Hyperbolic Dynamics - 1^{st} lecture

A discrete dynamical system is a pair (M, f), where M is a space of states (that can be a manifold, a metric space, etc) and $f : M \to M$ is an evolution law. One is interested in study the evolution of some initial state when the law is applied repeatedly.

We define some of the main objects of the dynamics as follows. For $n \in \mathbb{N}$, denote by f^n the composition $f \circ \cdots \circ f$, n times, f^0 is the identity and $f^{-n} = f^{-1} \circ \cdots \circ f^{-1}$, n times (in the case where f^{-1} is well defined).

Definition 1. *The forward orbit of a point $p \in M$ is the set $\mathcal{O}^+(p) = \{f^n(p) : n \in \mathbb{N}\}$. If f is invertible, we define the orbit of $p \in M$ by $\mathcal{O}(p) = \{f^n(p) : n \in \mathbb{Z}\}$.*

Definition 2. *A point $p \in M$ is said to be a fixed point if $f(p) = p$, and it is said to be periodic if there exists $n > 0$ such that $f^n(p) = p$. The least positive integer n such that $f^n(p) = p$ is the period of p (or the minimum period of p).*

The main goal of the study of a specific dynamical system is to describe the orbit of all points in M. It is not always possible, but there are some situations where it is easy to describe some of the orbits. For instance, there are some results that give the existence of fixed or periodic points.

1. If $f : [a, b] \to [a, b]$ is a continuous function from the compact interval $[a, b]$ to itself, then there is a fixed point in $[a, b]$.

2. If $f : I \to I$ is a continuous function from the interval I to itself, and there exists a compact interval $J \subset I$ such that $f(J) \supset J$ (denote this situation by $J \to J$), then there exists a fixed point in J.

3. If $f : I \to I$ is a continuous function from the interval I to itself, and there exist a sequence of compact intervals $J_1, \ldots, J_n$ such that $f(J_1) \supset J_2$, $f(J_2) \supset J_3, \ldots$, $f(J_n) \supset J_1$ ($J_1 \to J_2 \to \cdots \to J_n \to J_1$), then there exists a point $p \in J_1$ such that $f^n(p) = p$ and $f^i(p) \in J_{i+1}$ for all $i = 1, \ldots, n-1$.

These results have many consequences in the context of dynamics of maps defined in intervals. One of the most famous is the Theorem of Sharkovsky, that we state here in a simplified form.

Theorem 3. *Let $f : I \to I$ be a continuous function defined in the interval $I \subset \mathbb{R}$. If there is a periodic point with period 3, then there is a periodic point with period k, for every $k \in \mathbb{N}$.*

Proof. Let x_1 be a point of period 3, and let $f(x_1) = x_2$ and $f(x_2) = x_3$. Assume that $x_1 < x_2 < x_3$ (the other possibilities lead to analogous proofs). Let $I_1 = [x_1, x_2]$ and $I_2 = [x_2, x_3]$. Them we have that $f(I_1) = I_2$ and $f(I_2) = I_1 \cup I_2$.

Since $I_2 \to I_2$, there is a fixed point p_1 in I_2 (period 1). We also have that $I_1 \to I_2 \to I_1$, then there is a periodic point p_2 of period 2 in I_1. These two points are clearly different from x_1, x_2 and x_3, since their periods are less than 3. We also have, for a given integer $n > 3$, the following sequence:

$$I_1 \to I_2 \to I_2 \to \cdots \to I_2 \to I_1,$$

where the interval I_2 appears $n-1$ times. This implies that there exists a point p_n in I_1 such that $f^n(p_n) = p_n$, and $f^i(p_n) \in I_2$ for all $0 < i \le n-1$. Notice that $p_n \neq x_2$, because $f^2(x_2) \in I_1 \setminus I_2$. Then we have that $f^i(p_n) \neq p_n$ for all $0 < i \le n-1$, and the period of p_n is exactly n. □

The Sharkovsky Theorem tells us that maps that appear to be simple can have a very complicated dynamical behavior. In the opposite direction, there are some situations that imply a very simple dynamics.

Theorem 4. *If (X, d) is a complete metric space, and $f : X \to X$ is a contraction[1], then there is a unique fixed point $p \in X$, and $\lim f^n(x) = p$ for all $x \in X$.*

Proof. Let $0 \le \lambda < 1$ be such that $d(f(x), f(y)) \le \lambda d(x, y)$ for every $x, y \in X$. Let $x \in X$. Then we have, for $0 < n < m$,

$$\begin{aligned} d(f^n(x), f^m(x)) &\le \lambda^n d(x, f^{m_n}(x)) \\ &\le \lambda^n (d(x, f(x)) + d(f(x), f^2(x)) + \cdots + d(f^{m-n-1}(x), f^{m-n}(x))) \\ &\le \lambda^n (d(x, f(x)) + \lambda d(x, f(x)) + \cdots + \lambda^{m-n-1} d(x, f(x))) \\ &\le \lambda^n (1 + \lambda + \cdots + \lambda^{m-n-1}) d(x, f(x))) \\ &\le \frac{\lambda^n}{1-\lambda} d(x, f(x)) \to_{n\to\infty} 0. \end{aligned}$$

Then $f^n(x)$ is a Cauchy sequence, and, since X is a complete metric space, it must converge for some $p \in X$. Now, since $\lim f^n(x) = p$, and contractions are continuous maps, we have that $f(p) = f(\lim f^n(x)) = \lim f^{n+1}(p) = p$. It remains to show that p is the only fixed point. In fact, if q is also fixed, then $d(p, q) = d(f(p), f(q)) \le \lambda d(p, q)$. Since $0 \le \lambda < 1$, we have $p = q$. □

In other words, contractions have simple dynamics. By applying repeatedly the evolution law, every initial condition converges to a fixed state.

Definition 5. *Let $f : M \to M$ be a map defined in the metric space M. A fixed point p is said to be an attractor if there is a neighborhood U of p in M such that $\lim f^n(x) = p$ for all $x \in U$. The basin of attraction of p is the set $B(p) = \{x \in M : \lim f^n(x) = p\}$.*

The point p in the statement of theorem 4 is a particular case of attractor, where the basin of attraction is the whole space. There are no fixed nor periodic points in the basin of attraction of an attractor.

If $f : I \to I$ is C^1 map[2] and $|f'(x)| < \lambda < 1$, then f is a contraction. If I is a closed interval, then the theorem 4 can be applied, and we conclude that there is a unique fixed

[1]there exists a constant $0 \le \lambda < 1$ such that $d(f(x), f(y)) \le \lambda d(x, y)$ for every $x, y \in X$.
[2]f has a continuous derivative at every $x \in I$

point in I for f. This is an example of how the derivative of a map gives information about the dynamics.

Definition 6. *Let $f : I \to I$ be a differentiable function defined in the interval $I \subset \mathbb{R}$. A point $p \in I$ is said to be a hyperbolic fixed point if $f(p) = p$ and $|f'(p)| \neq 1$. If $f^n(p) = p$ and n is the period of p, then p is said to be a hyperbolic periodic point if $|(f^n)'(p)| \neq 1$. If $f'(p) = 0$, it is said to be a critical point.*

As a consequence of the definition and the results above, if p is a fixed point such that $|f'(p)| < 1$, then p is an attractor.

Exercises

1. If $f : I \to I$ is a continuous function from the interval I to itself, and there exists a compact interval $J \subset I$ such that $f(J) \supset J$, then there exists a fixed point in J.
2. If $f : I \to I$ is a continuous function from the interval I to itself, and there exists a compact interval $J \subset I$ such that $f(J) \supset J$, then there exists an interval $J_0 \subset J$ such that $f(J_0) = J$.
3. Let $f : [0,1] \to [0,1]$ be defined by

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 \leq x < 1/2 \\ 1 - 2x & \text{if} \quad 1/2 \leq x \leq 1. \end{cases}$$

   Prove (without using Sharkovsky's Theorem) that there are periodic points of all positive periods.