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Conference on Cold Materials, Hot Nuclei, and Black Holes: Applied Gauge/Gravity Duality

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Relativistic heavy ion collisions, entropy and thermalization

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Heavy Ion Collisions, Thermalization and Entropy Production

Thanks to:	*Vijay Balasubramanian*
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Berndt Müller

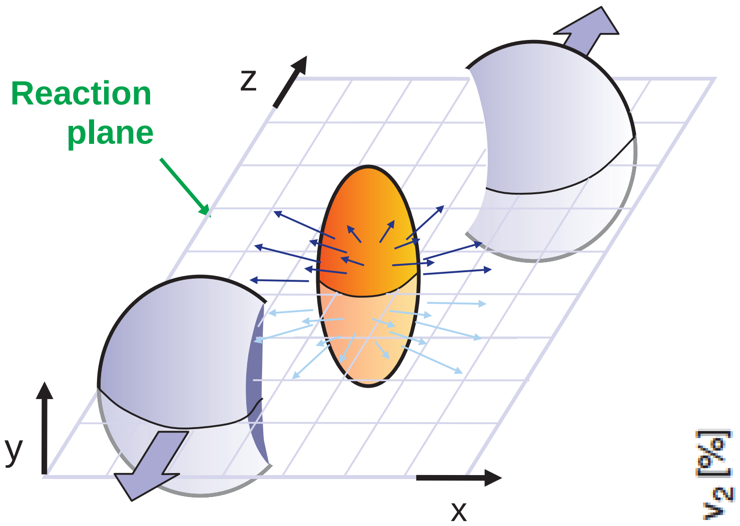
Conference on Cold Matter, Hot Nuclei, and Black Holes

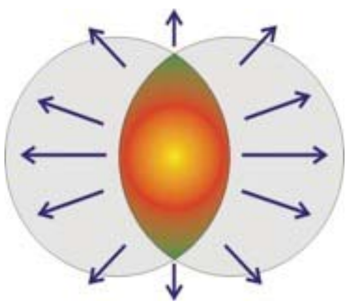
ICTP, Trieste - 19 August 2011

Overview

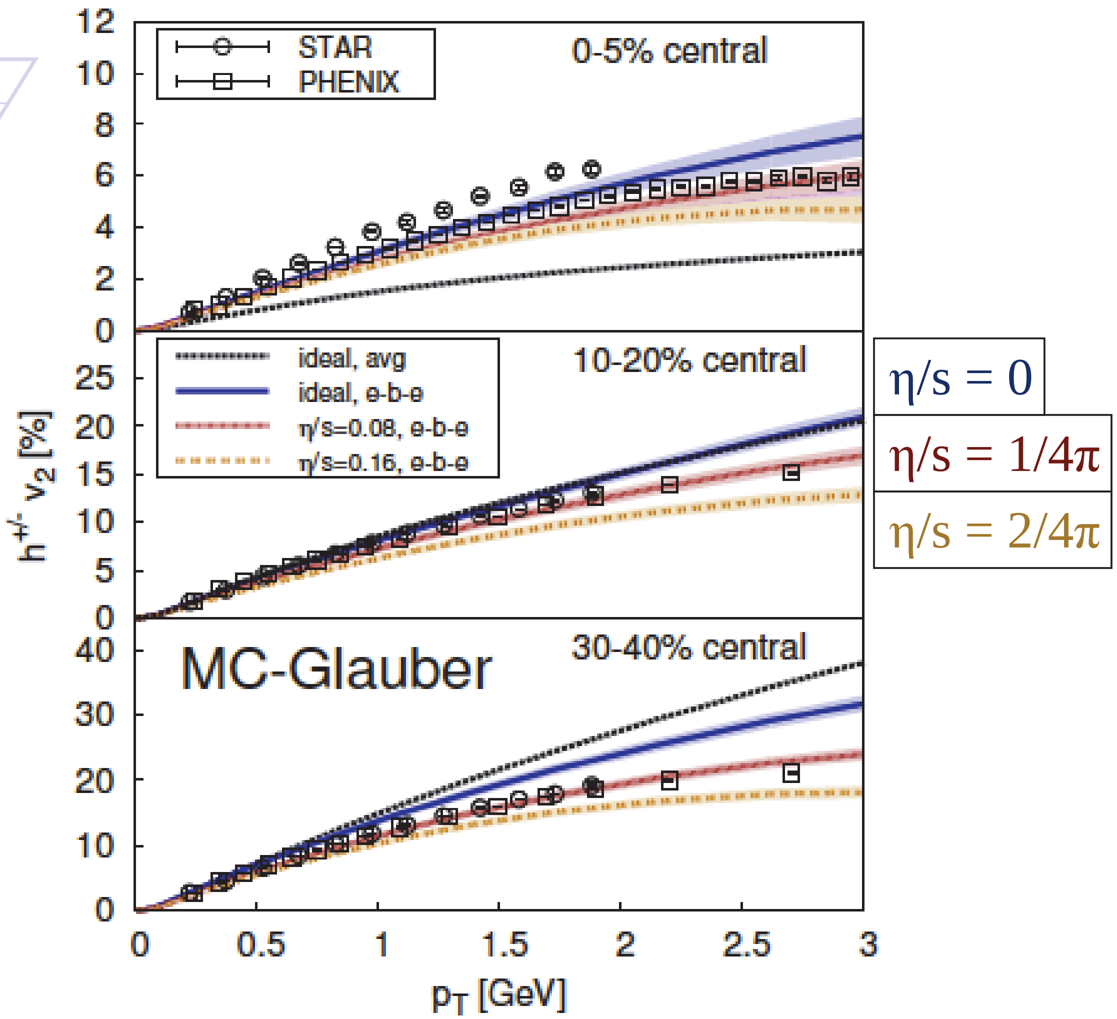
- Flow correlations from RHIC & LHC
- Entropic history of a heavy ion collision
- von Neumann entropy
- Coarse grained entropy
- Nakajima-Zwanzig projection formalism
- Relevant entropy
- Entanglement entropy
- Husimi-Wehrl coarse graining
- Eigenstate thermalization hypothesis
- Examples (YMQM, LQCD)

Thermalization & perfect fluid



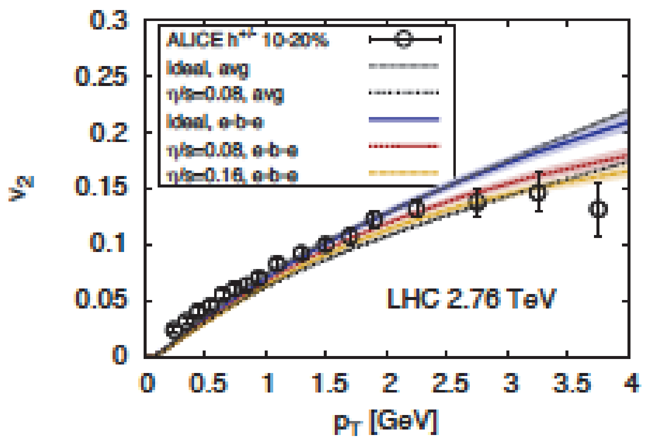


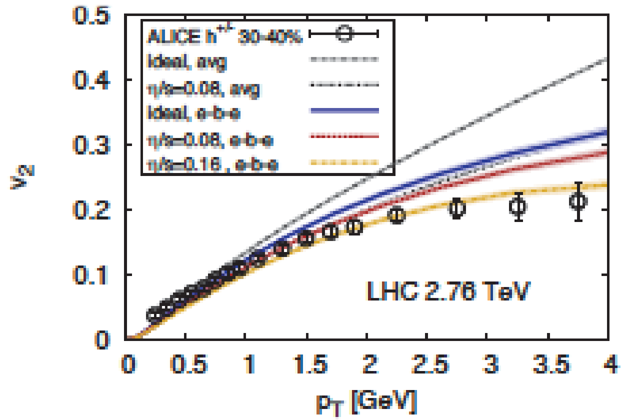
∇P(↔) > ∇P(↕)

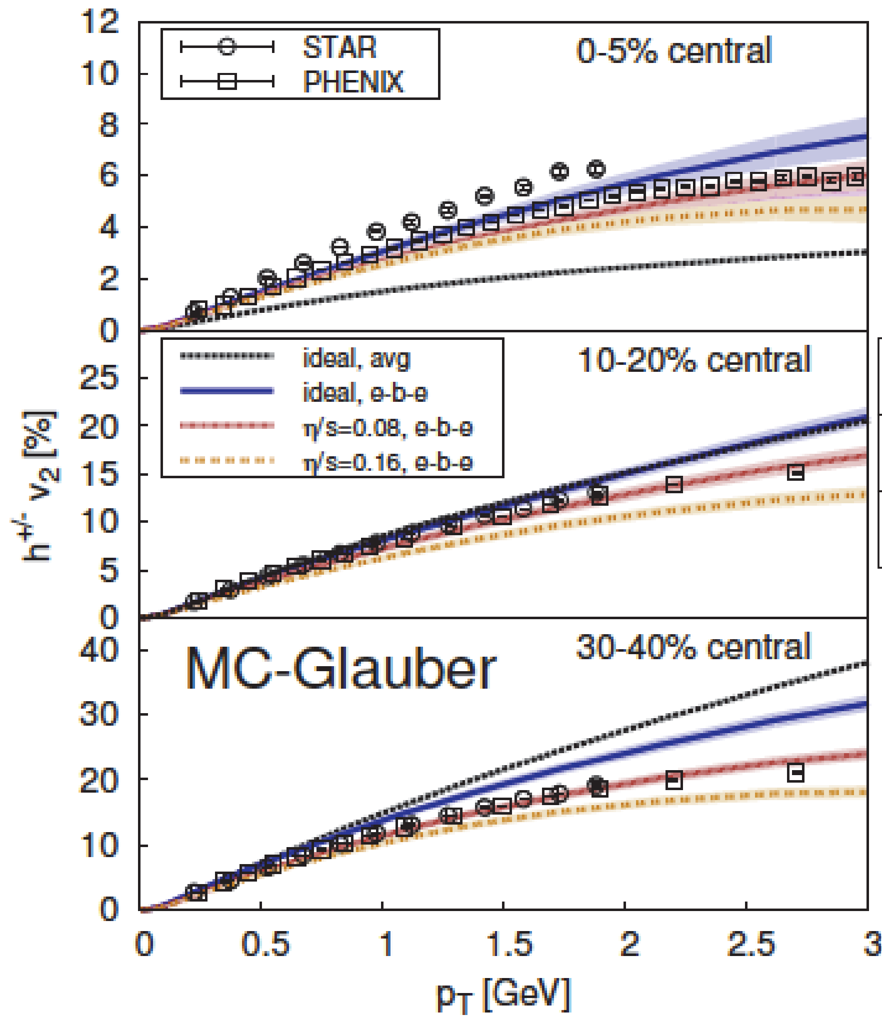


Thermalization & perfect fluid

LHC results agree almost perfectly with RHIC







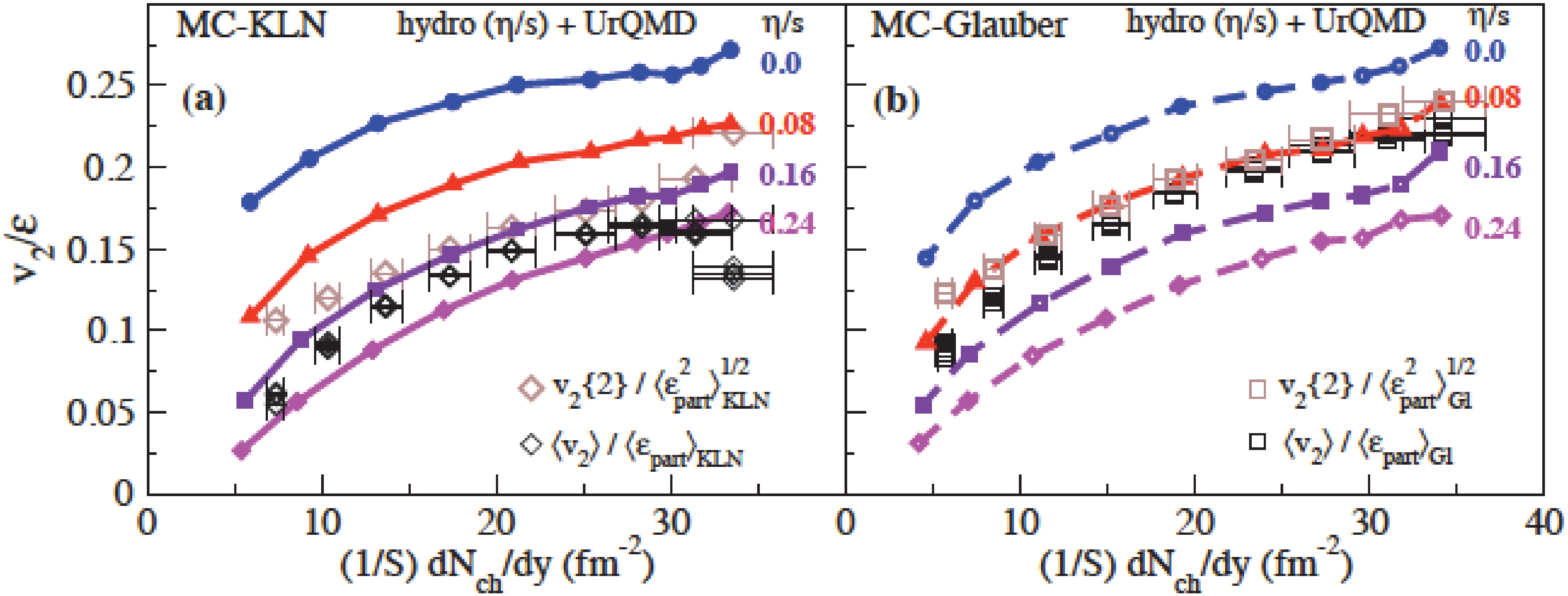
$\eta/s = 0$

$\eta/s = 1/4\pi$

$\eta/s = 2/4\pi$

Shear viscosity

Song, Bass, Heinz, Hirano, Shen, PRL 106 (2011) 192301 [Duke - OSU - Tokyo]

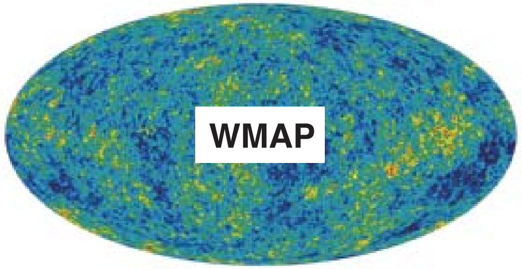


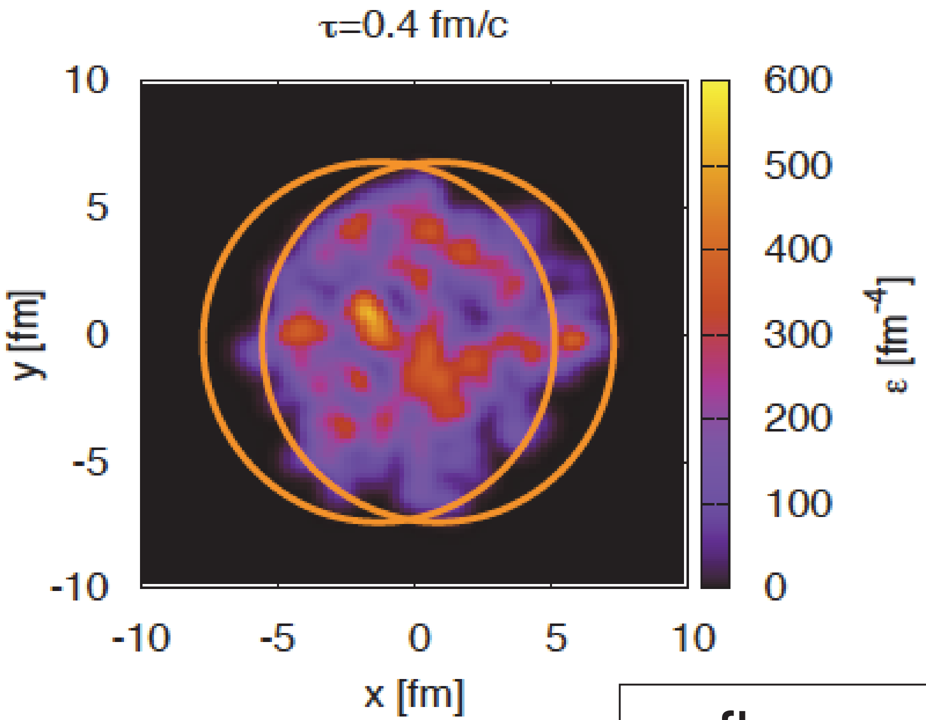
Conclusion: $1 \leq 4\pi\eta/s \leq 2.5$

Remaining uncertainty mainly due to initial density profile

Event-by-event fluctuations

Initial state generated in A+A collision is grainy
event plane ≠ reaction plane
⇒ eccentricities ε_1, ε_2, ε_3, ε_4, etc. ≠ 0

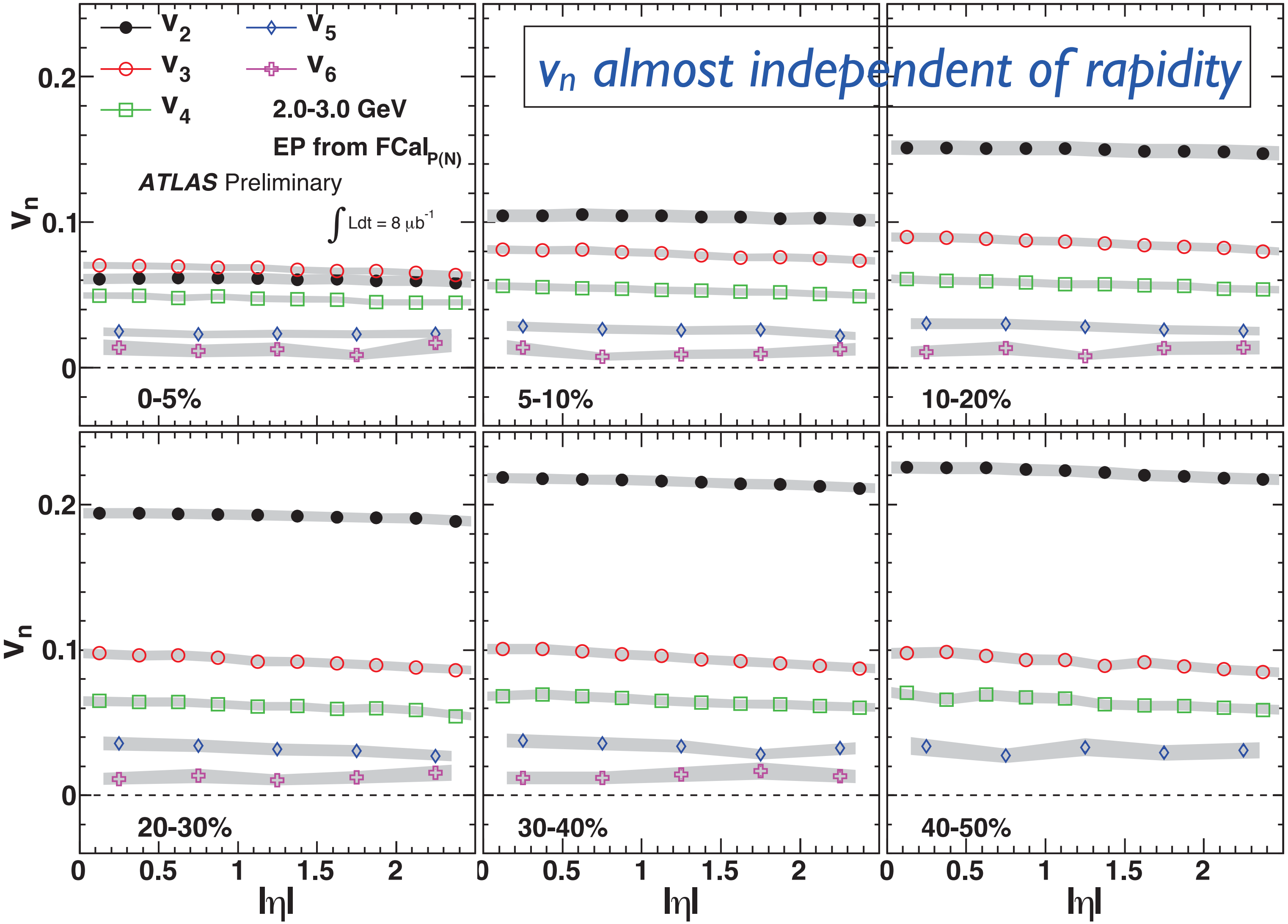




Idea: Energy density fluctuations in transverse plane from initial state quantum fluctuations. These thermalize to different temperatures locally and then propagate hydrodynamically to generate angular flow velocity fluctuations in the final state.

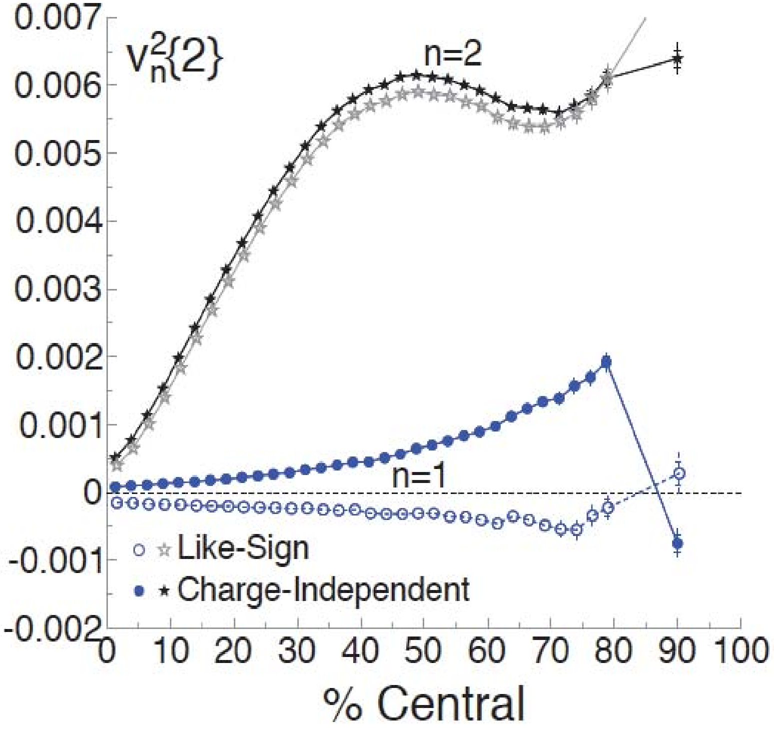
⇒ flows v_1, v_2, v_3, v_4,...

v_n (n = 2,...,6)

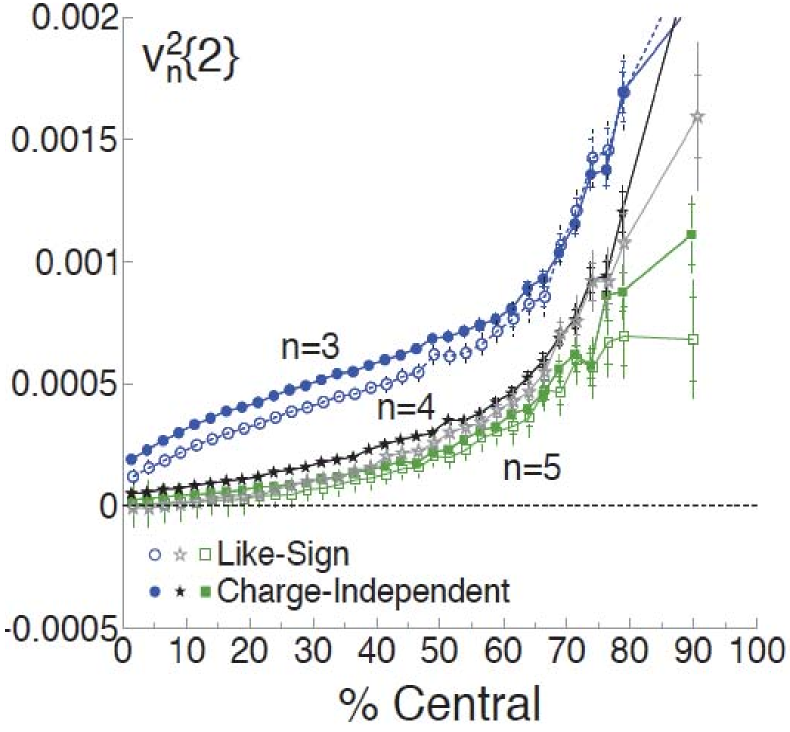


v_n almost independent of rapidity
v_2
v_3
v_4
v_5
v_6
2.0-3.0 GeV
EP from $FCal_{P(N)}$
ATLAS Preliminary
$\int Ldt = 8\ \mu b^{-1}$
v_n
0-5%
5-10%
10-20%
20-30%
30-40%
40-50%
|η|

Centrality dependence

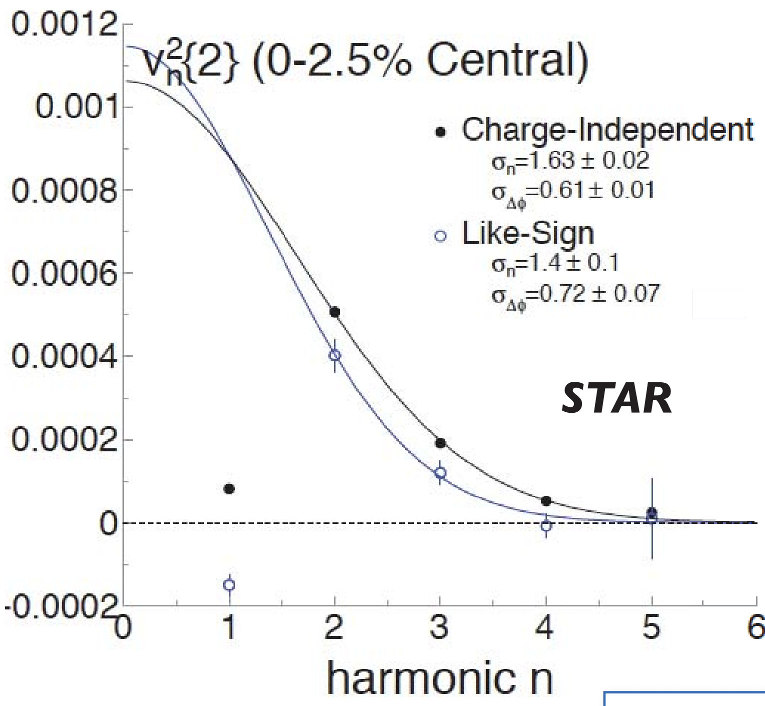


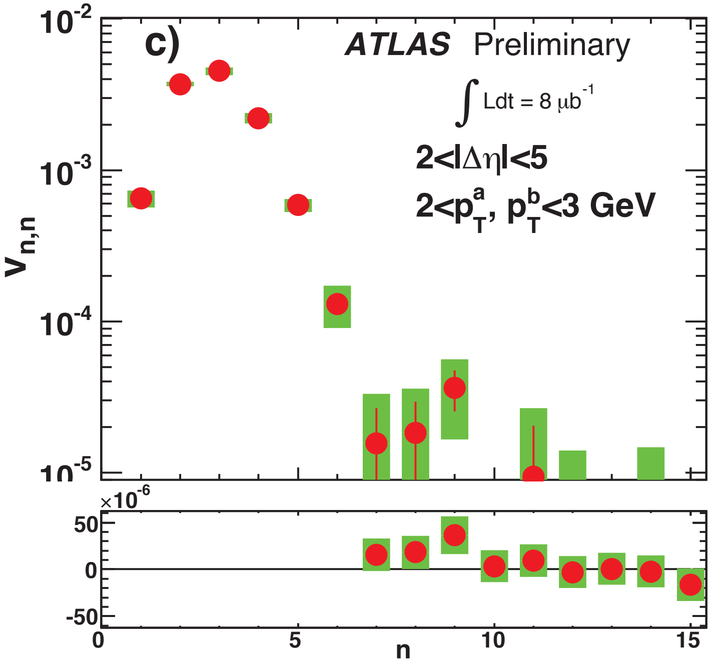
Impact parameter *b*

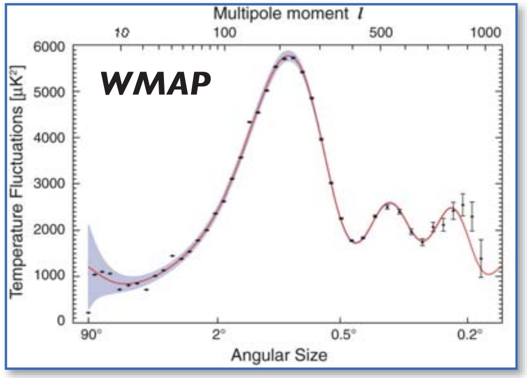


Impact parameter *b*

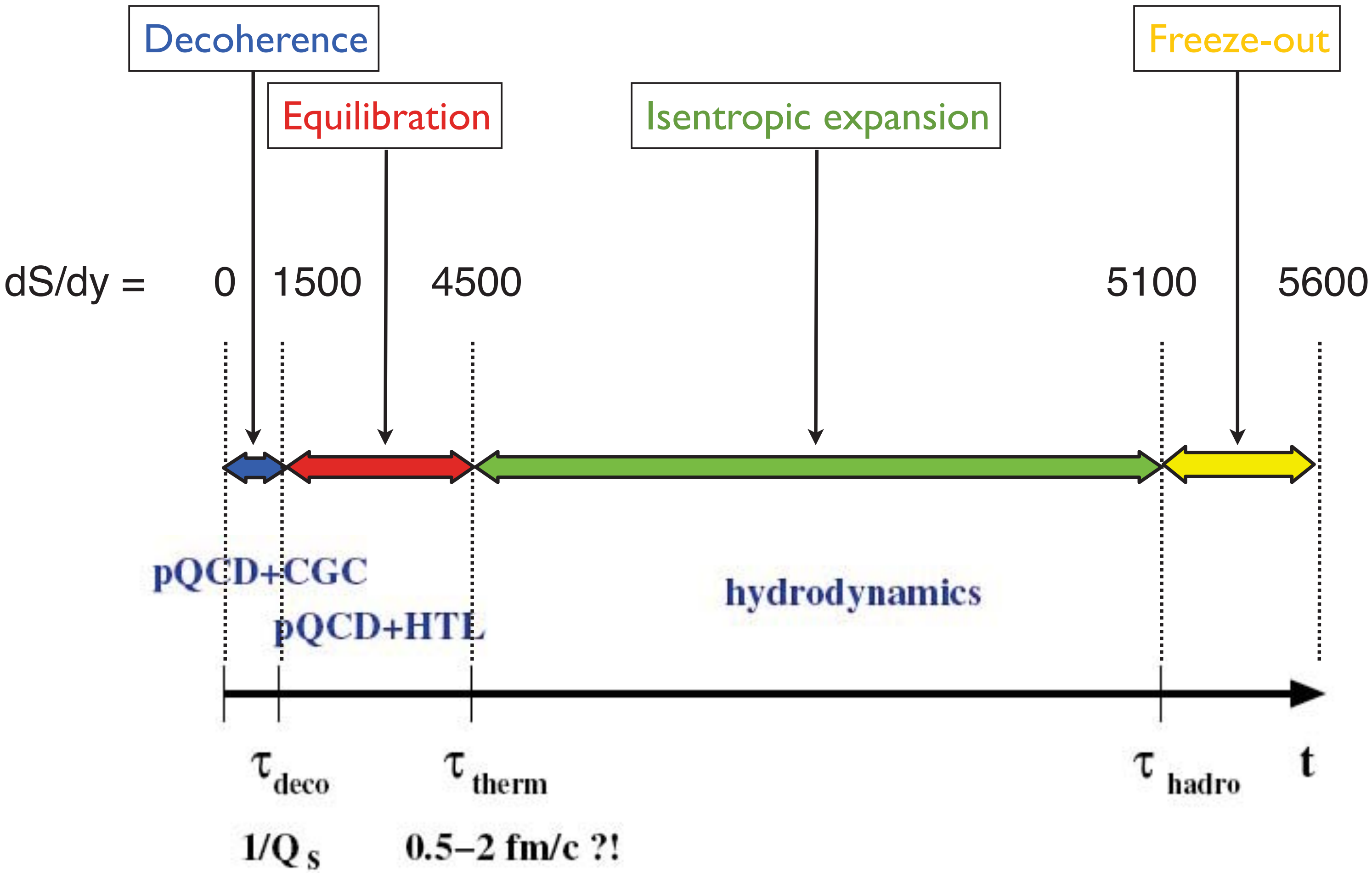
Power spectrum







Entropic history of a HI collision



Density matrix

The state of a system (ensemble) is specified by the density matrix:

$$\rho = \rho^{\dagger}; \quad \mathrm{tr}(\rho) = 1; \quad \mathrm{tr}(\rho^2) \leq 1; \quad \langle \Psi | \rho | \Psi \rangle \geq 0 \ \ \forall | \Psi \rangle$$

The density matrix evolves according to the von Neumann equation:

$$i\hbar \frac{\partial}{\partial t} \rho = [H, \rho] \qquad \rightarrow \qquad \rho(t) = e^{-iHt/\hbar} \rho(0) e^{iHt/\hbar}$$

The unitary time evolution implies that the von Neumann entropy

$$S_{\mathrm{vN}} = \mathrm{tr}(\rho \ln \rho)$$

does not change with time: Information about the quantum system is never lost.

However, not all information about the quantum system may be recoverable
by an observer, in principle or in practice: “coarse graining” or “entanglement”.

Equilibration

Yukalov [Laser Phys. Lett. 8 (2011) 485] gives the following definition:

A system ***equilibrates*** from an initial state ρ(0), if the expectation value of all observables reaches a steady-state limit:

$$\lim_{t\to\infty}\langle A(t)\rangle = \mathrm{tr}\left[\rho^*_{(\rho(0))}A\right]$$

A system ***thermalizes***, if the expectation value of all observables reaches a steady-state limit, which is independent of the initial state ρ(0):

$$\lim_{t\to\infty}\langle A(t)\rangle = \mathrm{tr}\left[\rho^* A\right]$$

A thermalized system is in a steady state and has “forgotten” all information about its initial state (except conserved quantum numbers).

Thermalization

Thermalization means that a system loses all information about its history.

This can happen in two ways:

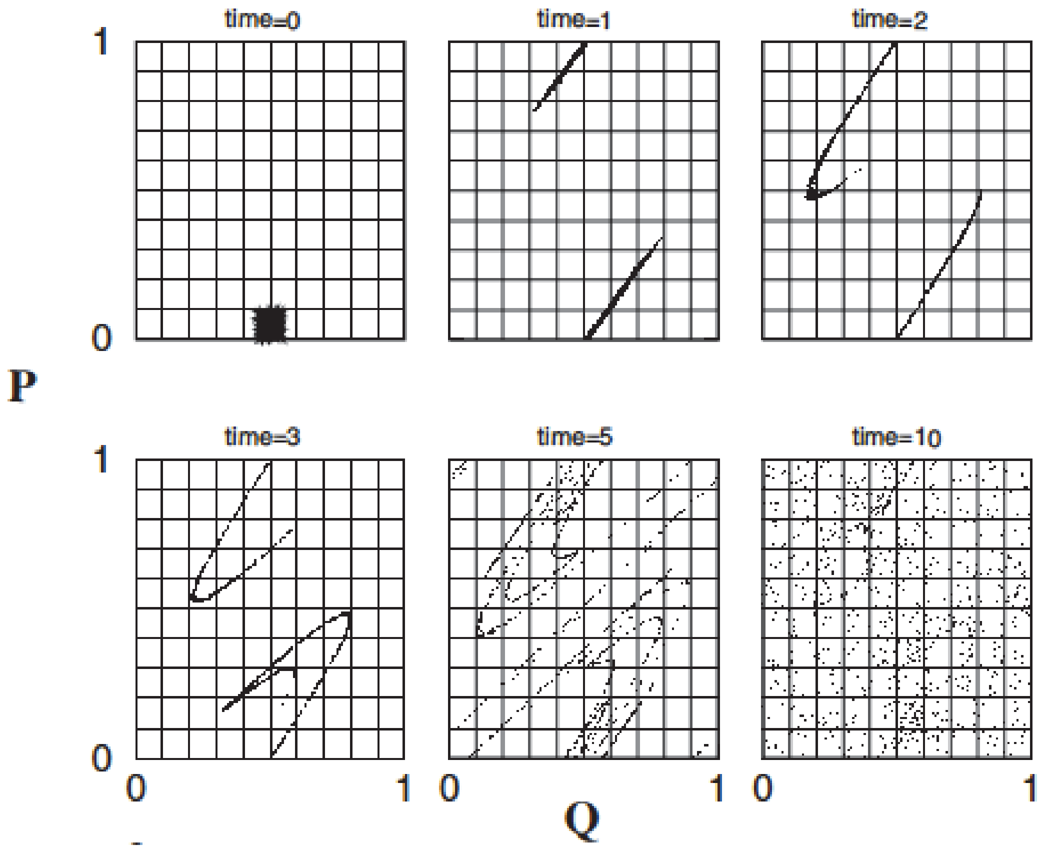
1. The system exchanges information with its environment (heat bath). This is true thermalization. The thermal state of the system is characterized by a density matrix, which only depends on the conserved quantum numbers (energy, particle number, charge, etc.). The entropy of the system is a measure of its information loss to the environment. In this case, the quantum state of the system becomes *entangled* with the quantum state of its environment.

2. The state of the system evolves by itself into a complicated superposition of components that cannot be distinguished by any practical measurements. This is apparent thermalization, implied by the *coarse graining* inherent in physical observations. A single eigenstate of the system can appear thermal (*eigenstate thermalization*). The physical mechanism by which a system can evolve into such complex states under its own dynamics is called *quantum chaos*.

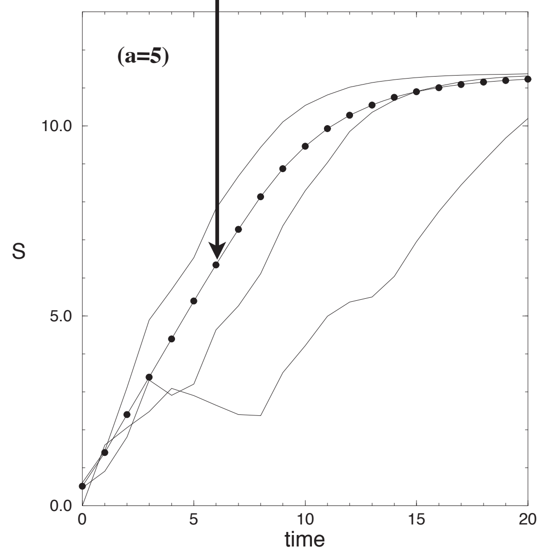
Coarse-grained entropy

Evolution of the "standard map" system: $p_{n+1} = p_n + \frac{a}{2\pi}\sin(2\pi q_n); \quad q_{n+1} = q_n + p_n \quad (\text{mod } 1)$

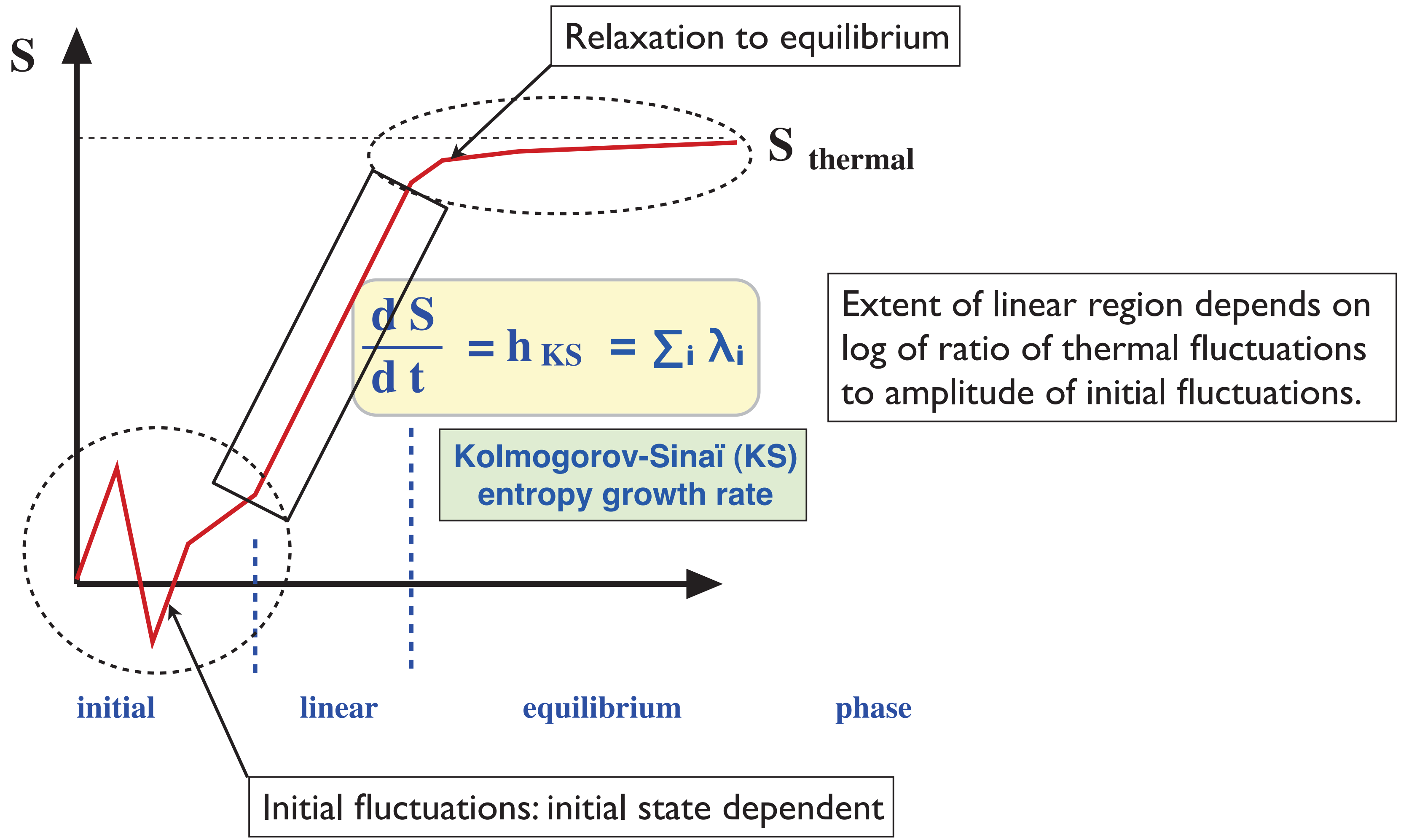
[M. Baranger, V. Latora, A. Rapisarda, *Chaos, Solitons and Fractals* 13 (2002) 471]



Coarse-grained entropy grows linearly after averaging over initial conditions: dS/dt = λ



General picture



Nakajima-Zwanzig theory

For a highly complex system (many degrees of freedom) usually only simple, slowly varying observables (few-body, low resolution, etc.) can be measured.

Split the density matrix into a relevant part ρ_R that determines the value of the observable A and an irrelevant part ρ_I that has no influence on the value of A:

$$\rho = \rho_R + \rho_I \quad \text{with} \quad \langle A \rangle = \mathrm{tr}\left(\rho A\right) = \mathrm{tr}\left(\rho_R A\right); \quad \mathrm{tr}\left(\rho_I A\right) = 0$$

Define a projection operator *P* such that: $\rho_R = P\rho$

Then:

$$\frac{\partial}{\partial t}\rho_R = -PL\rho_R(t) - iPL\,e^{-i(1-P)L}\rho_I(0) - \int_0^t d\tau\, G(\tau)\rho_R(t-\tau)$$

where $L = \frac{1}{\hbar}\left[H, \circ\right] \qquad G(\tau) = PL\,e^{-i(1-P)L\tau}\left(1-P\right)LP$ (memory kernel)

[For a review, see e.g.: J. Rau, BM, *Physics Reports* 272 (1996) 1]

Time scales

An important question is which observables A should be considered to define the relevant part ρ_R of the density matrix. These should be experimentally measurable quantities, which implies that they should vary only on observable time scales: they must be slowly varying observables.

In many cases the memory kernel, which describes the feedback from the irrelevant degrees of freedom, decays much faster than the characteristic time scale on which the value of the observables change. The evolution equation for ρ_R then becomes effectively Markoff.

Any analysis of the problem of entropy creation and thermalization in the Nakajima-Zwanzig formalism thus starts from an analysis of time scales.

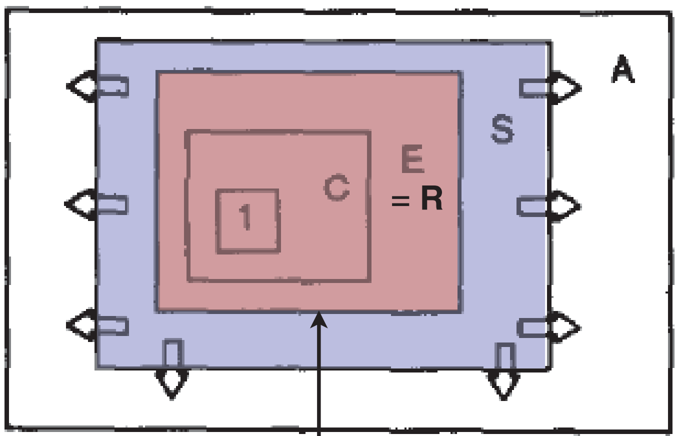
Note: The projector P ensuring $\mathrm{tr}(P\rho A) = \mathrm{tr}(\rho A)$ is called the Kawasaki-Gunton projector; the resulting evolution equation for ρ_R is called the Robertson equation. Because ρ is time dependent, P depends on time.
An alternative formulation is due to Mori, who defined the projector such that $\mathrm{tr}(P_M \rho_{eq} A) = \mathrm{tr}(\rho_{eq} A)$, which makes P_M time independent.

Relevant entropy

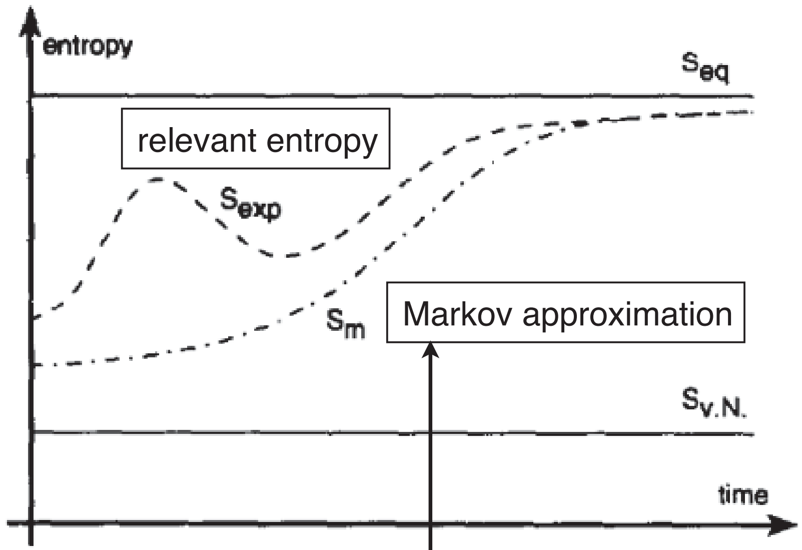
The relevant entropy $S_R = \mathrm{tr}(\rho_R \ln \rho_R)$ generally increases with time (but not necessarily monotonously), because information gets transferred into irrelevant degrees of freedom. Special case: $1-P$ = projector on the environment.

The relevant entropy is “in the eye of the beholder”.

C = conserved observables
E = experimentally relevant observables
S = “slowly varying” observables
A = all observables



“Level of description” of the system



Good if large separation of time scales

Omphaloskepsis?

The relevant entropy is "in the eye of the beholder".

Does this mean that it is not a property of the system?
Or can an isolated system "observe itself" ?

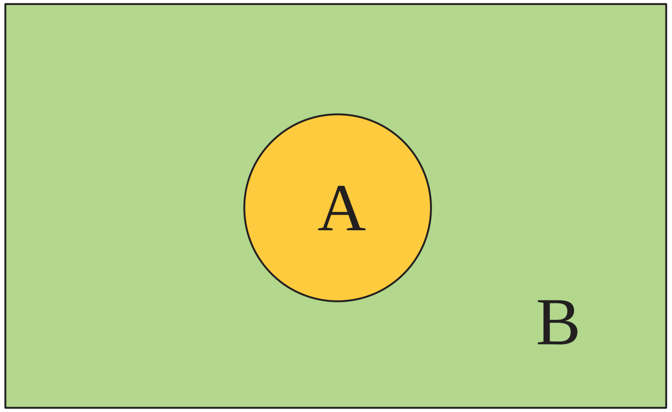
In a certain sense, yes: A system "observes itself" via the Hamiltonian, which governs its time evolution. But the Hamiltonian generally has special properties:

- Only 2-body interactions for non-relativistic many-body systems;
- Only local interactions for quantum field theories.

➡ The short-term evolution of a quantum system is thus oblivious to many-body or long-range correlations. From the "point of view of the system itself" they form part of the *irrelevant* density matrix ρ_I.

☛ The parts of a system hidden behind an event horizon are always *irrelevant* for its observable time evolution.

Entanglement entropy - I



Consider a vacuum QFT in a large box.

An observer restricted to subvolume *A* will experience a reduced density matrix

$$\rho_A = \mathrm{Tr}_B(\rho) = \mathrm{Tr}_B\big(|0\rangle\langle 0|\big)$$

Special case of Nakajima-Zwanzig projection!

The *entanglement entropy* between *A* and *B* is defined as $\boxed{S_A = -\mathrm{Tr}_A\big(\rho_A \ln \rho_A\big)}$

It measures the loss of information to the observer from not knowing exactly what the state of the field in the subvolume *A* is, if she does not know the state in *B.*

S_A is a useful measure of how entangled the wave function of the ground state $|0\rangle$ is between *A* and *B*. Naïvely, one would expect that any mode component in *A* with wave number *k* “knows” about the presence of *B* if it is located within distance $\hbar/k$ of the boundary.

Entanglement entropy - II

Therefore one expects (Srednicki, 1993): $$S_A \sim \int(\partial A)\sum_k^{k_{\max}} \frac{\hbar}{|k|} \sim \kappa \|\partial A\| k_{\max}^2$$

The entanglement entropy is thus proportional to the *surface area* of *A*. If one chooses $k_{\max} \sim M_{Pl}$, S_A becomes the Bekenstein entropy of a black hole with surface area ll∂All. Black hole entropy is thus a form of entanglement entropy.

Interactions introduce finite corrections to the UV divergent entanglement entropy. These provide a measure of the range of quantum correlations in the ground state wave function.

Another variant is when the QFT is not considered in the vacuum state, but at finite temperature *T*. The entanglement entropy then receives a contribution proportional to ll*A*ll = Vol(*A*), which is precisely the thermal equilibrium entropy.

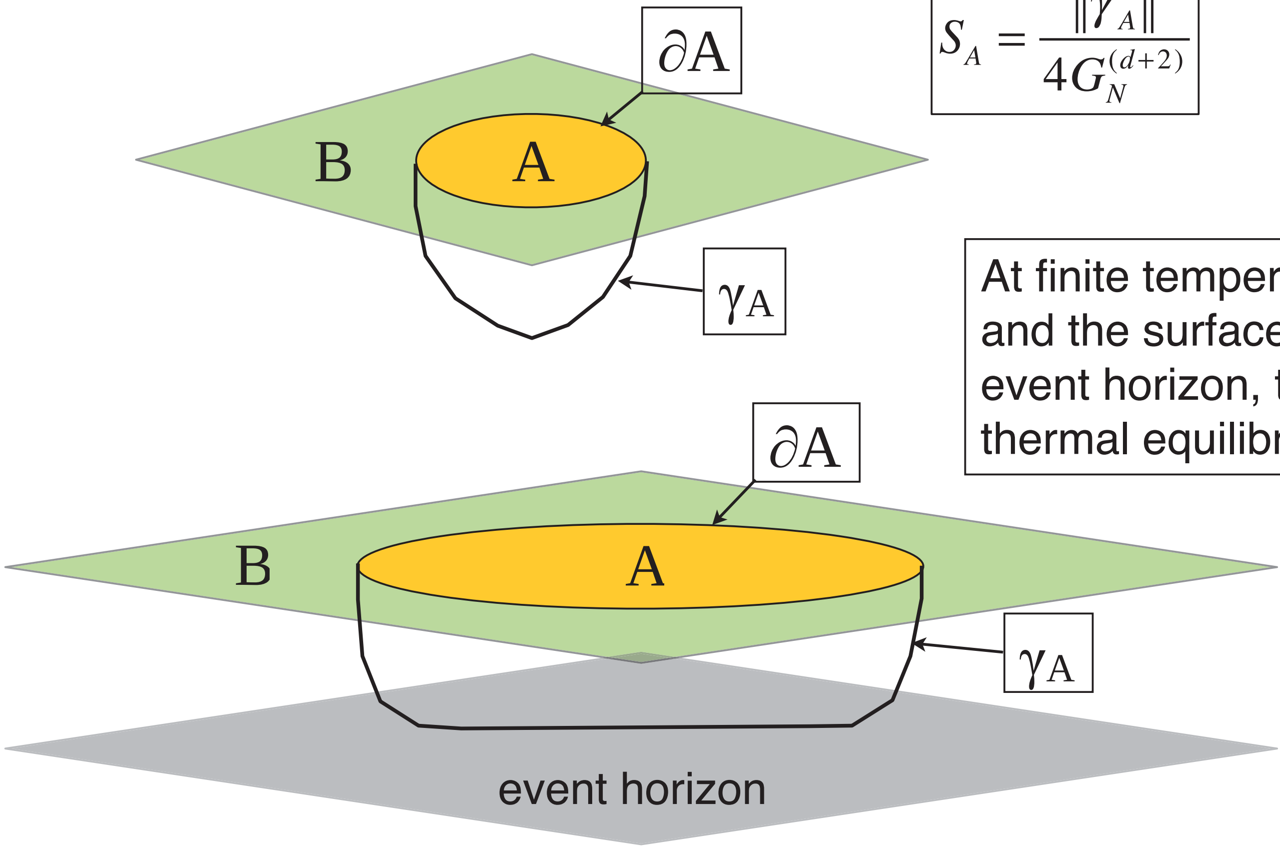
In QFT, S_A can be generally calculated with the *replica trick*:

$$S_A = -\mathrm{Tr}_A\left(\rho_A \ln \rho_A\right) = -\lim_{n\to 1} \frac{\partial}{\partial n}\left(\mathrm{Tr}_A\, \rho_A^n\right) = -\lim_{n\to 1} \frac{\partial}{\partial n}\left[\frac{Z_{nA}}{\left(Z_A\right)^n}\right]$$

AdS/CFT

For a (*d*+1)-dimensional QFT with a holographic gravity dual, S_A can be calculated in the dual theory from the area of the extremal surface γ_A in the bulk, with has the same boundary ∂A as *A*: $\partial(\gamma_A) = \partial A$.

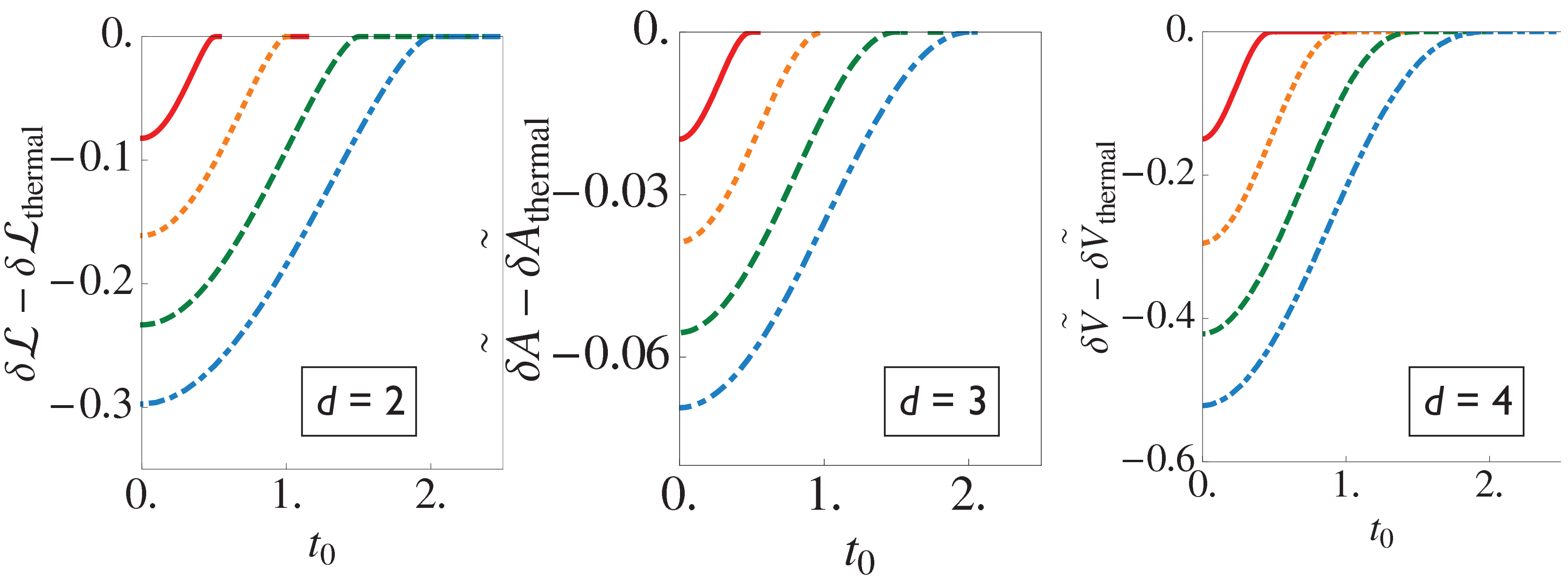
$$S_A = \frac{\|\gamma_A\|}{4G_N^{(d+2)}}$$



At finite temperature, a BH is present, and the surface γ_A picks up a part of the event horizon, thus accounting for the thermal equilibrium entropy of *A*.

see review by: Nishioka, Ryu, Takayanagi, arXiv:0905.0932

Entanglement entropy



For details, see Ben Craps' talk

Husimi coarse graining

A minimal coarse-graining of a quantum system is achieved by projecting its density matrix on a coherent state (Husimi ["*Fushimi*"] 1940):

$$H(x,p) = \langle z|\rho|z\rangle \quad \text{with} \quad z = \frac{x}{\sqrt{\hbar\Delta^{-1}}} + \frac{p}{\sqrt{\hbar\Delta}} \quad \Rightarrow \quad \rho_H = |z\rangle H \langle z|$$

The Husimi phase-space density is positive semi-definite and can be used to define a coarse grained entropy (Wehrl, 1978):

$$S_H = -\mathrm{Tr}\left[\rho_H \ln \rho_H\right] = -\int \frac{dx\,dp}{2\pi\hbar} H(x,p) \ln H(x,p)$$

As opposed to the von Neumann entropy S = -Tr(ρlnρ), the Husimi-Wehrl entropy is not conserved by unitary evolution. Its value depends on Δ, but its growth rate at large times is independent of the smearing Δ (Kunihiro *et al.* [KMOS], 2008). Far off equilibrium it is equal to the *Kolmogorov-Sinaï* (KS) entropy growth rate:

$$\frac{dS_H}{dt} \xrightarrow{t\to\infty} \sum_{\alpha}^{\lambda_\alpha > 0} \lambda_\alpha = \dot{S}_{\mathrm{KS}}$$

Husimi II

Husimi density can be understood as smearing of the Wigner function with a Gaussian minimum-uncertainty wave packet:

$$H_\Delta(p,x;t) \equiv \int \frac{dp'\,dx'}{\pi\hbar} \exp\left(-\frac{1}{\hbar\Delta}(p-p')^2 - \frac{\Delta}{\hbar}(x-x')^2\right) W(p',x';t)$$

Special case of Gaussian smearing with $\sigma_p \sigma_x = \hbar/2$:

$$H_{\sigma_p \sigma_x}(p,x;t) = \int \frac{dp'dx'}{2\pi\sqrt{\sigma_p \sigma_x}} \exp\left(-\frac{(p-p')^2}{2\sigma_p^2} - \frac{(x-x')^2}{2\sigma_x^2}\right) W(p',x';t)$$

Formally, the Husimi transformation of the density matrix is of the form:

$$\rho_H = \Gamma_H \rho = \Gamma\left(\sigma_p, \sigma_x\right)\rho$$

with $\sigma_p^2 = \hbar\Delta/2$, $\sigma_x^2 = \hbar/2\Delta$. Note that Γ is not a projection operator:

$$\Gamma\left(\sigma_p, \sigma_x\right)^2 = \Gamma\left(\sqrt{2}\sigma_p, \sqrt{2}\sigma_x\right)$$

$\rho_H \nexists \{\rho_R\}$: Formal proof

Can the Husimi coarse graining be understood as a projection on “coarse”, relevant degrees of freedom, i.e. can we find a projector P_H so that

$$\rho_H = P_H \rho \quad \text{with} \quad P_H^2 = P_H \text{ ?}$$

This implies: $$\det \rho_H = (\det P_H) \det \rho$$

Now: $$(\det P_H)^2 = \det P_H \quad \rightarrow \quad \det P_H = 0 \quad \text{or} \quad \det P_H = 1$$

Since the eigenvalues of P_H are either 0 or 1:

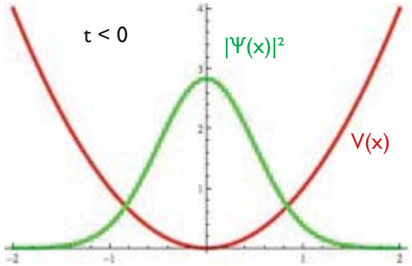
$$\det P_H = 1 \quad \text{if} \quad P_H = I; \qquad \det P_H = 0 \quad \text{otherwise.}$$

Because P_H cannot be I, we conclude that $\det \rho_H = 0$. But ρ_H is diagonal in the coherent state basis; this implies that one diagonal element must vanish.

But $\langle z | \rho_H | z \rangle = \langle z | \rho | z \rangle$, thus unless ρ has a zero diagonal element, i.e. in general, such a projector P_H cannot be found.

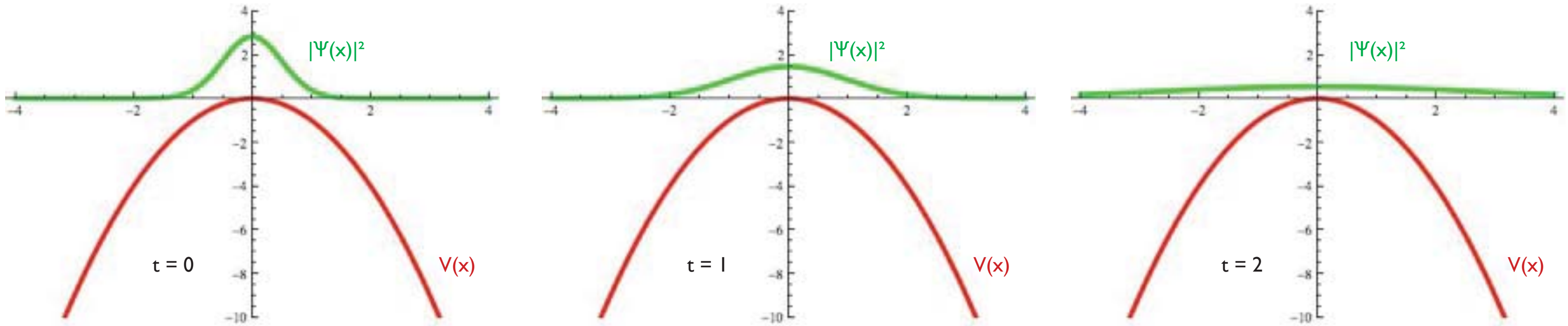
Quantum quench

The decay of an unstable vacuum state is a common problem, e.g., in cosmology and in condensed matter physics. Paradigm case: inverted oscillator.



$$\hat{H}(t) = \frac{p^2}{2} + \frac{m(t)^2}{2} x^2$$

with $\quad m(t)^2 = \omega^2 \theta(-t) - \lambda^2 \theta(t)$



Wigner function: $\quad W(q,p;t) = \int du\, e^{-ipu} \langle q + \frac{1}{2}u |\, \hat{\rho}(t)\, | q - \frac{1}{2}u \rangle$

Wigner vs. Husimi

Wigner function

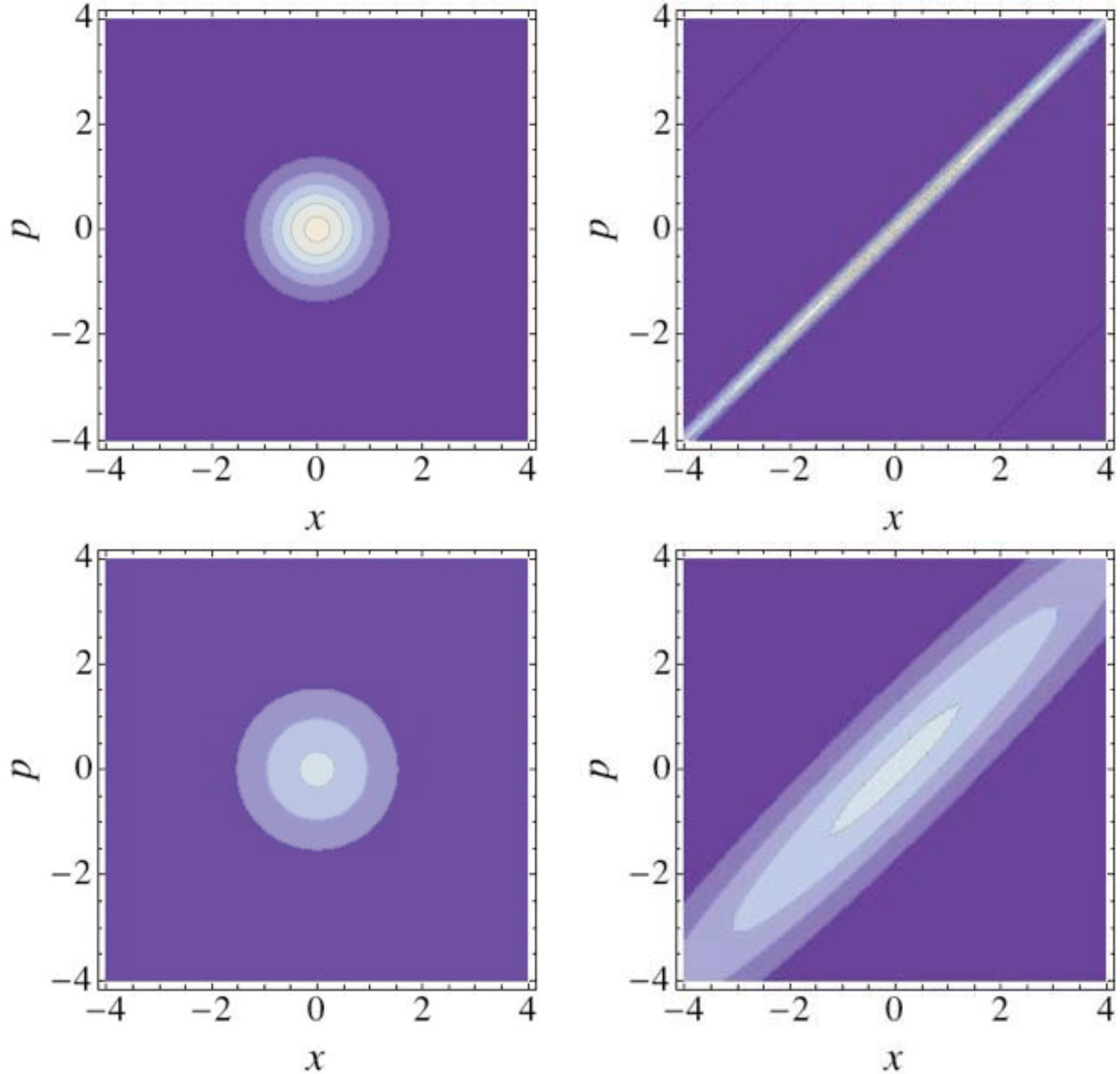
t = 0

t = 2

Husimi function

t = 0

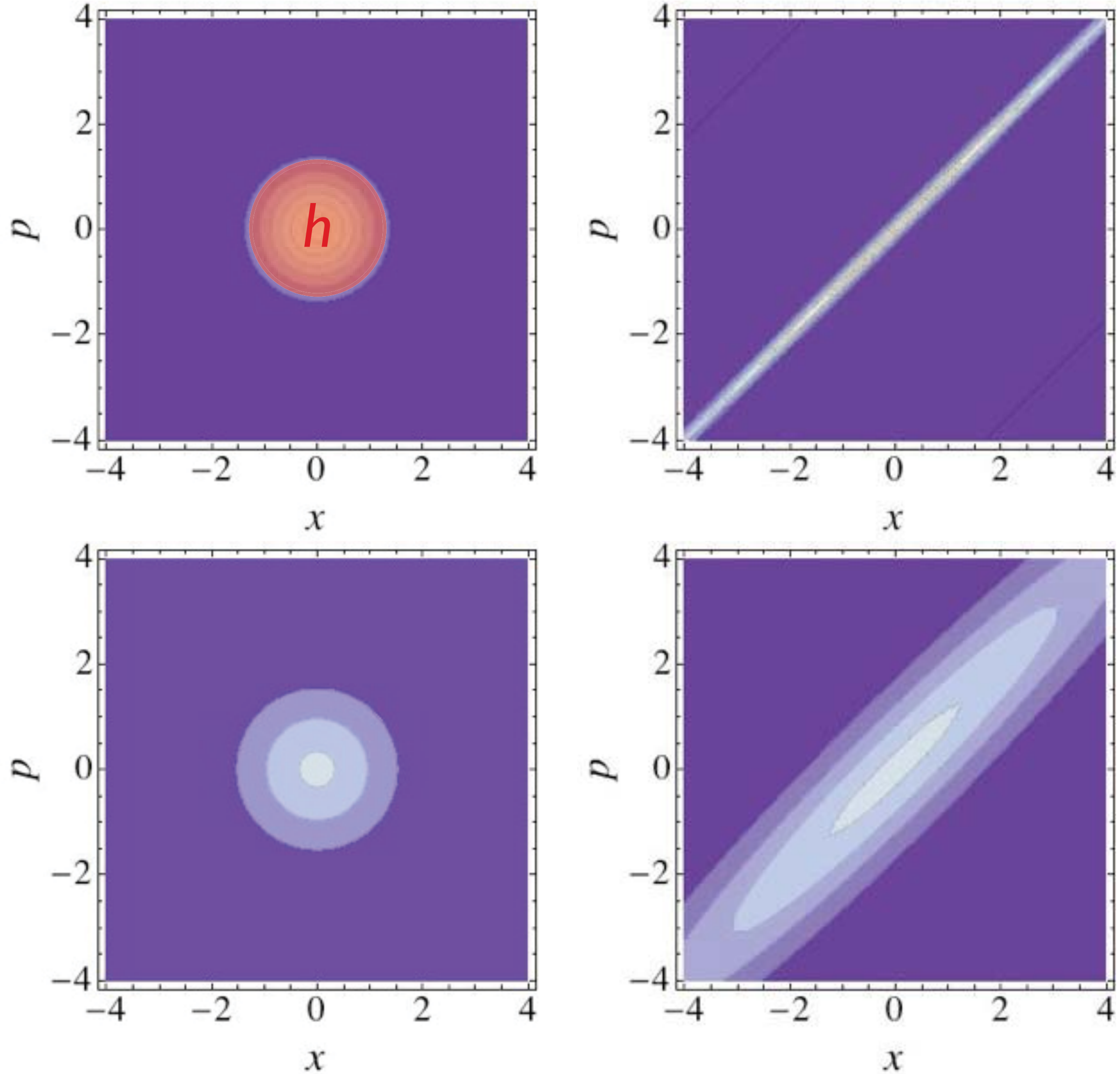
t = 2



Wigner vs. Husimi

Wigner function

$t = 0$



$t = 2$

Husimi function

$t = 0$

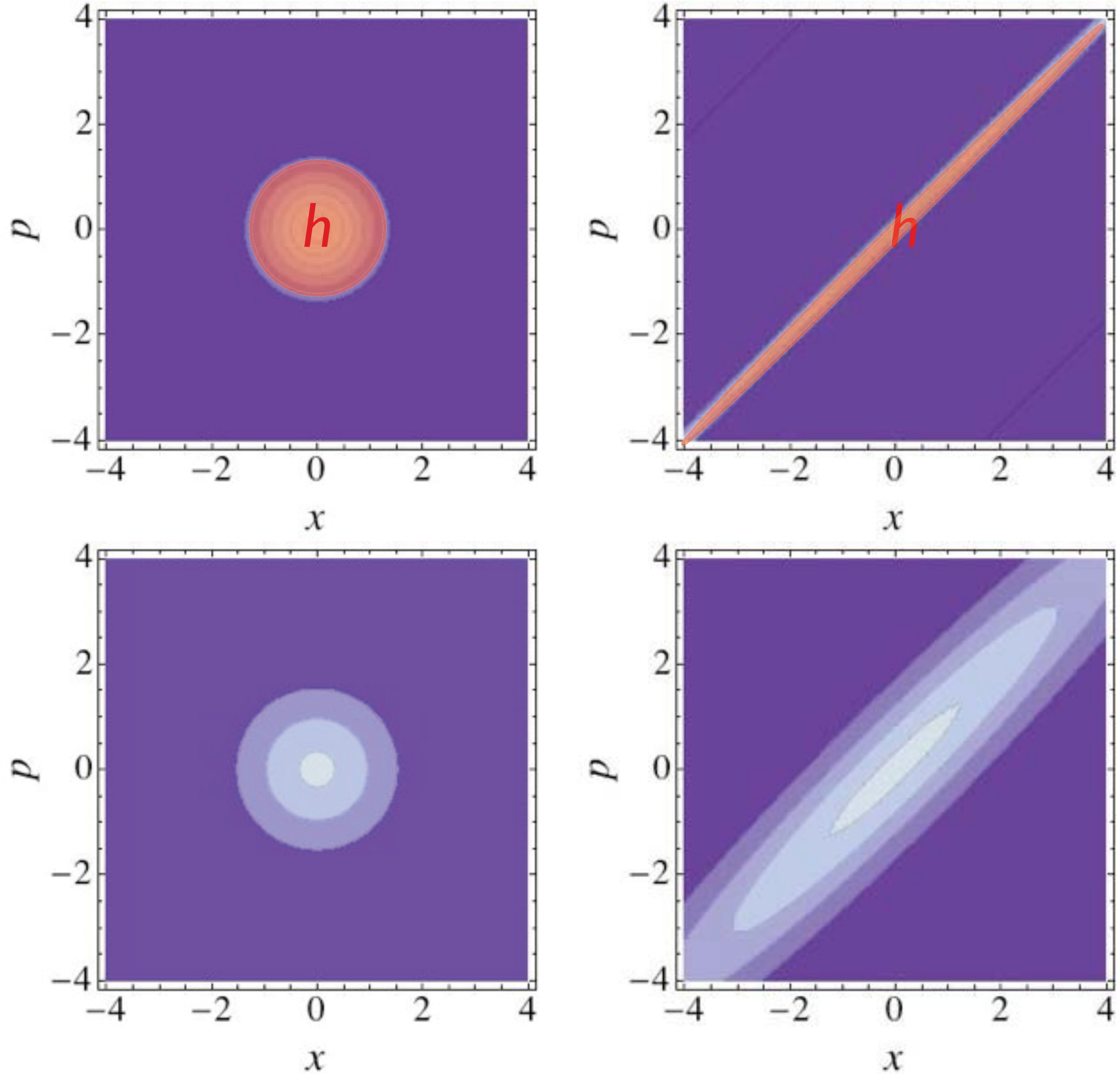
$t = 2$

Wigner vs. Husimi

Wigner function

t = 0

t = 2

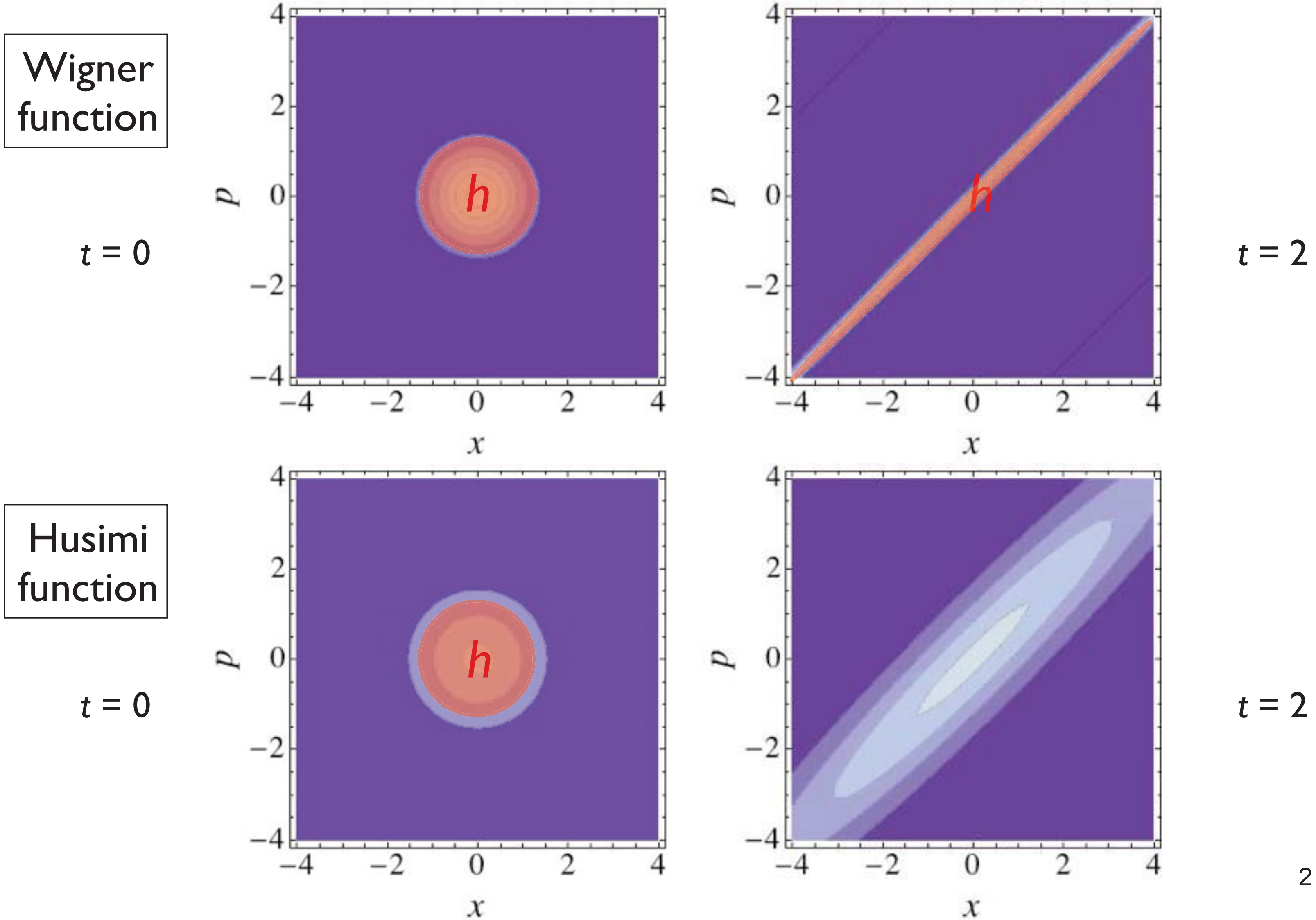


Husimi function

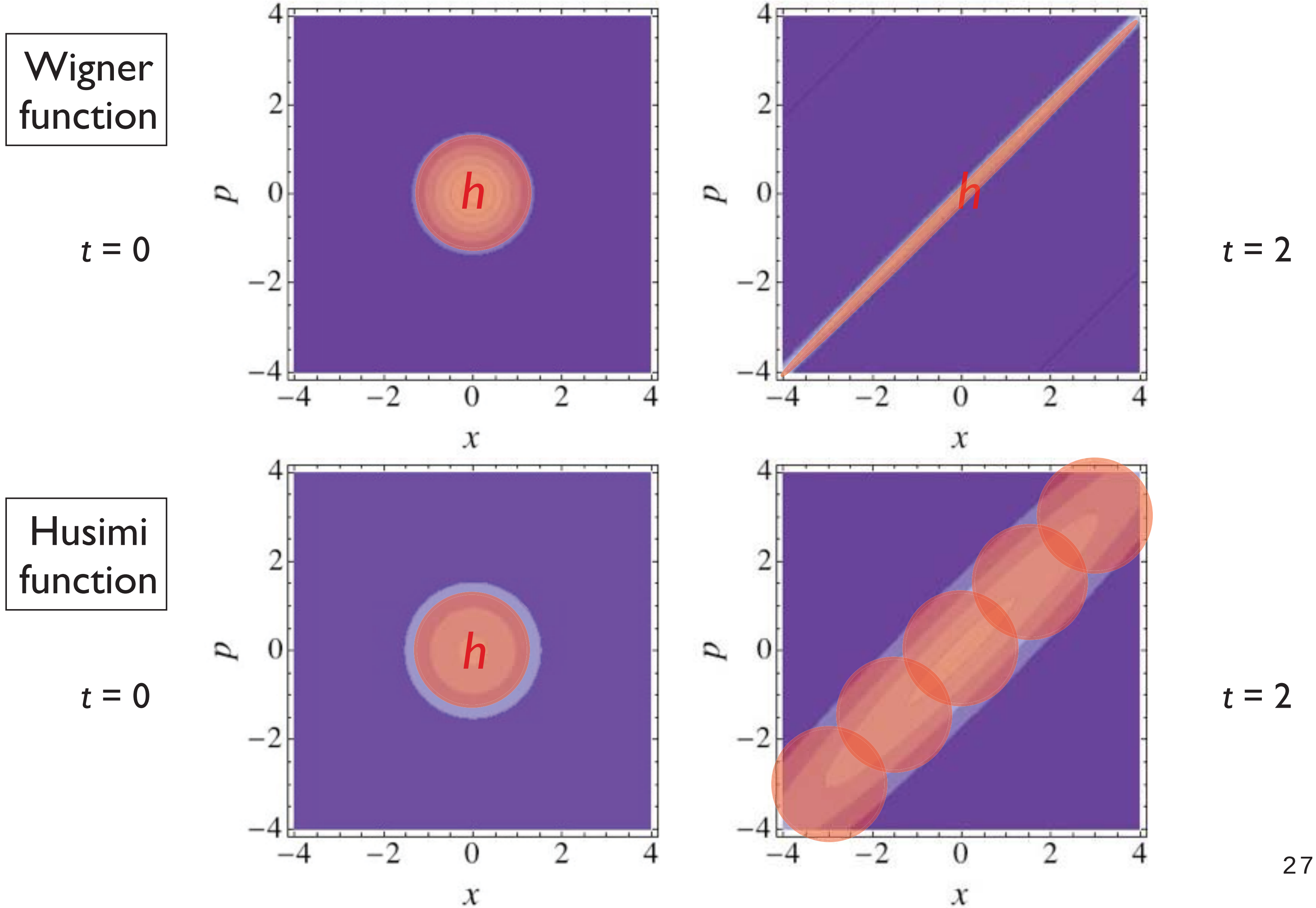
t = 0

t = 2

Wigner vs. Husimi



Wigner vs. Husimi



The ETH

The *Eigenstate Thermalization Hypothesis* (ETH - Deutsch '91, Srednicki '94):

$$\left\langle \Psi_E \middle| A \middle| \Psi_E \right\rangle = \left\langle A \right\rangle_{E,\,\mathrm{mc}}$$

for any complex many-body system and few-body operator *A* or other slowly varying observable.

Corollary: The eigenstate of a few-body operator *A* is superposition of energy eigenstates with thermally distributed probabilities:

$$\Psi_A = \sum_E c_E \Psi_E \quad \text{with} \quad \left| c_E \right|^2 \sim e^{-E/T}$$

Over time, phases of different components diverge from each other; eventually the pure state becomes indistinguishable from a mixed state:

$$\left| \Psi_A(t) \right\rangle \left\langle \Psi_A(t) \right| = \sum_{E,E'} c_E c_{E'}^* e^{i(E'-E)t/\hbar} \left| \Psi_E \right\rangle \left\langle \Psi_{E'} \right| \xrightarrow{t\to\infty} \sum_E \left| c_E \right|^2 \left| \Psi_E \right\rangle \left\langle \Psi_E \right|$$

$$S \to -\sum_E \left| c_E \right|^2 \ln\left(\left| c_E \right|^2 \right)$$

Entropy growth

$$S(t) \to -\sum_{E_\alpha} \left|\tilde{c}_{E_\alpha}\right|^2 \ln\left(\left|\tilde{c}_{E_\alpha}\right|^2\right) \qquad \text{where} \qquad \tilde{c}_{E_\alpha} = \sum_{\left|E-E_\alpha\right| \le \frac{1}{2}\Delta E(t)} c_E$$

with amplitudes remaining coherent within energy bands Δ*E*(t) ~ ħ/t.

Consider again the quantum quench to the inverted oscillator:

$$\Psi_E(x) \approx \sqrt{\frac{2}{\pi\hbar}} \frac{\cos\frac{1}{\hbar}\int_0^x dx'\, p_E(x')}{\sqrt{p_E(x)}} \quad \text{with} \quad p_E(x) = \sqrt{2E + \lambda^2 x^2}$$

De-phasing leads to a block structure of the density matrix:

$$\rho(E_\alpha, x_\beta) \approx \Theta\left(\Delta x - \left|x_\beta - x_0 e^{\lambda t}\right|\right) \Delta E \frac{e^{-2E_\alpha/\hbar\omega}}{\sqrt{2\pi\hbar\omega E_\alpha}} \quad \text{with} \quad \Delta x \approx \frac{\hbar}{2\lambda x}$$

T = ½ħω is the “temperature” determined by the initial wave packet.

$$S(t) \approx \sum_{E_\alpha} \sum_{x_\beta} \rho(E_\alpha, x_\beta) \ln\left(\rho(E_\alpha, x_\beta)\right) = \lambda t$$

YM-QM model

A simple example of a non-trivial chaotic quantum system is given by the infrared limit of SU(2) gauge theory (*Yang-Mills Quantum Mechanics*):

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{3} p_i^2 + \frac{g^2}{4}\sum_{i \neq k}^{3} x_i^2 x_k^2$$

Further simplification: $x_1 = x$, $x_2 = y$, $x_3 = 0$ (*x-y model*): $\mathcal{H} = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{g^2}{2} x^2 y^2$

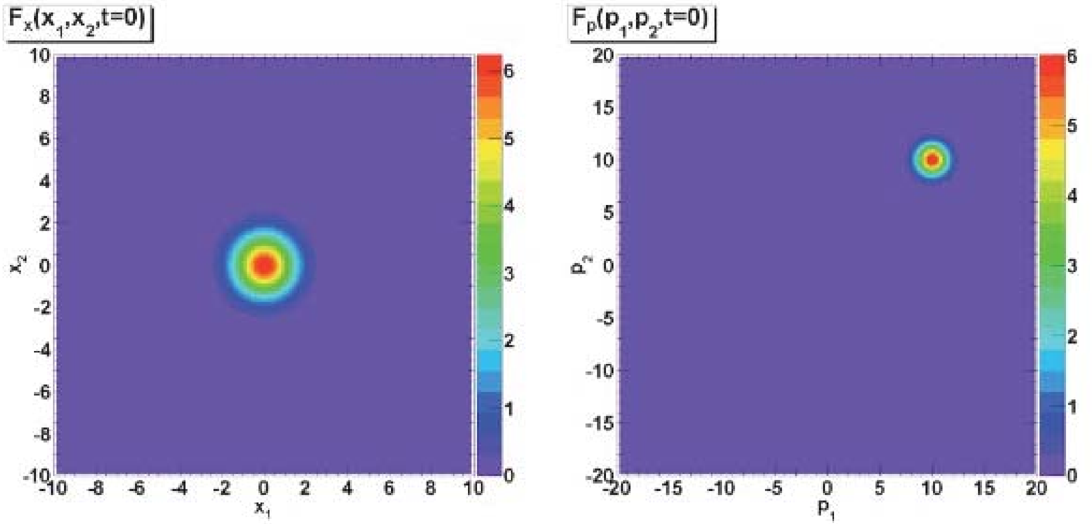
Solve equation of motion for Husimi density $H(x,y,p_x,p_y,t)$ using superposition of Gaussians with time-dependent positions and widths:

$$H(\xi_i,t) = \sum_{\alpha} \exp\left[-\sum_{ij} c_{ij}(t)\left(\xi_i - \xi_i^{(\alpha)}(t)\right)\left(\xi_j - \xi_j^{(\alpha)}(t)\right)\right] \quad \text{with} \quad \xi_i = (x, y, p_x, p_y)$$

Evolution conserves the coarse grained Hamiltonian

$$\mathcal{H}_H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{g^2}{2} x^2 y^2 - \frac{g^2 \hbar}{4\Delta}\left(x^2 + y^2\right) + \frac{g^2 \hbar^2}{8\Delta^2} - \frac{1}{2}\hbar\Delta$$

YM-QM - the movie

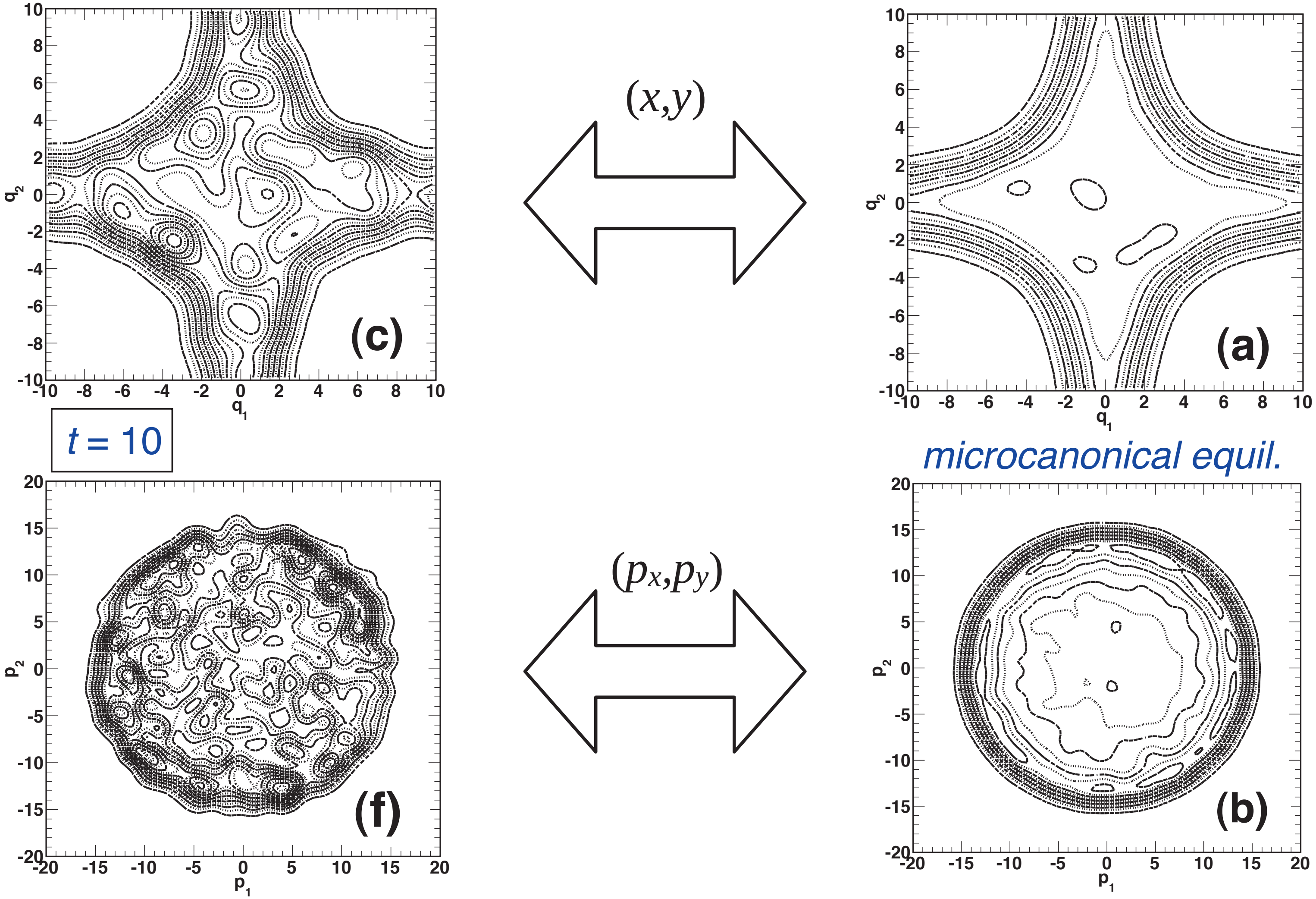


Position space x_1, x_2

Momentum space p_1, p_2

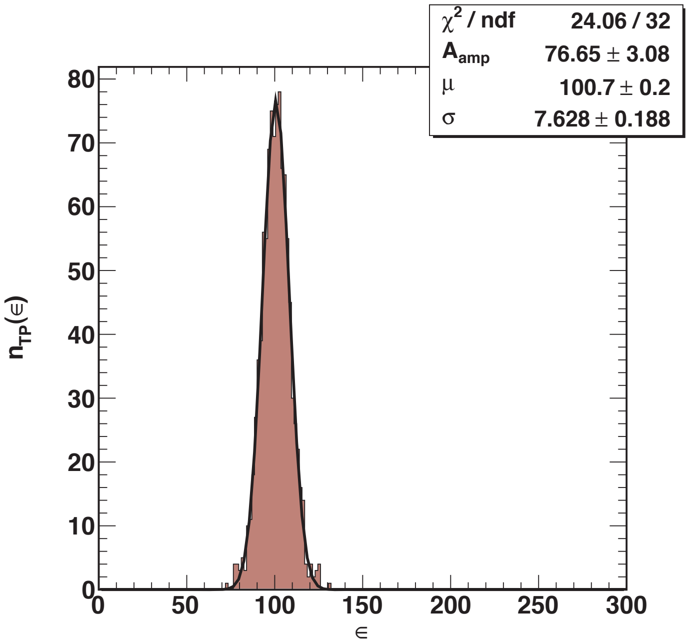
Hung-Ming Tsai & BM, arXiv:1011.3508

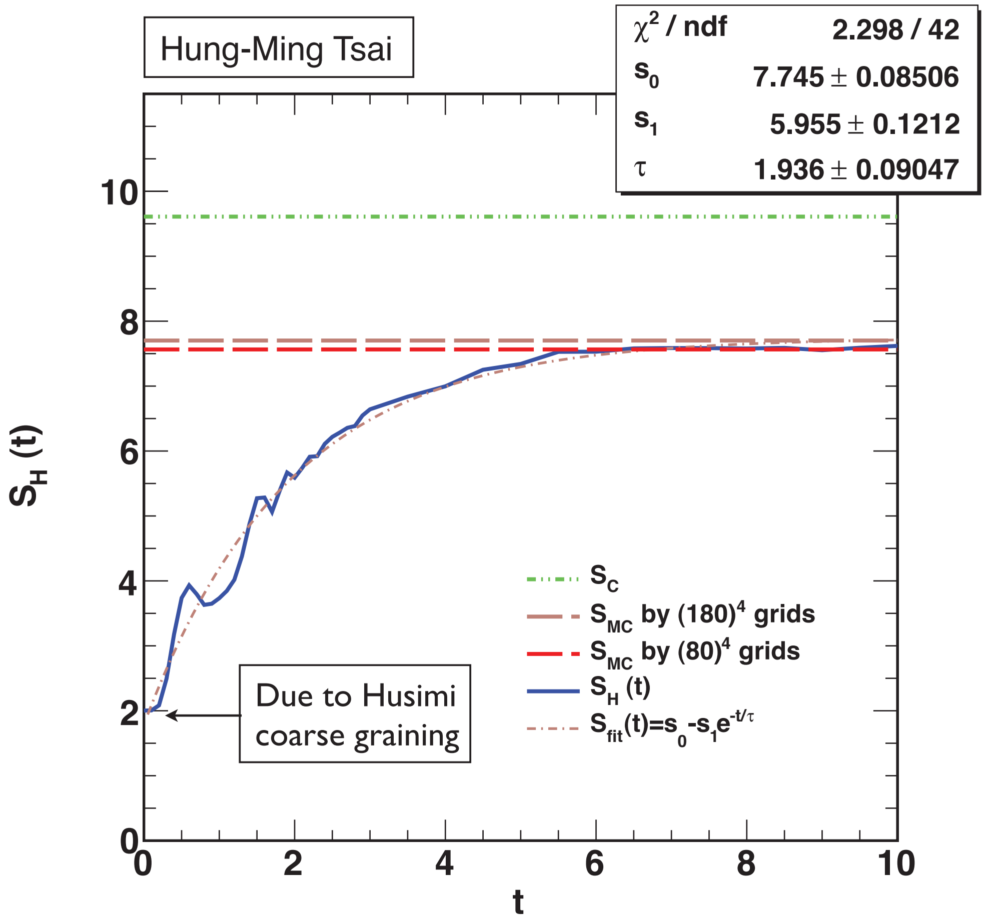
YM-QM equilibration



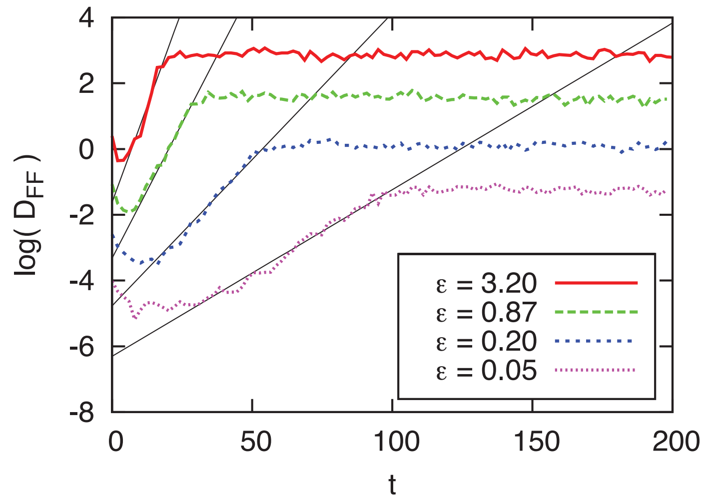
YM-QM equilibration - II

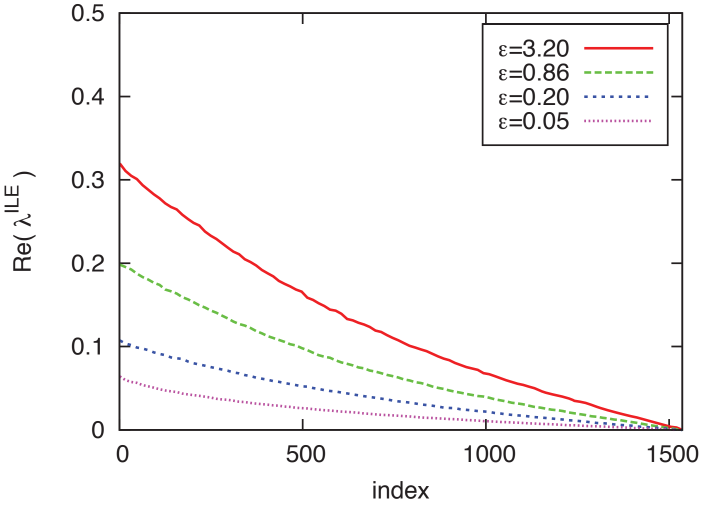
Initial Husimi density is approximately a narrow microcanonical distribution peaked around $E \approx 100$. Final Wehrl entropy should agree with micro-canonical equilibrium, and it does!





Classical lattice SU(3)





T. Kunihiro, BM, A. Ohnishi, A. Schäfer, T. Takahashi & A. Yamamoto, PRD 82 (2010) 114015

LLE = Local Lyapunov exponents:
= Eigenvalues of the Hesse matrix

ILE = Intermediate Lyapunov exponents:
= Growth rate of distance between neighboring gauge field config's

GLE = Global Lyapunov exponents:
= Asymptotic divergence rate of neighboring gauge field config's
= Standard definition of LE's

Summary

- Different definitions of relevant entropy:
  - ☐ Coarse graining, Husimi transform, (ETH) dephasing
  - ☐ Projection, relevant density matrix, entanglement
- Linear growth of coarse grained entropy determined by sum of Lyapunov exponents: KS entropy growth rate
- What are the holographic dual descriptions?
  - ✓ Entanglement entropy
  - ☐ How is coarse graining implemented in holography?
  - ☐ How are Lyapunov exponents manifested?
  - ☐ Is there an analogy to KS entropy growth rate?