Every mathematical theory, however abstract, is inspired by some idea coming in our mind from the observation of nature, and has some application to our world, even if very unexpected ones and lying centuries ahead. *V.I.Arnold*

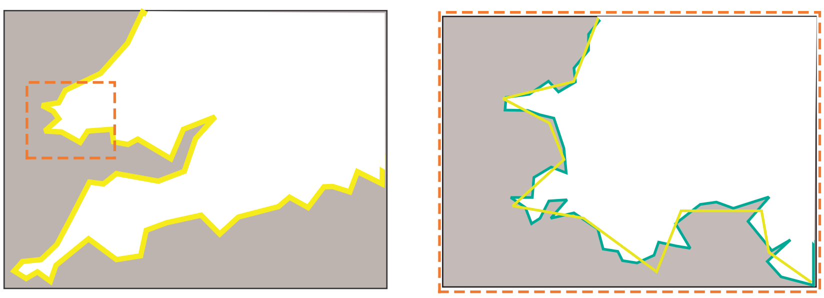
INTRODUCTION TO FRACTAL GEOMETRY

FRANCESCA AICARDI

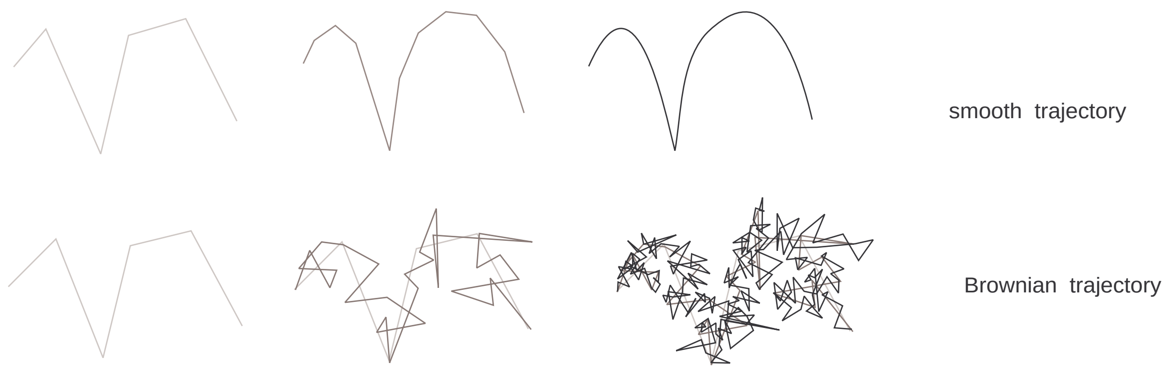
I would like to dedicate these lessons to my master V.I.Arnold, who left us this year. So, I introduce the 'Fractal Geometry' starting from different phenomena of nature, whose description in terms of integer dimensional geometry posed some problems. Then I will present some easy geometrical models to explain the basic ideas underlying the theory of non integer dimension and at the end I will present the mathematical settlement of the theory of the Hausdorrf measure.

1. Phenomena

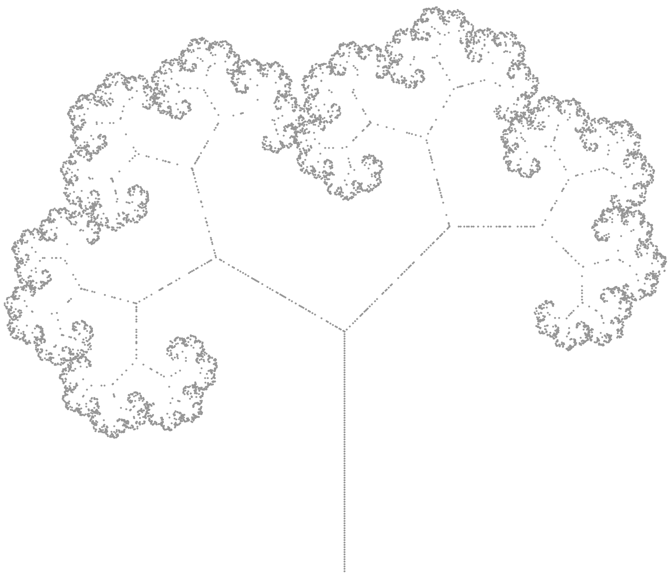
1) *Geography.* At the beginning of the past century Richardson noticed that the length of the England coast, according to different Encyclopedias, varied of about twentyfive percent. It is evident that the measure of a coast indented as the Great Britain coast depends on the scale at which we measure it.



2) *Physics.* A trajectory of an object in space is described by a continuous curve, which can cease to be smooth at some points (for instance, when the object encounters an obstacle and its trajectory changes direction without continuity). However, the trajectories of the Brownian motion (motion of a small particle suspended in a fluid) changes direction 'every time'.



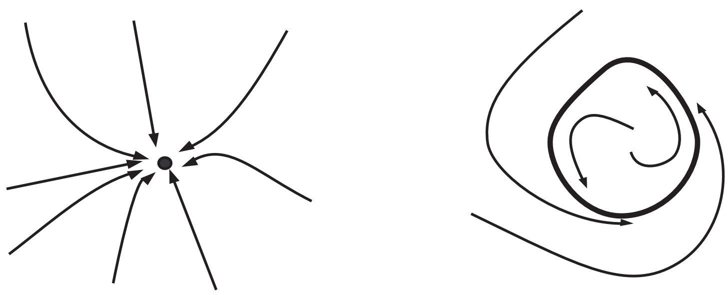
3) *Botanic, Crystallography.* There are forms of composite leaves and in inflorescences, each part of which reproduces the shape of the entire leaf or flower. Also in some crystal growth, each adjoint detail reproduces the form of the entire crystal.



Fractal inflorescence

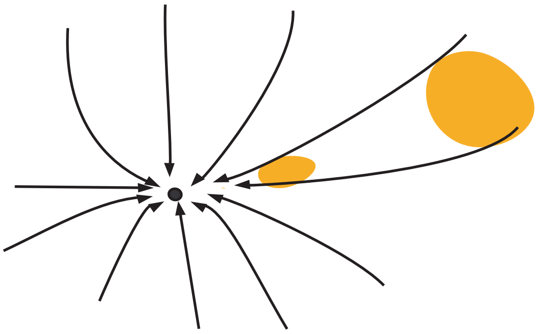
Fractal composite leaf

4) *Dynamical systems.* The temporal evolution of a continuous dynamical system is described by a (smooth) curve in the phase space, say $\mathbb{R}^n$. The equations define a vector field, i.e., a field of 'velocities': it determines at every point the velocity of the trajectory passing through that point. If this field depends continuously on the coordinates, the theorem of existence and uniqueness of the solutions of Ordinary Differential Equations guarantees that the trajectory passing trough any point is unique, i.e., different trajectories cannot have intersections. (Observe that the same trajectory may pass twice trough a point, but in this case it is a periodic trajectory, which passes periodically trough all its points). It may happen that all trajectories, beginning at any point of the space, go towards a unique set of points, which is called for this reason a 'global attractor'. In a two-dimensional space a theorem says that the global attractor may be a fixed point or a closed curve, or a set of fixed points and curves connecting them (These curves are trajectories which tends to the fixed points for time going to infinity and for time going to minus infinity). Observe that the 'volume' of these limit sets (the area, in this case) is zero.

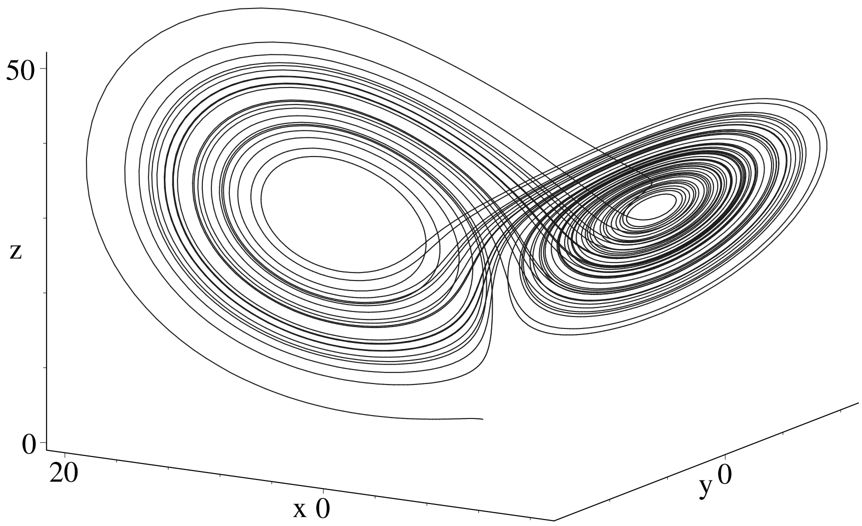


Attracting fixed point (left) and attracting cycle (right).

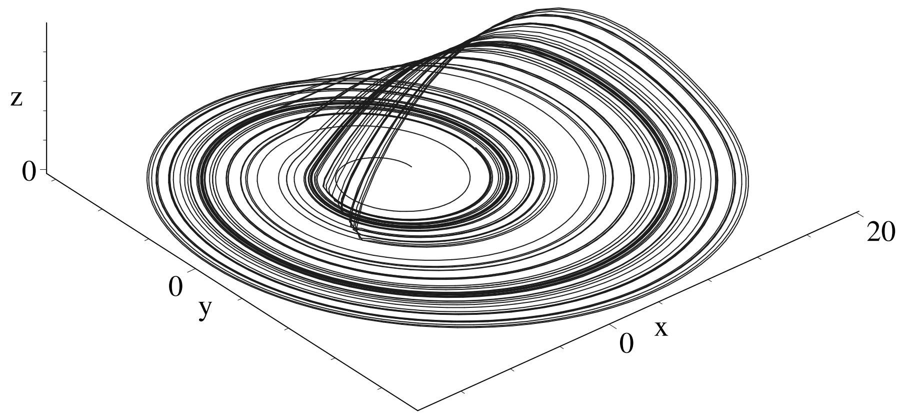
There is a criterion to determine if the flow generated by a dynamical system is contracting. One says in this case that the flow contracts the volume (of the phase space) in the following meaning. Let us choose any ball in the space and consider the time evolution of this ball, i.e., all trajectories with starting points in such a ball. As the time grows to infinity, all these trajectories become closer to the attractor, say a point. Therefore, after a time sufficiently long the trajectories carry the initial points in the ball to a smaller set near the attracting point. So the volume of the initial ball is contracted and tend to zero when the time grows to infinity.



In dimension 3 there are systems for which the flow is contracting, and the attracting set is either a point (zero-dimension), or a closed curve (a one-dimensional set). But it may happen also the following: the flow is contracting, but the attracting set is not of the types typical of two-dimensional systems. The attracting set may be a non periodic trajectory, i.e. a curve which never passes twice at the same point. Such a curve, however, does not lie on a two-dimensional closed surface. (Observe that on a genus-one surface a non periodic curve necessarily intersects itself). It turns out that such attractors have a non integer dimension, bigger than 2.



Lorenz attractor



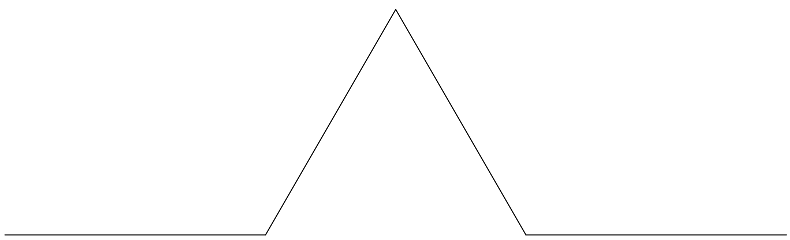
Rössler attractor

2. Geometrical Examples

2.1. **Curves.**

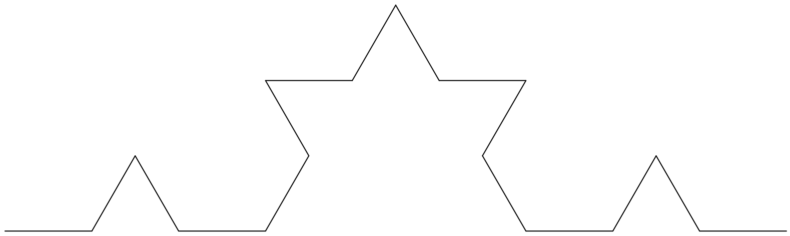
The Koch[1] curve is defined as the limit of the following recurrence procedure.

The curve C_0

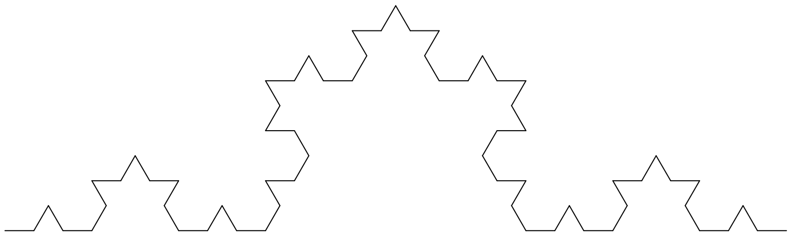


The curve C_1

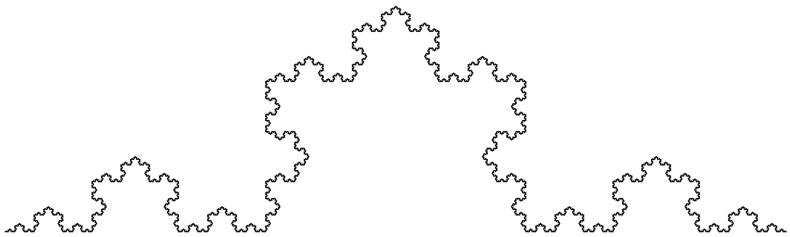
[1]Niels Fabian Helge von Koch (Stockholm, January 25, 1870 ibidem, March 11, 1924) was a Swedish mathematician



The curve C_2



The curve C_3



The curve C_7

Let us suppose that the starting segment, C_0, has length one. We want to measure the length of the Koch curve, as the limit of the lengths of the curves constructed by the recurrence. We get that C_1 has length $4/3$ the length of C_0, and at step n the curve C_n has length $\frac{4}{3}$ the length of C_{n-1}, therefore the length of C_n is equal to $(\frac{4}{3})^n$, which diverges for $n \to \infty$.

The curve C_1, moreover, has 3 angular points (where it is not differentiable), the curve C_2 has $(3 + 4 \cdot 3)$ of such points, the curve C_3 has $(3(1 + 4 + 4^2))$ of such points, i.e. C_n has

$$3(\sum_{j=0}^{n-1} 4^j) = 3 \cdot (4^n - 1)/3 = 4^n - 1$$

of such points, where the derivative has a jump.

Since the length of the segments between such points decreases as $1/3^n$, the limit curve does not admit derivative at any point, and has infinite length.

We have therefore an example of a curve for which the lengths of the approximating curves diverge (like the coast of Great Britain), and which is 'full' of angular points, like the trajectory of the Brownian motion.

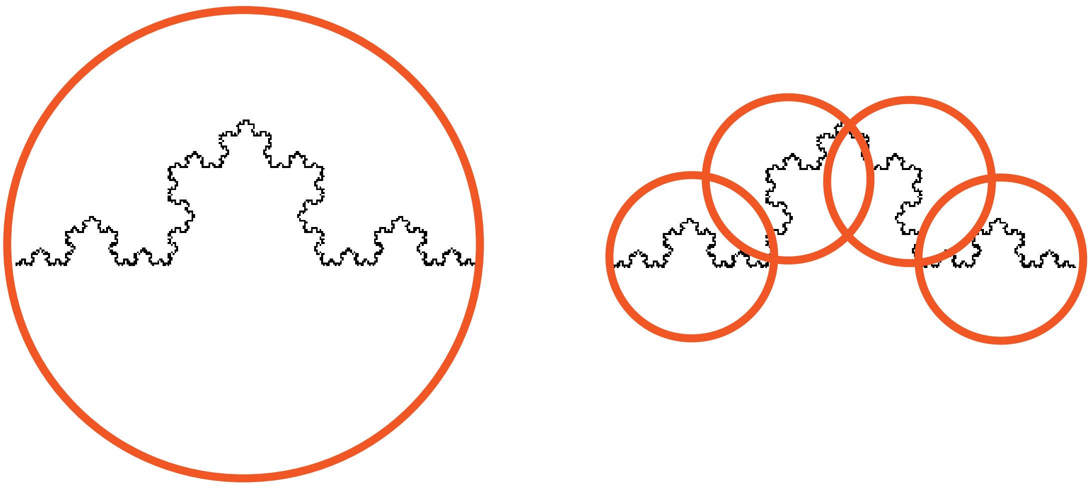
NOTE. A remarkable property of the Koch curve, which is typical of the fractals, is the *self-similarity*: any small part of the curve contains, by a rescaling, the whole curve.

But is it a curve? A curve is a topological object with topological dimension one. Recall the definition of topological dimension of a topological space X.

Let $\mathcal{U}$ a cover of X by open sets U_i. A refinement $\mathcal{V}$ of $\mathcal{U}$ is another cover of X by open sets V_j such that every V_j is contained in some U_i. The Lebesgue covering (or topological) dimension of X is then defined to be the minimum value of n, such that every cover of X has a refinement in which no point of X is in more than $n+1$ elements. If no such minimal n exists, the space is said to have an infinite covering dimension.

For a line such a dimension n satisfies evidently $n+1=2$, for a surface $n+1=3$:

For the Koch curve we also get $n+1=2$, so that its topological dimension is one.



Two covers of the Koch curve, the cover at right is a refinement of the cover at left.

For the Koch curve we also get $n+1=2$, so that its topological dimension is one.

We introduce now the definition of *fractal* dimension. We will give later the exact formal definition of the Hausdorff dimension (1919)[2].

Let X be a metric space. Consider a cover of X, by simplicity, by open spheres of diameter r. Let $N(r)$ be the minimal number of such spheres covering X. If, reducing r, the number $N(r)$ of spheres necessary to cover X grows according to the rate

$$N(r)/N(r_0) = (r/r_0)^{-d},$$

for an arbitrarily small radius, d is called the fractal dimension of X. I.e.,

$$d = \lim_{r\to 0} \frac{\log(N(r))}{\log(1/r)}.$$

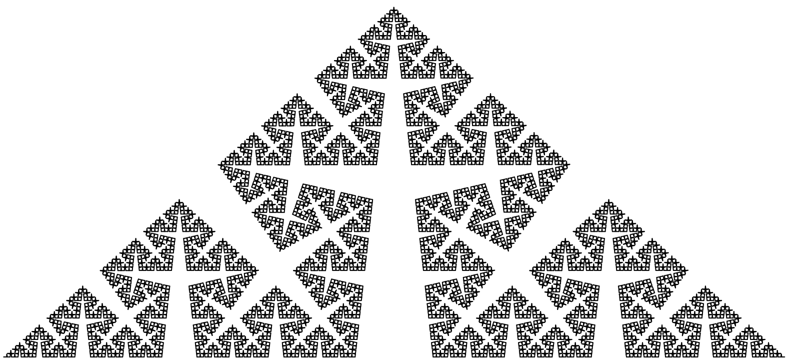
[2]Felix Hausdorff (November 8, 1868 January 26, 1942) was a German mathematician

We apply now this definition to the Koch curve. The self-similarity of the curve allows us to find, for $r = (1/3)^n$, $N(r) = 4^n$. Therefore

$$d = \lim_{n\to\infty} \frac{\log(4^n)}{\log(3^n)} = \frac{\log(4)}{\log(3)} \approx 1.2619$$

So the Koch curve has fractal dimension bigger than that of a smooth line.

To understand that such a fractal curve occupies in fact a 'bigger' portion of the plane than a smooth line, consider for instance a generalization of the Koch curve, where the length of the four segments replacing a segment is not 1/3 of the length of that segment, but .45. The obtained curve is

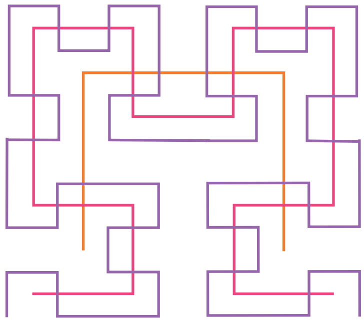


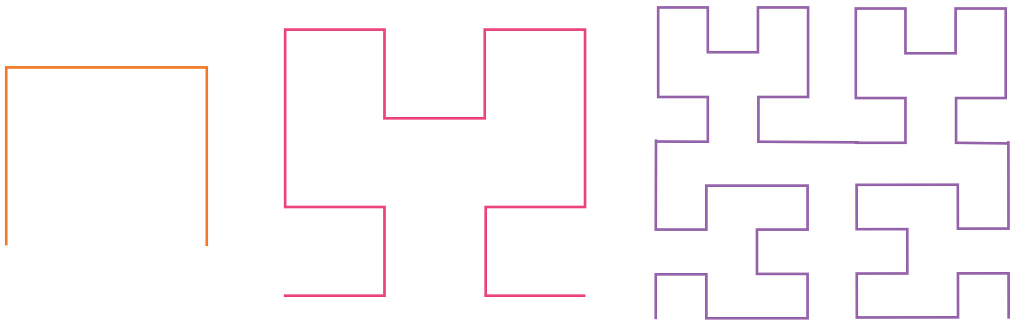
which has dimension $\frac{\log(4)}{\log 20/9} = 1.7361$, closer to 2 than the dimension of the Koch curve.

A plane fractal curve can have fractal dimension equal to 2, i.e., when it fills an entire planar domain. It is the case of the Peano [3] curve (1890). There are many different realizations of such curves, and they may have different properties.

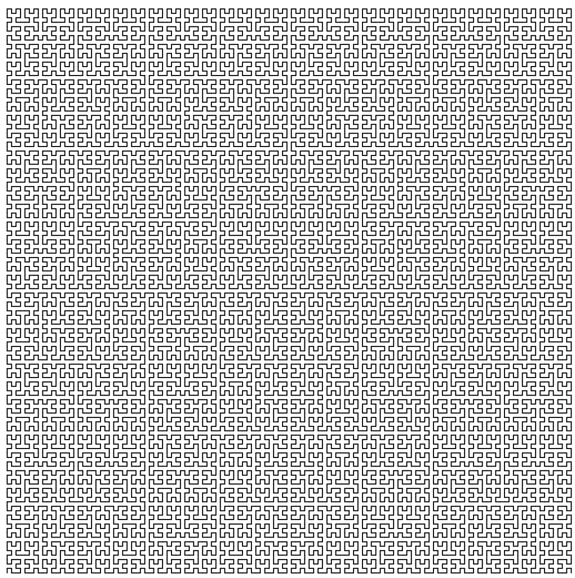
For instance, the curve, invented by Hilbert (1891), as the limit of this procedure

[3]Giuseppe Peano (27 August 1858 20 April 1932) was an Italian mathematician





is a curve which is not differentiable at any point.



Curve approaching the Hilbert curve at the seventh iteration

Remark. There are also curves filling the plane which are almost everywhere differentiable.

2.2. **Cantor set and Cantor dusts.**

The Cantor[4] set (1883) is defined as follows:

[4]Georg Cantor (1845-1918) was a German mathematician

Start from the unit segment. Then divide it into three segments and remove the (open) central segment. Then repeat this procedure indefinitely. The limit set is the Cantor set.

It is easy to see that the measure of the Cantor set is zero. Indeed, we can measure the length of all removed segments, which are, step by step:

$$\frac{1}{3}, \qquad 2\frac{1}{9}, \qquad 4\frac{1}{27}, \quad \dots \quad 2^n\frac{1}{3^{n+1}}.$$

Their summation is equal to the length of the entire segment:

$$\frac{1}{3}\sum_{n=0}^{\infty}(2/3)^n = \frac{1}{3}\frac{1}{(1-2/3)} = 1.$$

REMARK. The Cantor set is uncountable and nowhere dense.

By removing a central part of length 1/4, one obtain a similar Cantor set which is called Smith-Volterra-Cantor set. Also this set is uncountable and nowhere dense.

EXERCISE. Find its fractal dimension.

The Cantor set has by construction self-similarity property and has a non zero fractal dimension. Using the above definition, we obtain that its fractal dimension is

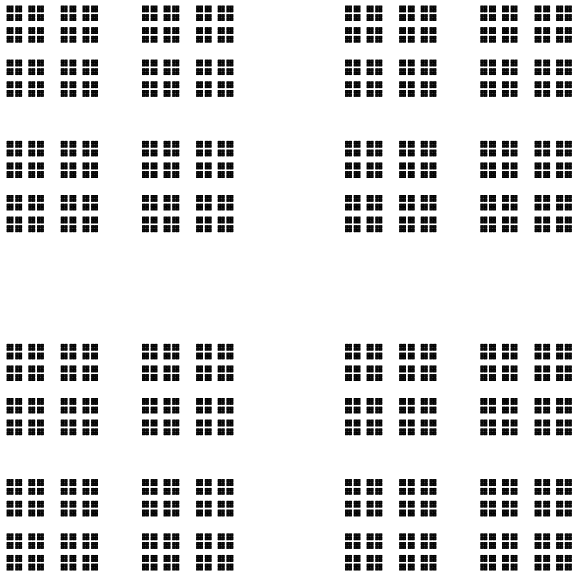
$$d = \lim_{n\to\infty}\frac{\log(2^n)}{\log(3^n)} = \frac{\log(2)}{\log(3)} \approx 0.6309.$$

A middle-α Cantor set is a set like the Cantor set where the ratio between the length of the removed segment and the length of the initial entire segment is α. A middle-α Cantor set is by construction self-similar.

EXERCISE. Find the fractal dimension of a middle-α Cantor set ($0 < \alpha < 1$) as function of α.

We may consider the Cartesian product of the Cantor set by a segment, or of a Cantor set by itself, or of a Cantor set by another Cantor set.

The Cartesian product of two Cantor sets in $\mathbb{R}^2$ or of 3 Cantor sets in $\mathbb{R}^3$ is called *Cantor dust*.



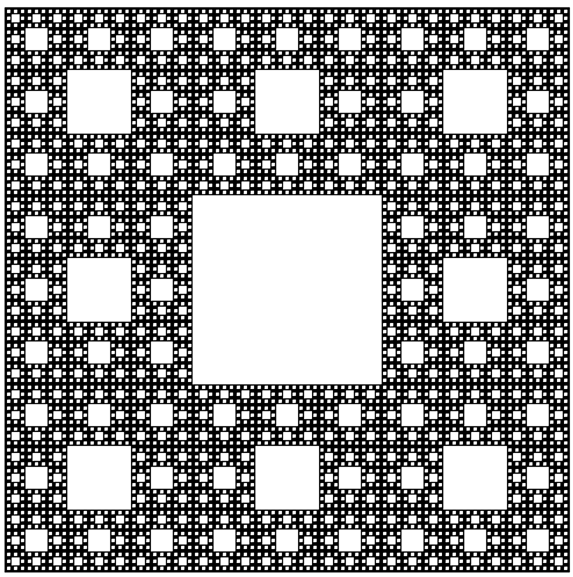
Cantor dust: it is the Cartesian product of 2 middle-α Cantor set ($\alpha = 1/5$).

EXERCISE. Prove that the fractal dimension of the Cartesian product is the sum of the fractal dimensions of its factors.

2.3. **Sierpinski carpet, Menger sponge.**

We may obtain self-similar sets by starting by a cube in any dimension, subdivide it into sub-cubes, remove someone of them, and successively repeat the procedure for all sub-cubes remained, and so on.

The simplest examples are the Sierpinski carpet in a square, and the Menger sponge in a cube.

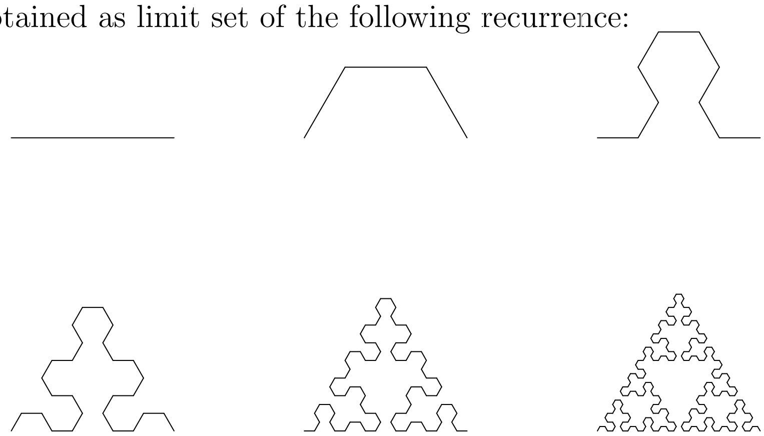


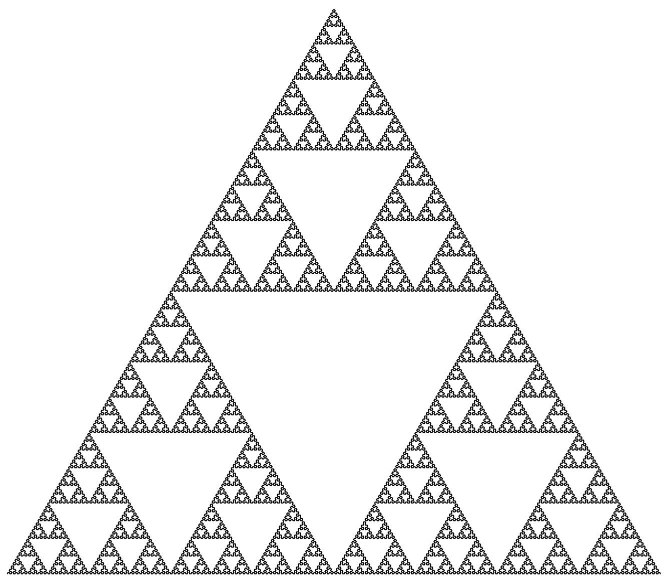
Sierpinski carpet

The fractal dimensions of these objects are respectively $\log(8)/\log(3) = 1.8928$ (we remove the central one of nine squares) and $\log(20)/\log(3) = 2.7268$ (we remove the central one of 27 sub-cubes and each sub-cube at the centre of the 6 faces).

REMARK. Fractal sets with the same dimension may be topologically different (i.e., may have different topological dimension).

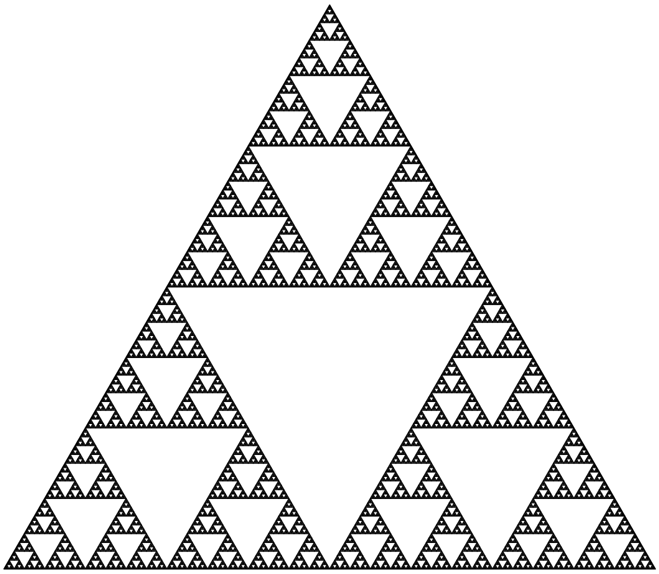
Observe that the fractal dimension of an object is independent from the procedure by which we define it. Consider for instance the following objects. The first one is defined as the curve obtained as limit set of the following recurrence:





Fractal curve

The second one is defined, like the Sierpinski carpet, as a triangle from which a smaller triangle is removed.



Triangular Sierpinski carpet

EXERCICE. Prove that they have the same fractal dimension $\log(3)/\log(2) = 1.5850$

CURIOSITY. Consider the infinite Pascal Triangle, i.e., for each $n = 0, 1, 2, \ldots$, put in sequence $\binom{n}{k}_{k=0}^{n}$ modulo 2. The obtained triangle of zeros (white) and ones (black) form the triangular Sierpinski carpet.

3. The theory of Hausdorff dimension

The fractal dimension of a set X is also called *box-dimension*, and is denoted by $d_b(X)$. Instead of spheres of radius r we may cover X by n-dimensional cubes of side r. Also, if $X \subset \mathbb{R}^n$, we may consider at the beginning the partition of $\mathbb{R}^n$ by hyperplanes into cubes of side r, and then count $N(r)$ as the minimal number of such cubes covering X. These definitions coincide.

As before, we suppose $X \subset \mathbb{R}^n$.

Let U be a nonempty subset of $\mathbb{R}^n$. The *diameter* of U is defined as

$$|U| = \sup\{|x - y| : x, y \in U\}.$$

Let $\delta > 0$. A δ-cover of X is a cover of X by sets U_i which have diameter less or equal to δ.

We define

$$\mathcal{H}^p_\delta(X) = \inf \sum_{i=1}^{\infty} |U_i|^p.$$

where the infimum is taken over all countable δ-covers. Observe that the U_i are different, only their diameter does not exceed δ.

The *Hausdorff p-dimensional measure* of X is defined, for every $p \geq 0$ as

$$\mathcal{H}^p(X) = \lim_{\delta \to 0} \mathcal{H}^p_\delta(X)$$

The number $\mathcal{H}^p_\delta(X)$ increases as δ diminishes, therefore this limit exists but can be equal to infinity.

The *Hausdorff dimension of* X is d if

$$\mathcal{H}^p(X) = \infty \quad \forall p < d, \quad \mathcal{H}^p(X) = 0 \quad \forall p : d < p \leq \infty.$$

It is denoted by $\dim_H(X)$.

Observe that if $X \subset \mathbb{R}^n$ is a compact set, the Hausdorff n-dimensional measure of X coincides with the Lebesgue measure of X. Moreover:

$$\dim_H(X) \leq \dim_b(X) \leq n.$$

We have to remark that the theory of the Hausdorff measure can be applied to any Borel set[5] in $\mathbb{R}^n$, which is not necessarily a 'nice' fractal.

A Cantor set C is defined more generally as a subset of a topological space such that is totally disconnected (the connected components are single points), perfect (is closed and every point $x \in C$ is the limit of points $x_n \in C$ different from x), and compact (each open cover has a finite subcover).

Therefore, there are Cantor sets with positive Lebesgue measure. This can be reached removing a portion $\alpha = \alpha(n)$ at the step n which is a decreasing function of n such that $\sum_{n=1}^{\infty} \alpha(n) < \infty$. In this case the self-similarity is lost.

4. The Mandelbrot set

The Mandelbrot set[6] is a mathematical set of points in the complex plane, the boundary of which forms a fractal. Consider the following iterate map in the complex plane:

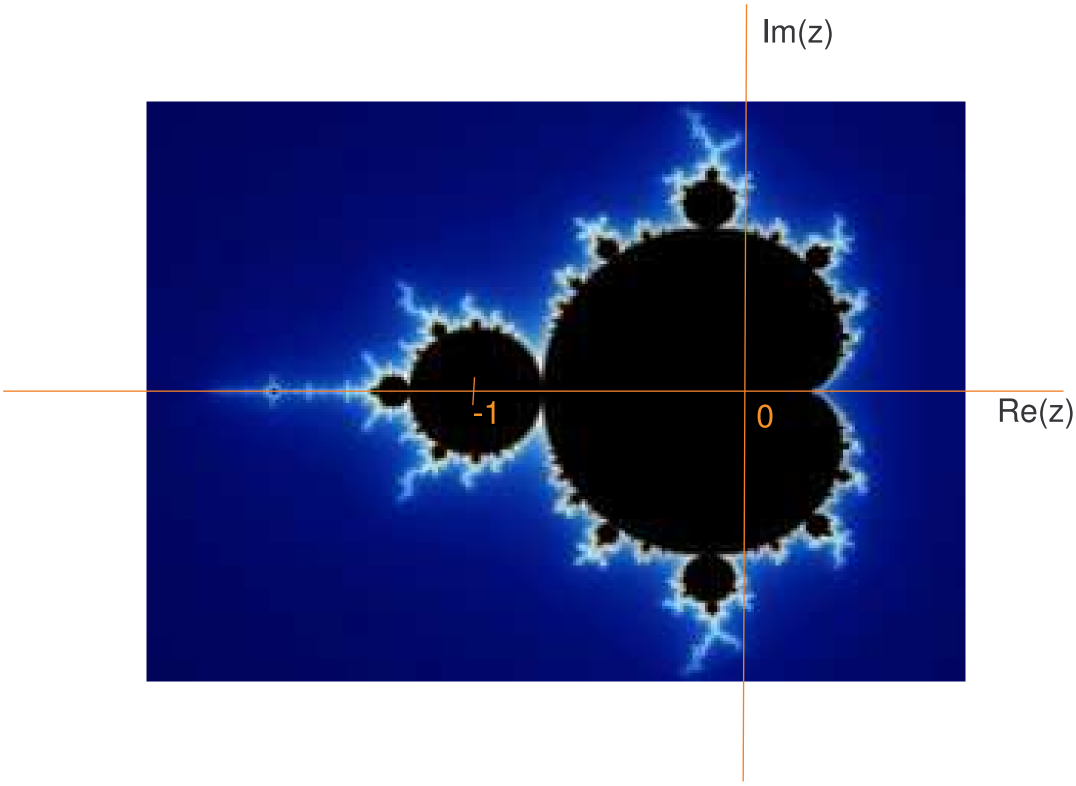
$$z_{n+1} = z_n^2 + c$$

where $c \in \mathbb{C}$. Starting from $z_0 = 0$, one gets a series of complex numbers: this series is bounded or not. If this series is bounded, then the value of c is put in the diagram as a black dot. For example, letting $c = 1$ gives the sequence 0, 1, 2, 5, 26,, which tends to infinity. As this sequence is unbounded, 1 is not an element of the Mandelbrot set.

On the other hand, $c = i$ (where i is defined as $i^2 = -1$) gives the sequence $0, i, (-1 + i), -i, (-1 + i), -i, ...$, which is bounded and so i belongs to the Mandelbrot set. The boundary of the black set in figure is very complicated, and has the self-similarity property. The white part helps to understand how complicate is this boundary, which is actually a fractal curve. Shishikura (1994) proved that the boundary of the Mandelbrot set is a fractal with Hausdorff dimension 2, the same dimension as the total set.

[5]a Borel set is a set in a topological space that can be obtained from open sets (or, equivalently, from closed sets) by the operations of countable union, countable intersection, and relative complement. Borel sets are named after mile Borel (1871-1956), French mathematician

[6]The Mandelbrot set is named after Benoît Mandelbrot, who studied and popularized it.



The Mandelbrot set is black.

5. Final Remarks

Fractals are commonly defined as sets with self-similarity. For this reason their fractal dimension is very easy to compute: we do not need to control the coverings for r infinitely small, since the portrait is invariant under a scale changing.

In nature, self-similarity may happens at many levels of rescaling, but of course is not true *at every level*, therefore the fractal dimensions that are attributed to real things, as the coast of Great Britain, or the surface of a cavoli-flower, must be intended as 'finite step' approximations of abstract fractal geometric models.

In the case of strange attractors, it may happen that the invariant set (the attractor) has a self-similar structure, and a non integer Hausdorff dimension. For the Lorenz attractor this dimension exceeds 2 by a very little amount: 0.06, for the Rössler attractor by 0.02.

EXERCICES

1) Let $X = \{0\} \cup \{1/k : k \in \mathbb{N}\}$. Prove that $\dim_b(X) = 1/2$.

2) Given any $0 < d < 1$, find the middle-α Cantor set with Hausdorff dimension d. Prove that its Lebesgue measure is zero.

3) Construct a Cantor set in the line with box dimension equal to 1.