., positive) or complex conjugate with negative (resp., positive) real part.

Observe that there are many types of saddles in $\mathbb{R}^n$, depending on the number of positive and negative eigenvalues. The saddle-node bifurcation can be easily generalized to any dimension.

2.6.1. *Limit cycles in higher dimensional systems.* Closed trajectories, also called *periodic orbits*, exist in dynamical systems of any dimension. For dynamical system with more than two-dimensions, hyperbolic periodic orbit can be neither attractive nor attractive. To determine the behaviour of the system near a periodic orbit one again introduces the Poincaré $(n-1)$-dimensional section (transversal to the periodic orbit) and the Poincaré map or *first return map* for the flow. The idea is shown by the following picture, where the space is 3-dimensional and the $(n-1)$-dimensional section is a plane.

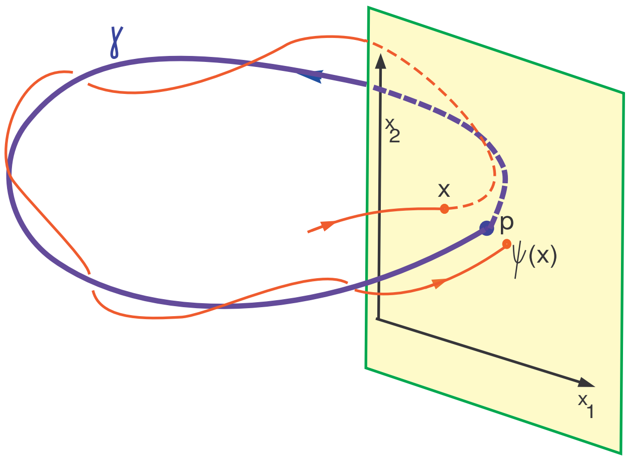

Figure 15. Poincaré section in $\mathbb{R}^3$: p is the fixed point of the map ψ

The map $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as shown in figure. A theorem guarantees that if the dynamical system in $\mathbb{R}^3$ is sufficiently smooth, then also the map ψ is smooth, and hence we can consider its Jacobian at the fixed point.

The eigenvalues of $J(p)$ determines the properties of the curve γ, namely:

γ is hyperbolic if all eigenvalues have modulus different from 1.

γ is an attracting (repelling) limit cycle if all eigenvalues have modulus less (greater) than 1.

γ is a saddle-periodic orbit if it is hyperbolic and is neither repelling nor attracting.

REMARK. To study a continuous dynamical system of dimension 2 (3) we have introduced a discrete dynamical system of dimension 1 (2). This shows the strict relation between the two types of dynamics (continuous and discrete), apparently so different. This technique allows to understand phenomena that - without reduction to smaller dimensions - should be very difficult. The phenomenon of period doubling bifurcation that we have seen in one-dimensional maps has a counterpart in continuous dynamical systems: varying a parameter, a limit cycle becomes unstable and a double cycle appears. See Figure 16

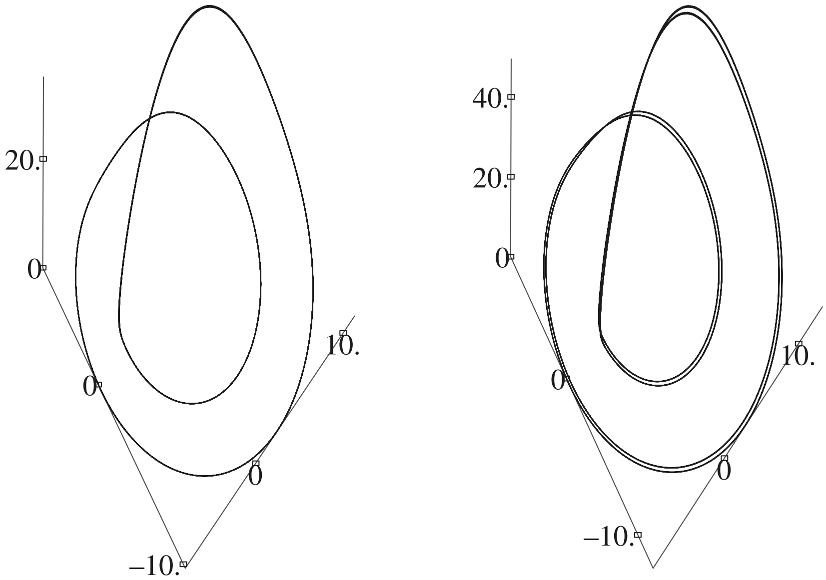

Figure 16. Period doubling bifurcation of a cycle in a 3-dimensional system