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Bigravity as a tool in massive gravity

L. Pilo

University of L'Aquila
Italy

Bigravity: a Tool for Massive Gravity

L. Pilo[1]

[1] Department of Physics
University of L'Aquila

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Outline

The Stuckelberg Trick in Massive GR: Bigravity I

Step 1: Recasting the mass term

- $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\alpha} h^{\nu}_{\alpha} + \cdots \quad \Rightarrow \quad g^{\mu\nu}\eta_{\mu\nu}$ represents a mass term
- To recover diff (gauge) invariance replace $\eta_{\mu\nu}$ by a dynamical (Stuckelberg) extra metric field $\tilde{q}_{\mu\nu}$

  $\eta_{\mu\nu} \to q_{\mu\nu}$
- Dynamical metric field in fictitious spacetime $\tilde{\mathcal{M}}$ pulled back in our spacetime$\mathcal{M}$

  $$q_{\mu\nu} = \frac{\partial \zeta^A}{\partial x^\mu} \frac{\partial \zeta^B}{\partial x^\nu} \tilde{g}_{AB} \qquad \zeta : \mathcal{M} \to \tilde{\mathcal{M}}$$

  New tensor from the two metric $X^{\mu}_{\nu} = g^{\mu\alpha} q_{\alpha\nu}$
- Typical mass terms are made out $\tau_n = \mathrm{Tr}(X^n)$

  $$a\,(\tau_1 - 4)^2 + b\,(\tau_2 - 2\tau_1 + 4) = \left(a\, h_{\mu\nu} h^{\mu\nu} + b\, h^2\right) + \cdots$$

The Stuckelberg Trick in Massive GR: Bigravity II

Step 2: Stuckelberg Dynamics

- The extra metrics is turned into a dynamical field

$$S_{MGR} = \int d^4x \left[\sqrt{g}\, M_{pl}^2\, R(g) + \kappa\, \tilde{M}_{pl}^2\, \sqrt{\tilde{g}}\, R(\tilde{g}) - 4(\tilde{g}g)^{1/4}\, V(X)\right]$$

- Matter couples only to $g_{\mu\nu}$
- Huge gauge symmetry: $\mathrm{Diff}_1 \times \mathrm{Diff}_2$

  with $\zeta \to f_2^{-1} \cdot \zeta \cdot f_1$, g and $\tilde{g}$ tensors
- Unitary gauge point wise identification of $\tilde{\mathcal{M}}$ with $\mathcal{M}$
  $\zeta = \mathrm{Id} \Rightarrow\ q_{\mu\nu} \equiv \tilde{g}_{\mu\nu}$
  $\mathrm{Diff}_1 \times \mathrm{Diff}_2$ broken down to $\mathrm{Diff}_{\mathrm{diagonal}}$
- When $\kappa \to \infty$, $\tilde{g}_{\mu\nu}$ gets nondynamical and flat: $\tilde{g}_{\mu\nu} = e^a_\mu\, e^b_\nu \tilde{\eta}_{ab}$
  $e^a = d\phi^a$ and $\tilde{g}_{\mu\nu} = \partial_\mu \phi^a \partial_\nu \phi^b\, \tilde{\eta}_{ab}$
- In the bigravity unitary gauge and $\kappa \to \infty$ the stuckelberg fields are ϕ^a

The Stuckelberg Trick in Massive GR: Bigravity III

- In the bigravity unitary gauge and $\kappa \to \infty$, the stuckelberg fields are ϕ^a
- Powerful formalism to treat in unified way both the Lorentz preserving and Lorentz breaking cases

  $X_{|bkg} = \text{Diag}(1,\,1,\,1,\,1)$ Lorentz preserving (LI) background

  $X_{|bkg} = \text{Diag}(a,\,b,\,b,\,b)$ Lorentz breaking (LB) background only rotational symmetry is present
- For any V the LI background is always present
- Modified Einstein equations (Bigravity Unitary gauge)

$$M_{pl}^2\, E^\mu_\nu + [Det(X)]^{1/4}\left[V\,\delta^\mu_\nu - 4(V'X)^\mu_\nu\right] = T^\mu_\nu$$

$$\kappa\, M_{pl}^2\, \tilde{E}^\mu_\nu + [Det(X)]^{-1/4}\left[V\,\delta^\mu_\nu + 4(V'X)^\mu_\nu\right] = 0$$

Exact Solutions, why ?

- In massive gravity perturbation theory can be tricky
- Check in a nonperturbative way the presence/absence of vDVZ discontinuity
- The spherically symmetric case in GR is the perfect benchmark

The Schwarzschild solution in a nutshell

- Adapted coordinates (t, r, θ, φ), the geometry is static (timelike Killing vector hypersurface orthogonal)

$$ds^2 = -J(r)\, dt^2 + K(r)\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- Eistein equations give

$$J = K^{-1} = 1 - \frac{2MG}{r} \qquad\qquad M_{pl}^2 = 16\pi\, G$$

  M integration constant to be determined the Kepler 3rd law or by ADM energy: total energy of the system.

- Leading PN effects measured by $J = 1 + 2\phi$ and $K = 1 - 2\psi$ GR gives $\psi = \phi = GM/r$

Modifying Schwarzschild I

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- Spherically symmetric ansatz

$$ds^2 = -J(r)\,dt^2 + K(r)\,dr^2 + r^2\,d\Omega^2$$

$$\tilde{ds}^2 = -C(r)\,dt^2 + A(r)\,dr^2 + 2D(r)\,dtdr + B(r)\,d\Omega^2$$

- Einstein equations

$$M_{pl}^2\,E^\mu_\nu + [Det(X)]^{1/4}\left[V\,\delta^\mu_\nu - 4(V'X)^\mu_\nu\right] = 0$$
$$\kappa\,M_{pl}^2\,\tilde{E}^\mu_\nu + [Det(X)]^{-1/4}\left[V\,\delta^\mu_\nu + 4(V'X)^\mu_\nu\right] = 0$$

Finding all solutions if very hard. Consider solutions with $D \neq 0$
Potential independent analysis: g_{μ_ν} is diagonal $\Rightarrow E^\mu_\nu$ diagonal

$\Rightarrow (V'X)^\mu_\nu$ diagonal $\Rightarrow \tilde{E}^\mu_\nu$ diagonal $\Rightarrow \tilde{E}^1_1 = \tilde{E}^2_2 \Rightarrow$ $K = J^{-1}$

- First result potential independent: $\psi = \phi$, leading PN physics same as in GR. Solar system tests are OK !

Modifying Schwarzschild II

Class of exact solvable potentials

If $\{\lambda_i,\ i = 0, \cdots, 3\}$ are the eigenvalues of X, the potentials

$$V_n = \sum_{i_1 > i_2 \cdots > i_n} \lambda_{i_1}^{-1} \lambda_{i_2}^{-1} \cdots \lambda_{i_n}^{-1} = \frac{e_{4-n}(X)}{e_4(X)}$$

lead to analytically solvable equations

Examples

$$V_1 = \frac{1}{6|\tilde{g}|}(\epsilon\,\epsilon\,\tilde{g}\,\tilde{g}\,\tilde{g}\,g) = \tau_{-1} = (6\mathrm{Det}(X))^{-1}(\tau_1^3 - 3\,\tau_2\tau_1 + 2\,\tau_3)$$

$$V_2 = \frac{1}{2|\tilde{g}|}(\epsilon\,\epsilon\,\tilde{g}\,\tilde{g}\,g\,g) = (\tau_{-1}^2 - \tau_{-2}) = \mathrm{Det}(X)^{-1}(\tau_1^2 - \tau_2)$$

$$V_3 = \frac{1}{|\tilde{g}|}(\epsilon\,\epsilon\,\tilde{g}\,g\,g\,g) = (\tau_{-1}^3 - 3\,\tau_{-2}\tau_{-1} + 2\,\tau_{-3}) = 6^{-1}\mathrm{Det}(X)^{-1}\,\tau_1$$

Solution I

$$J = \left(1 - \frac{2Gm_1}{r}\right) + 2G\,S\,r^{\gamma}\,, \qquad KJ = 1$$

$$C = c^2\omega^2\left(1 - \frac{2Gm_2}{\kappa\,r}\right) - \frac{2G}{c\,\omega^2\kappa}S\,r^{\gamma}\,, \qquad D^2 + AC = c^2\omega^4$$

$$B = \omega^2 r^2\,, \qquad A = \cdots$$

- Integration constants: m_1, m_2 and S. Determined by the parameters in V: c, ω
- When $\gamma < 2$, for $r \to \infty$
  $g \to \text{diag}(-1, 1, 1, 1)$ and $g \to \omega^2\,\text{diag}(-c^2, 1, 1, 1)$

  Lorentz Breaking asymptotics for $c \neq 1$
- When $S \neq 0$ nontrivial modification but still flat at infinity when $\gamma < 2$. When $-1 < \gamma < 2$ the large r behaviour is modified !
- In general the solution can by AdS or dS at infinity ($\gamma < 2$) not shown $\cdots$

Solution II

How m_1, m_2 and S can be detemined for a body ?

Match the interior and exterior solution for the body

Result: From the matching condition we have

$$m_1 = M(1 + \alpha_1\, m_g^2 R^2)\,, \qquad\qquad m_2 = -\alpha_1\, m_g^2 R^2 M/c\,\kappa^{\,2}$$

$$S = m_g^2 M R^{1-\gamma} \alpha_2$$

Body of radius R and "bare" mass $M = 4\pi\rho R^3/3$

m_g graviton mass scale, α_1 and α_2 numerical factors

- For low density object the deviation depends on size and not on the mass. Long hairs ......
- Sun: $m_g^2 R^2 \sim 10^{-10}$ with $m_g \lesssim (10^{-20}\,\text{eV}) \sim (100\text{AU})^{-1}$
- Deviations are important when $m_g^{-1} \sim R$
- For large objects $R \gtrsim 10^5 R_\odot$ (red giants, large gas clouds, galaxies. . .) the effect may be of important
- α can be negative and negative the interaction energy could cause large fluids to anti-gravitate

Potential falling slower than $1/r$?

Suppose we have a solution of Einstein equations with a static potential ϕ,
$$g_{tt} = -1 - 2\phi$$
that, at large distances, falls off slower than $1/r$

- The total energy of the system would infinite. According Newton, source's total mass is $\sim$ flux of $\nabla\phi$

$$E = \frac{1}{4\pi G} \int_{\mathcal{S}_2} d^2x \, \vec{\nabla}\phi \cdot \vec{n}$$

  Finite E only if $\phi \sim 1/r$
- No such a solution in perturbative GR: Green function goes as $1/r$
- Modified gravity is needed
- Why do we need a non-Newtonian potential ?

Total Energy

To be physical the solution must have finite total energy
Energy in GR is tricky

- Equivalence principle forbids localization of gravitational energy
  Hypothetical EMT of gravity: $T_{\mathrm{GR}}(x_0) \sim \mathcal{F}(\partial g)_{|x_0}$. But at each x_0
  $g(x_0) \equiv \eta$ and $\partial g(x_0) = 0 \Rightarrow T_{\mathrm{GR}}(x_0) = 0$
- Energy cannot be taken apart but must be considered as whole
  Locally there is no gravity !
- Energy in GR is the conserved charge associated with an arbitrary translation in time, diff generated by a timelike vector
- Equivalently, given a solution, its ADM energy is the value of the Hamiltonian
  Needed: a splitting of spacetime in space + time

Energy I

- In adapted coordinates (t, x^i), ADM energy measured by an observer with a clock ticking t

$$H_{tot} = \int_{t=\text{const}} d^3x \left[\mathcal{H}\, N + \mathcal{H}_i\, N^i\right] + \int_{S^2,\, r\to\infty} d^2x \mathcal{B}$$

$$\overset{\text{on shell}}{=} \int_{S^2,\, r\to\infty} d^2x \mathcal{B}$$

S^2 is 2-sphere bounding space ($t = \text{const}$) at infinity

- The value of $\mathcal{B}$ and then the total energy depends on the detailed asymptotics of $g_{\mu\nu}$
- For asymptotically flat spacetime, $h_{ij} \sim \delta_{ij}/r$ at large r, and using asymptotics Cartesian coordinates x^i

$$H_{\text{tot, on shell}} = \int_{S^2,\, r\to\infty} d^2x \sqrt{\sigma} \left(\frac{\partial h_{ij}}{\partial x^j} - \delta^{mn} \frac{\partial h_{mn}}{\partial x^i}\right) n^i$$

Energy II

- ADM limitations: derivatives of h_{ij} (extrinsic curvature) must fall-off at least as $1/r^2$ to be well defined
- Coordinates must be Cartesian at Infinity No good for our solution ! Large distances: $D \sim 1/\sqrt{r}$ (for $\gamma < -1$). Too slow
- Analogous to the Schwarzschild solution written in Painlevé coordinates: $dt = dT - f' dr$

$$ds^2 = -J\,dt^2 + J^{-1}\,dr^2 + r^2\,d\Omega^2$$
$$= -J\,dT^2 + 2f'J\,dTdr + dr^2 + 2f'J\,dTdr + r^2\,d\Omega^2$$
$$f'^2 = J^{-2} - 1$$

  ADM energy is zero in Painleve coordinates !! In reality is not defined in Painleve coordinates. Extrinsic curvature does not have the right fall-off
- We need a more general tool: Gravitational energy as a Noether charge

Energy as a Noether Charge

- Consider the Noether charge associated to timelike translations: $x^\mu \to x^\mu + \xi^\mu$, with $\xi^2 < 0$
- Choose a set boundary condition for dynamical variables, adjust boundary terms in the action so that the charge is a scalar (coordinate independent). NB a reference metric is needed. We use flat space
- Fixing the induced metric on the the 2-surface $t = const$, $r = \bar{r}$ with $\bar{r}$ large, we get the Nester expression for the energy

$$E = \frac{1}{32\pi G}\int_{S_t} d^2z\, \epsilon_{\rho\sigma\mu\nu}$$
$$\left(\xi^\tau \Pi^{\beta\lambda}\Delta\Gamma^\alpha_{\beta\gamma}\,\delta^{\mu\nu\gamma}_{\alpha\lambda\tau} + \bar{\nabla}_\beta\xi^\alpha\Delta\Pi^{\beta\lambda}\,\delta^{\mu\nu}_{\alpha\lambda}\right)\frac{\partial x^\rho}{dz^1}\frac{\partial x^\sigma}{dz^2}\,,$$

- For Schwarzschild , $E = M$, even in Painleve coordinates. Actually does not depend on coordinates ! Ideal tool fo us

Computation of the energy

- Boundary terms come only from the kinetic parts
  the potential has no role here
- Contribution of $R(g)$ $\quad E = M - S\,\bar{r}^{\gamma+1}$
- Contribution of $R(\tilde{g})$ $\quad \tilde{E} = \tilde{M}\,c^2 + S\,\bar{r}^{\gamma+1}$.
- Total energy, finite even when $\bar{r} \to \infty$!

$$E_{tot} = E + \tilde{E} = M + \tilde{M}\,c^2$$

- Beware ! Consider a the frozen $\tilde{g}$ theory, equivalent to $\kappa \to \infty$. The solution for g is similar, but there is no $\tilde{E}$ contribution. Energy is infinite !
- No decoupling effects of “heavy modes” of $\tilde{g}$, needed to account for all energy budget
- Effective field theories are tricky in gravity when energy is concerned, heavy modes warp spacetime and sometime cannot be neglected

Conclusions and Outlook

- A non-standard Newton potential calls for modified gravity.
- Bigravity is a great tool for studying massive deformation of GR
- No dDVZ discontinuity in a large class of bigravity solutions
- Spherically symmetric solution featuring:
  1. First nontrivial large distance modification of gravity
  2. Finite total energy

- Outlook
  - What happens when the ghost free potential is used ?
  - Cosmological impact of massive deformation