



The Abdus Salam
International Centre for Theoretical Physics



2265-17

**Advanced School on Understanding and Prediction of Earthquakes and
other Extreme Events in Complex Systems**

26 September - 8 October, 2011

**Recovery Times
as Indicators of Stability Loss**

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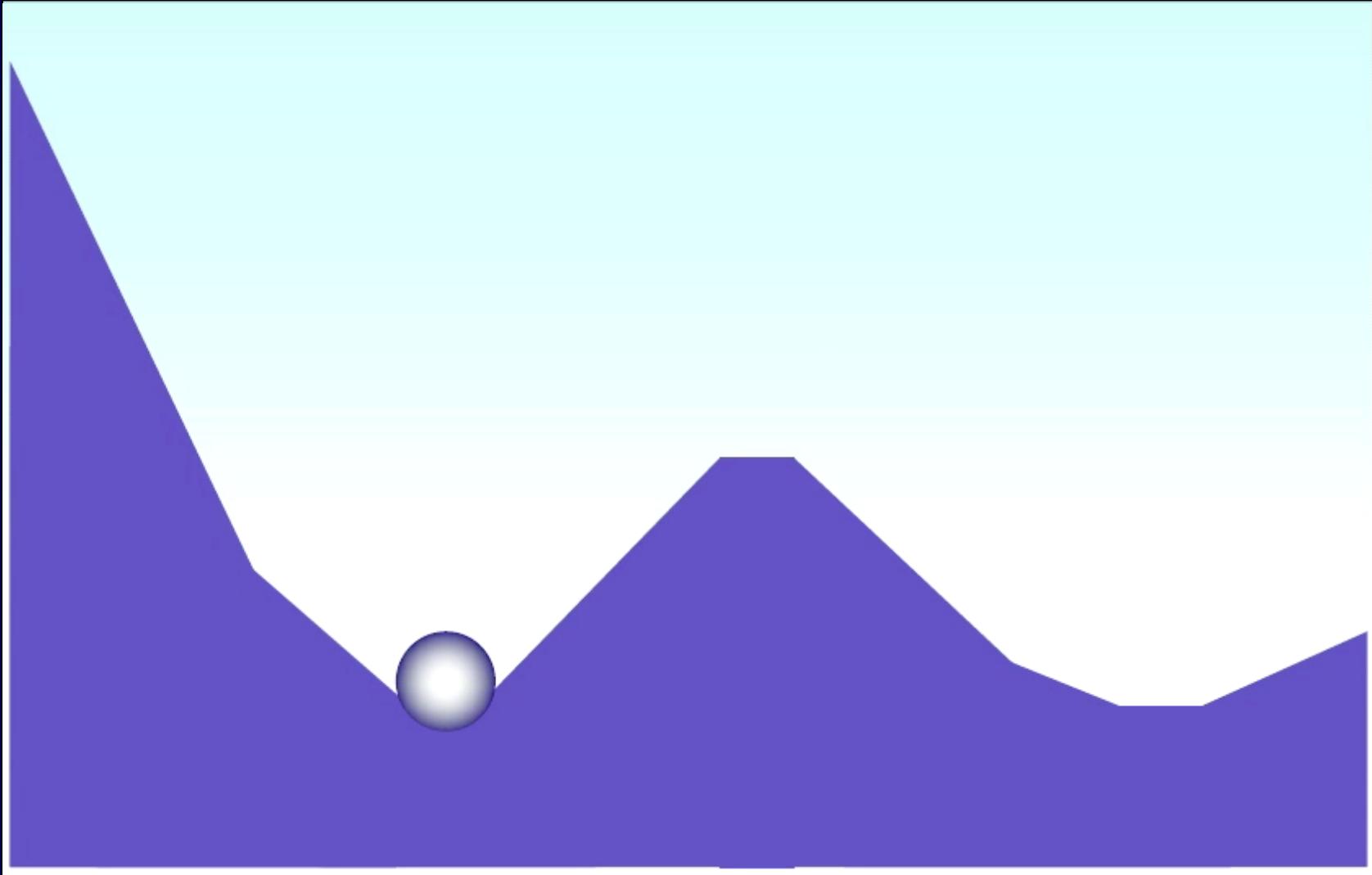
Arkady Kryazhimskiy

Recovery Times as Indicators of Stability Loss

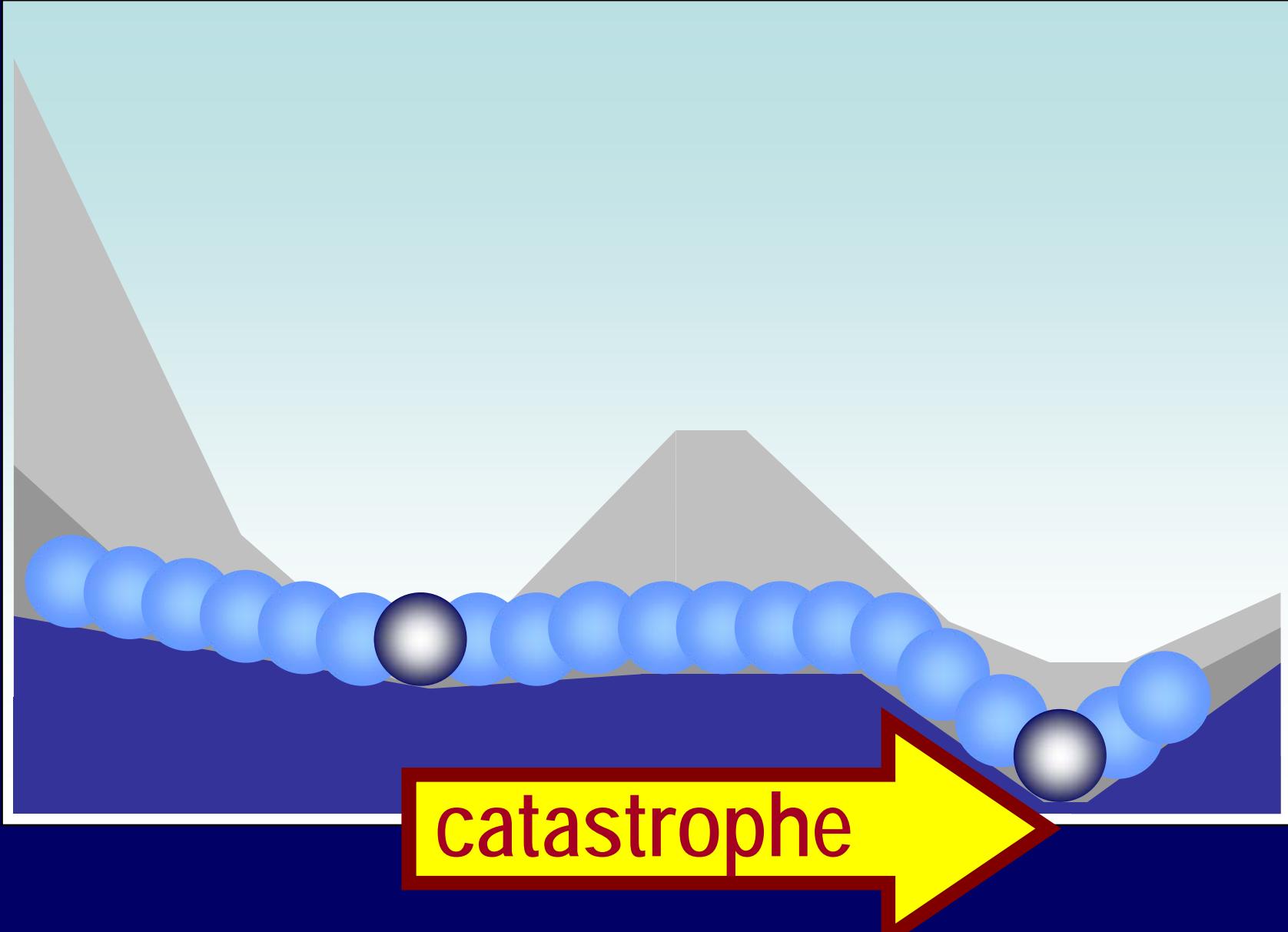
Trieste, October 4, 2011

Loss in stability

Loss in stability



Loss in stability



Loss in stability

Growth in recovery time

Loss in stability



Shift of steady state

Catastrophe



Mathematical interpretation

• Mathematical interpretation of the results

Mathematical interpretation

$$\dot{\xi} = c\xi$$

$$c = f'(x_0)$$

$$\dot{\xi} = f(x) - f(x^0) = f'(x^0)\xi$$

$$\xi = x - x^0$$

Rate of change



$$\dot{x} = f(x)$$

State



Steady state



$$f(x^0) = 0$$

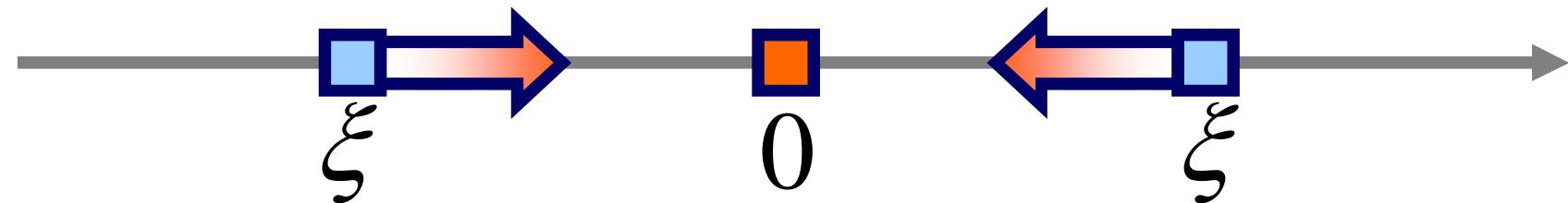
Mathematical interpretation

$$\dot{\xi} = c\xi$$

$$c = f'(x_0)$$

$$c < 0$$

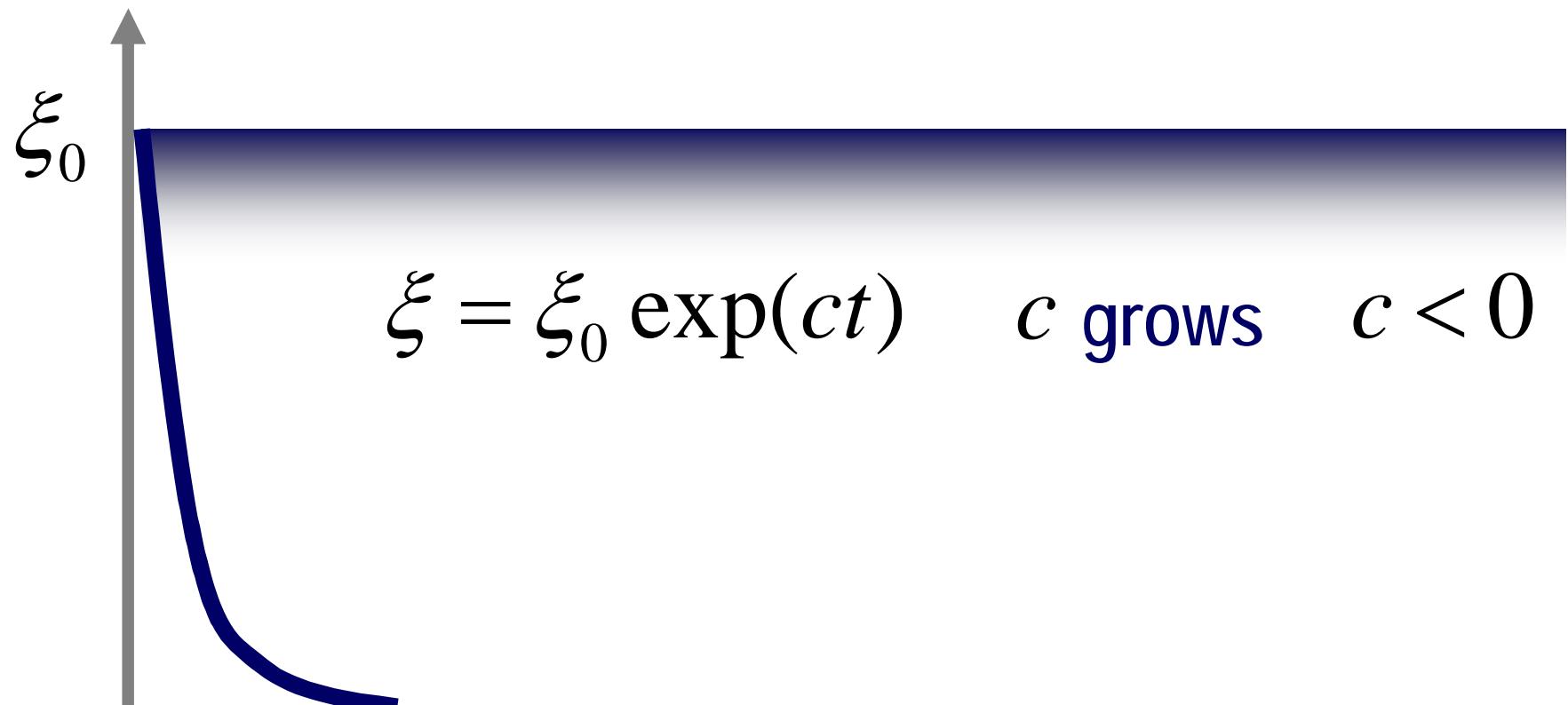
Stability



Mathematical interpretation

$$\dot{\xi} = c\xi$$

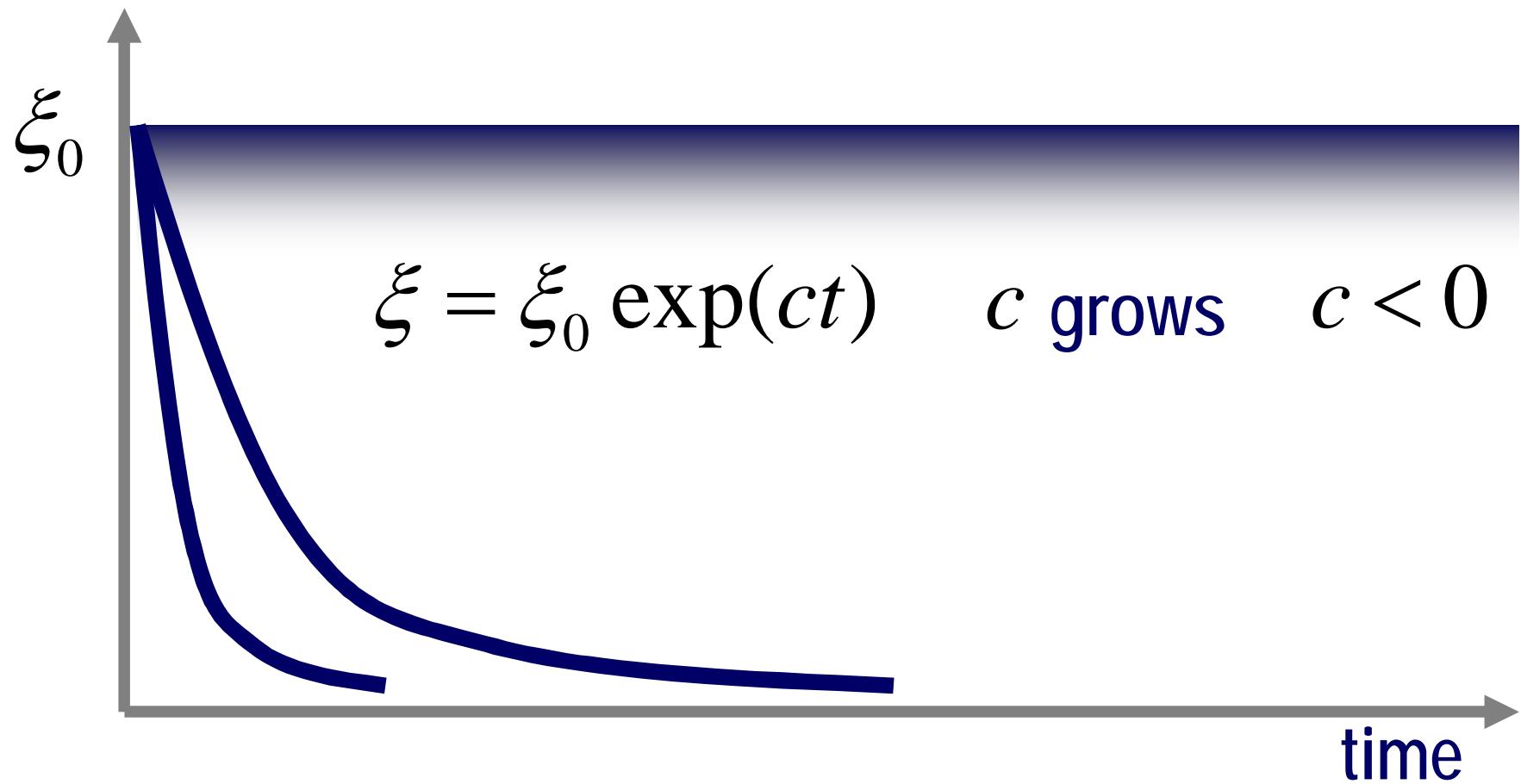
$$c = f'(x_0)$$



Mathematical interpretation

$$\dot{\xi} = c\xi$$

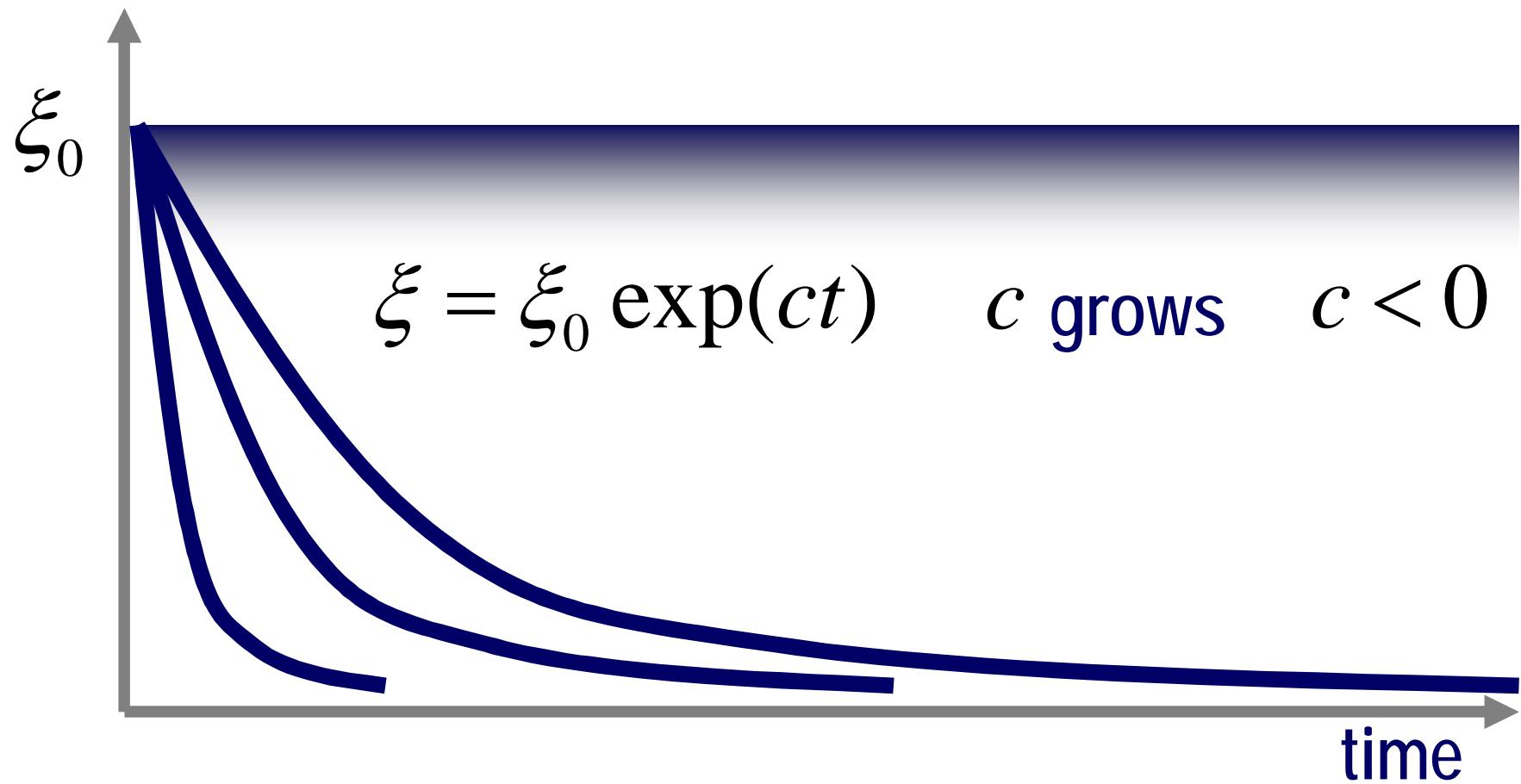
$$c = f'(x_0)$$



Mathematical interpretation

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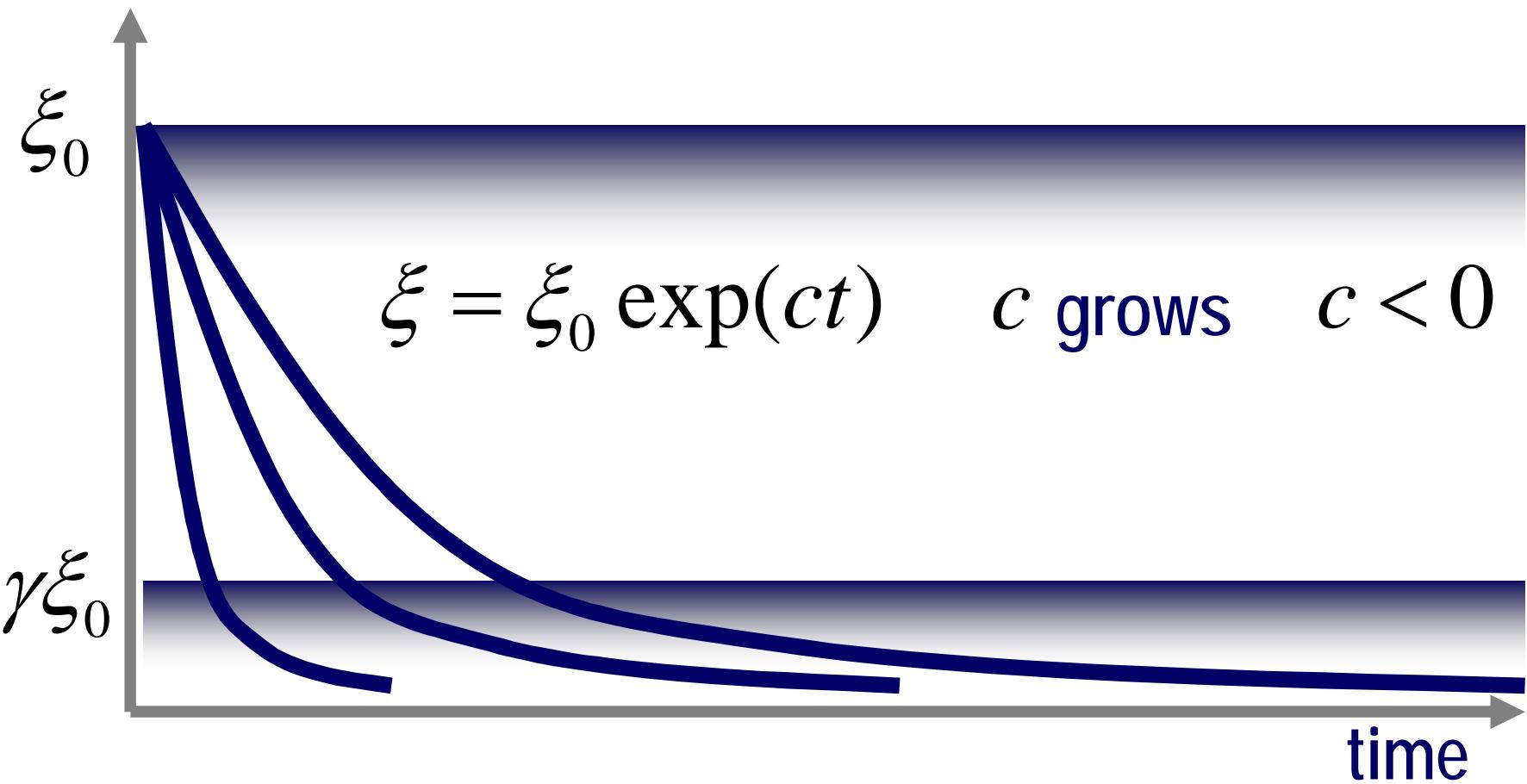
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Mathematical interpretation

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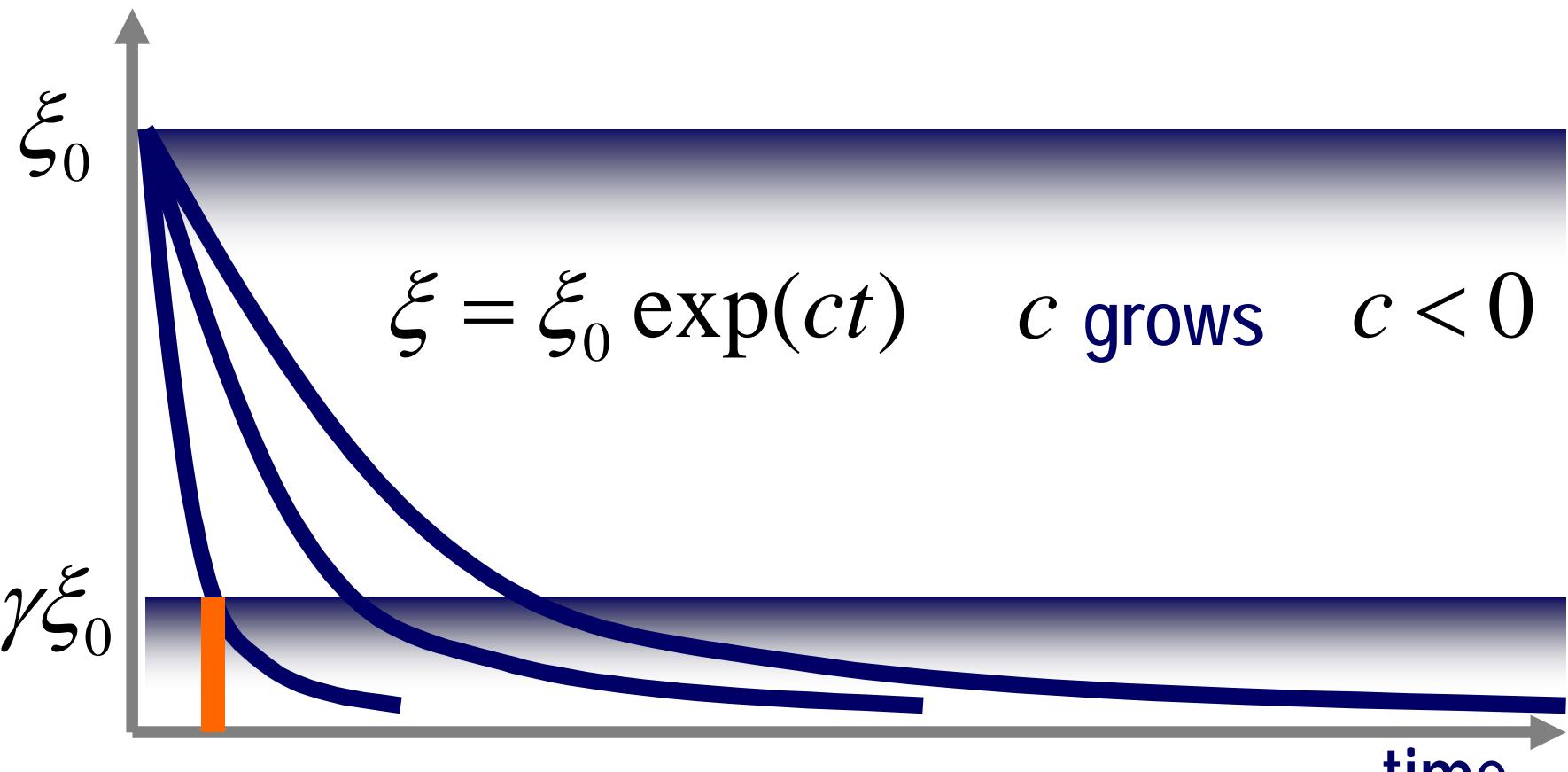
$$c = f'(x_0)$$



Mathematical interpretation

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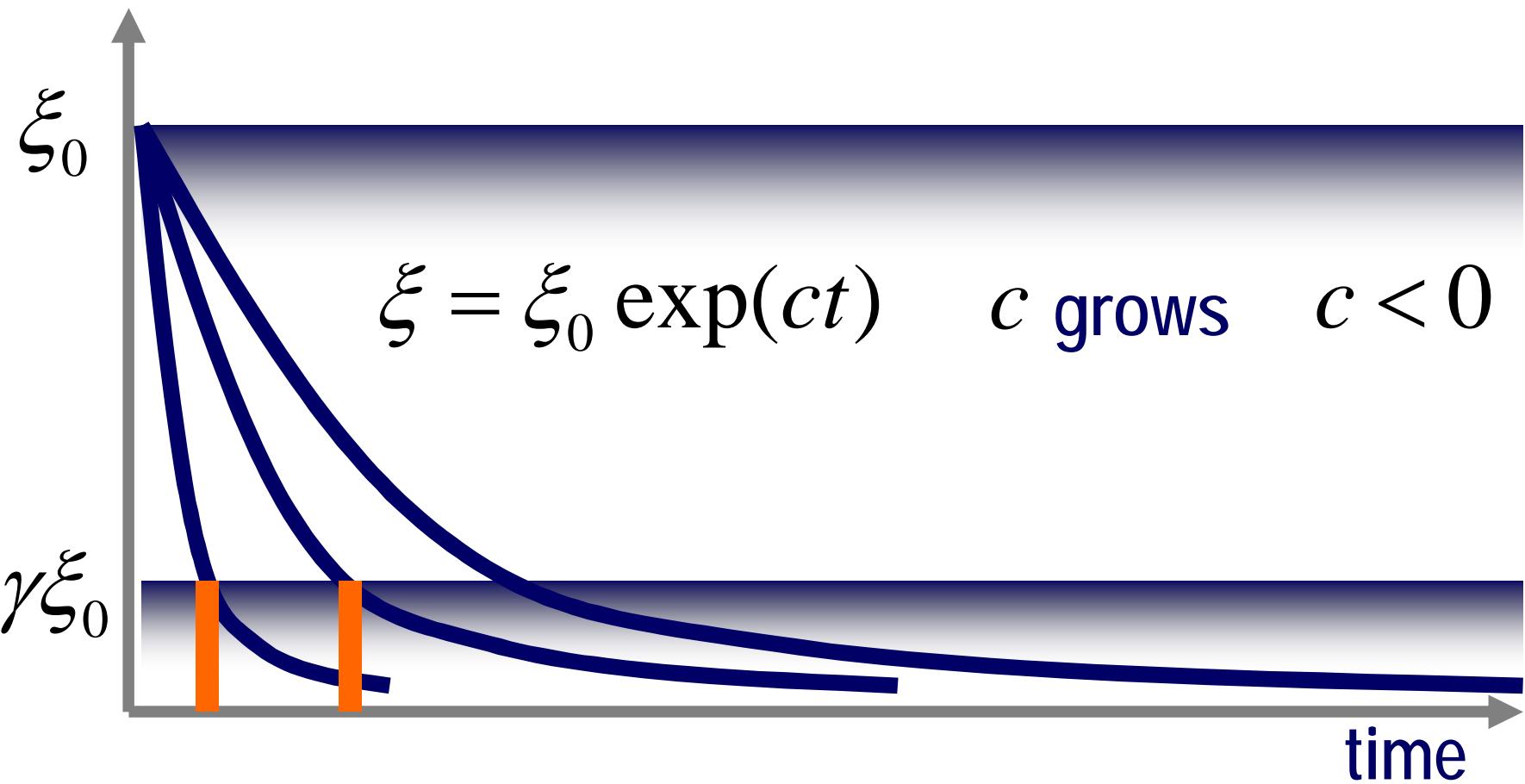
$$c = f'(x_0)$$



Mathematical interpretation

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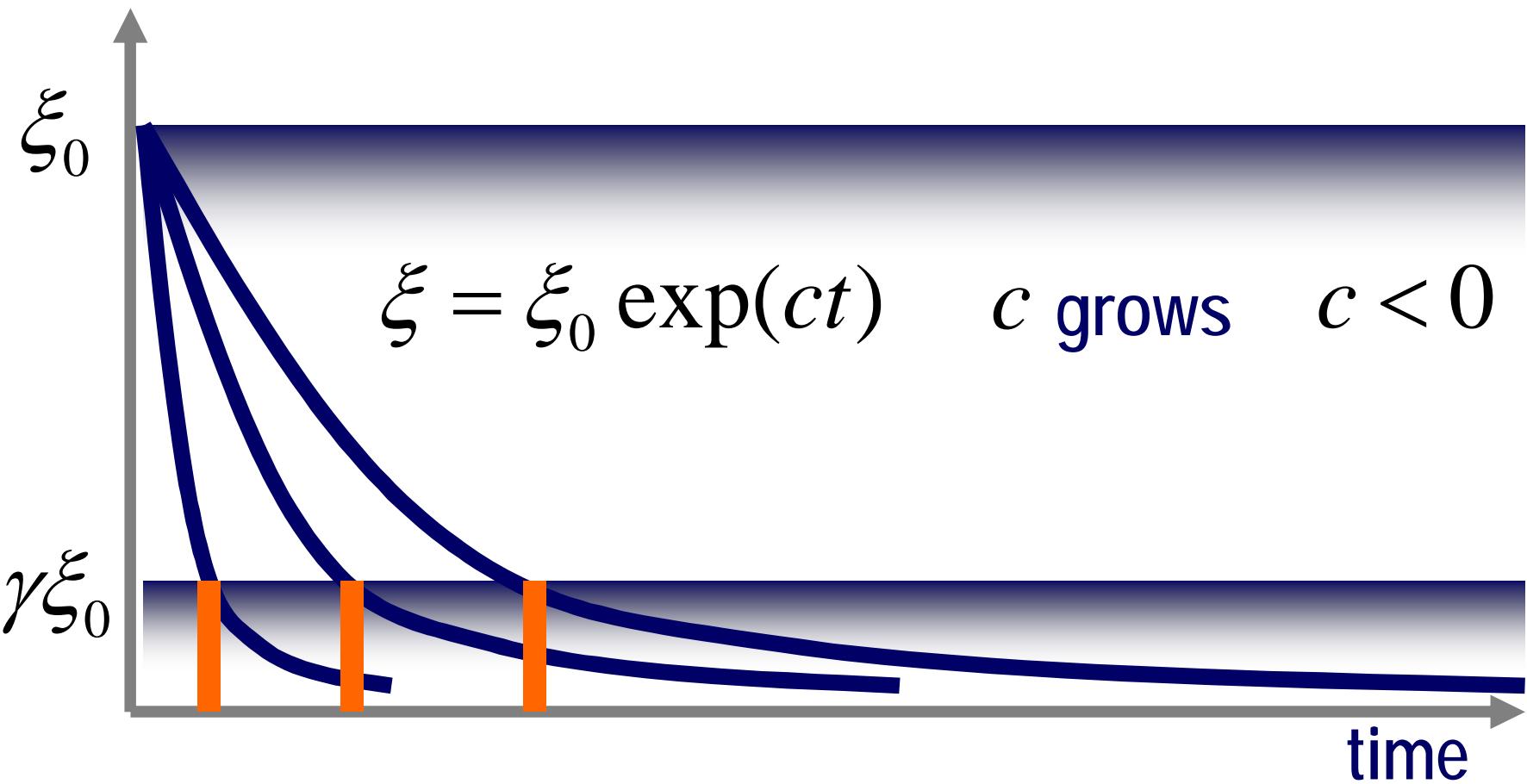
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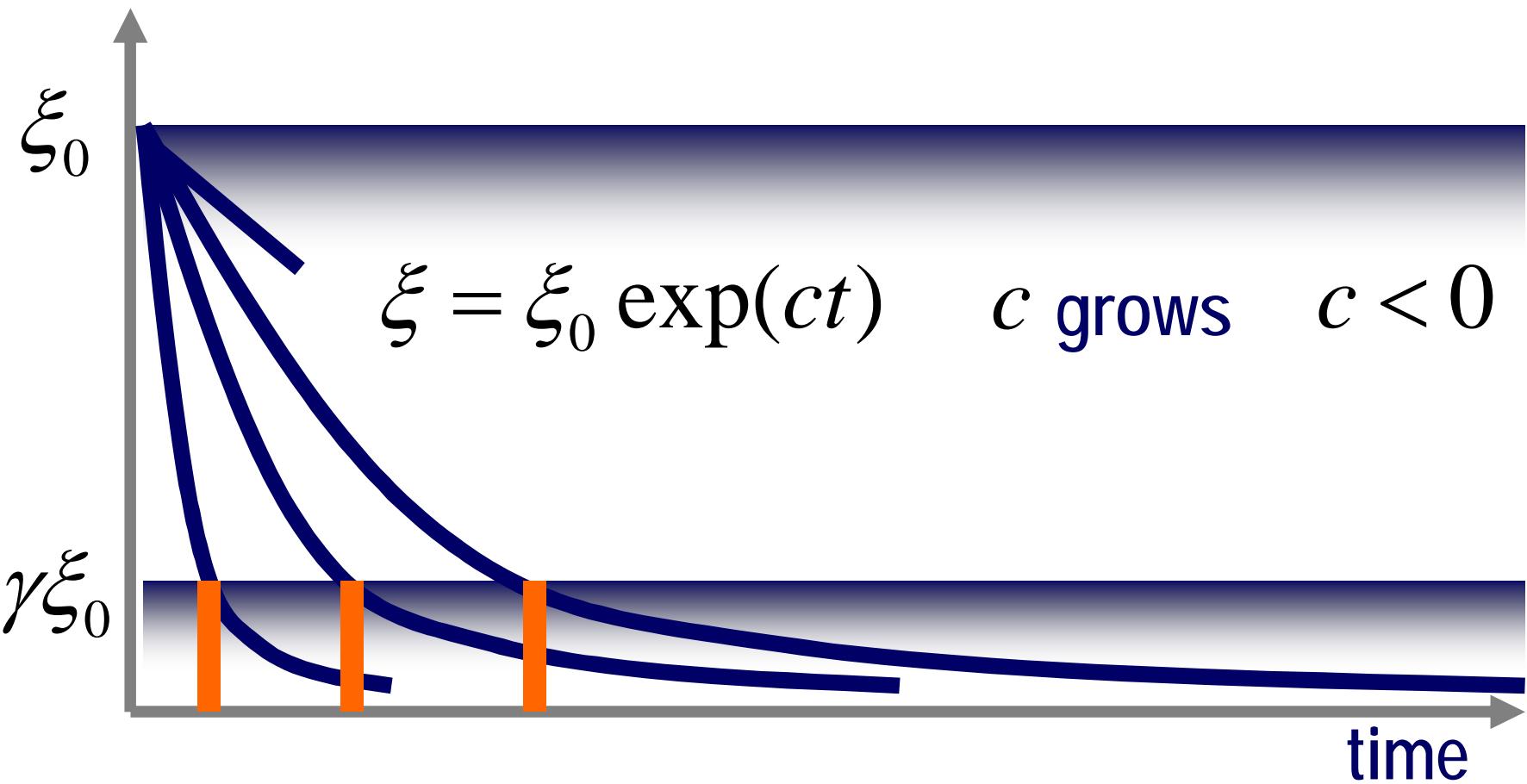
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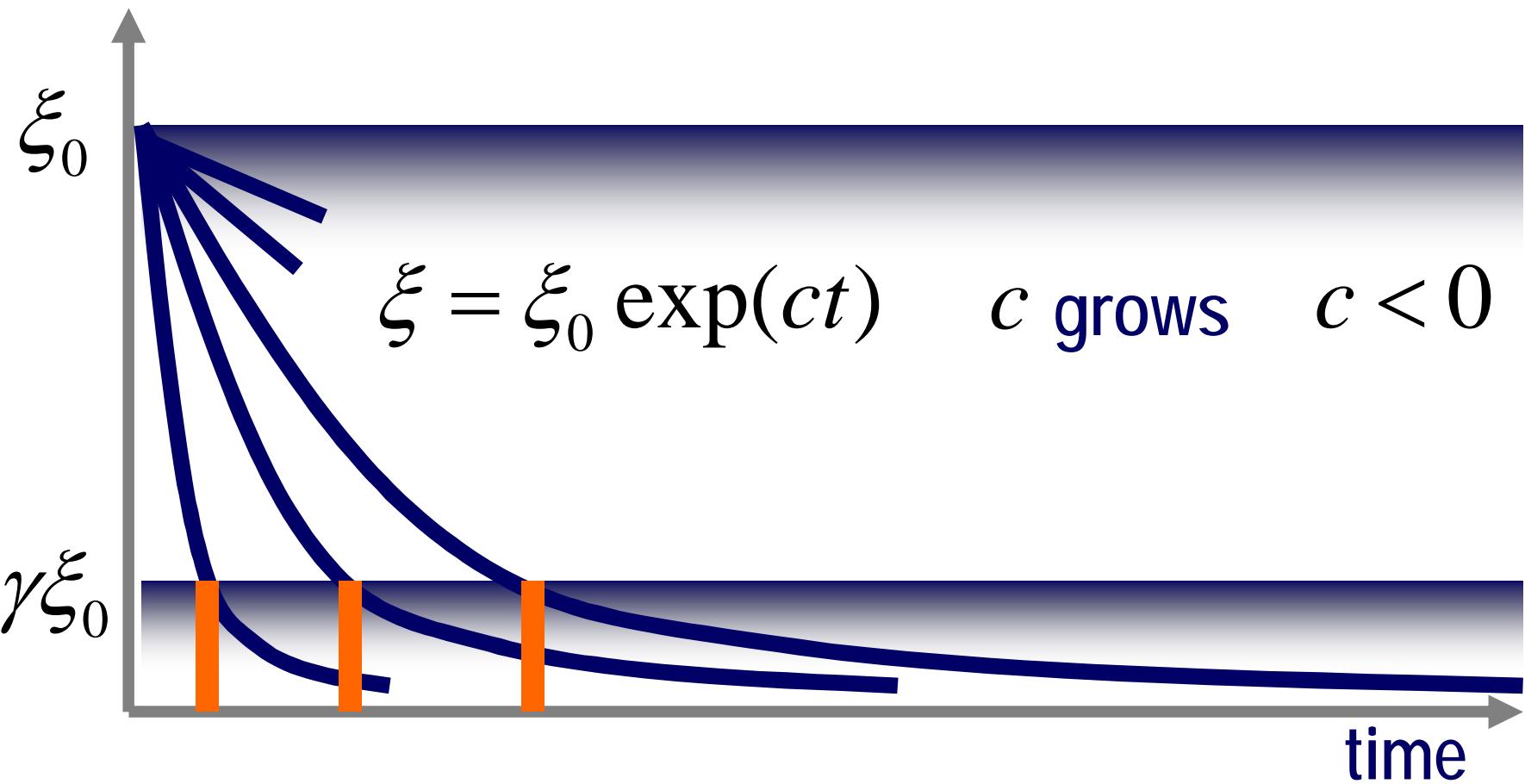
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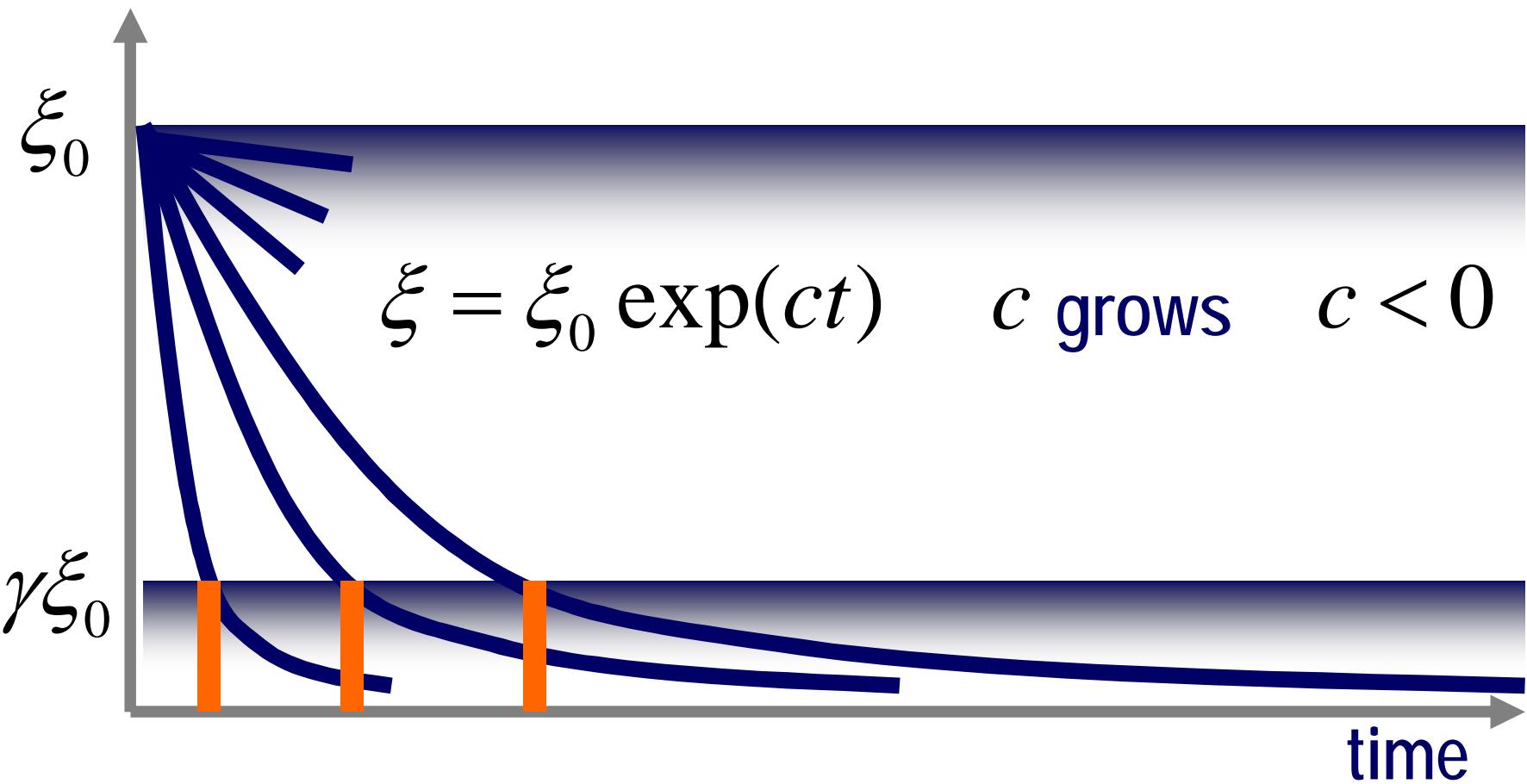
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Mathematical interpretation

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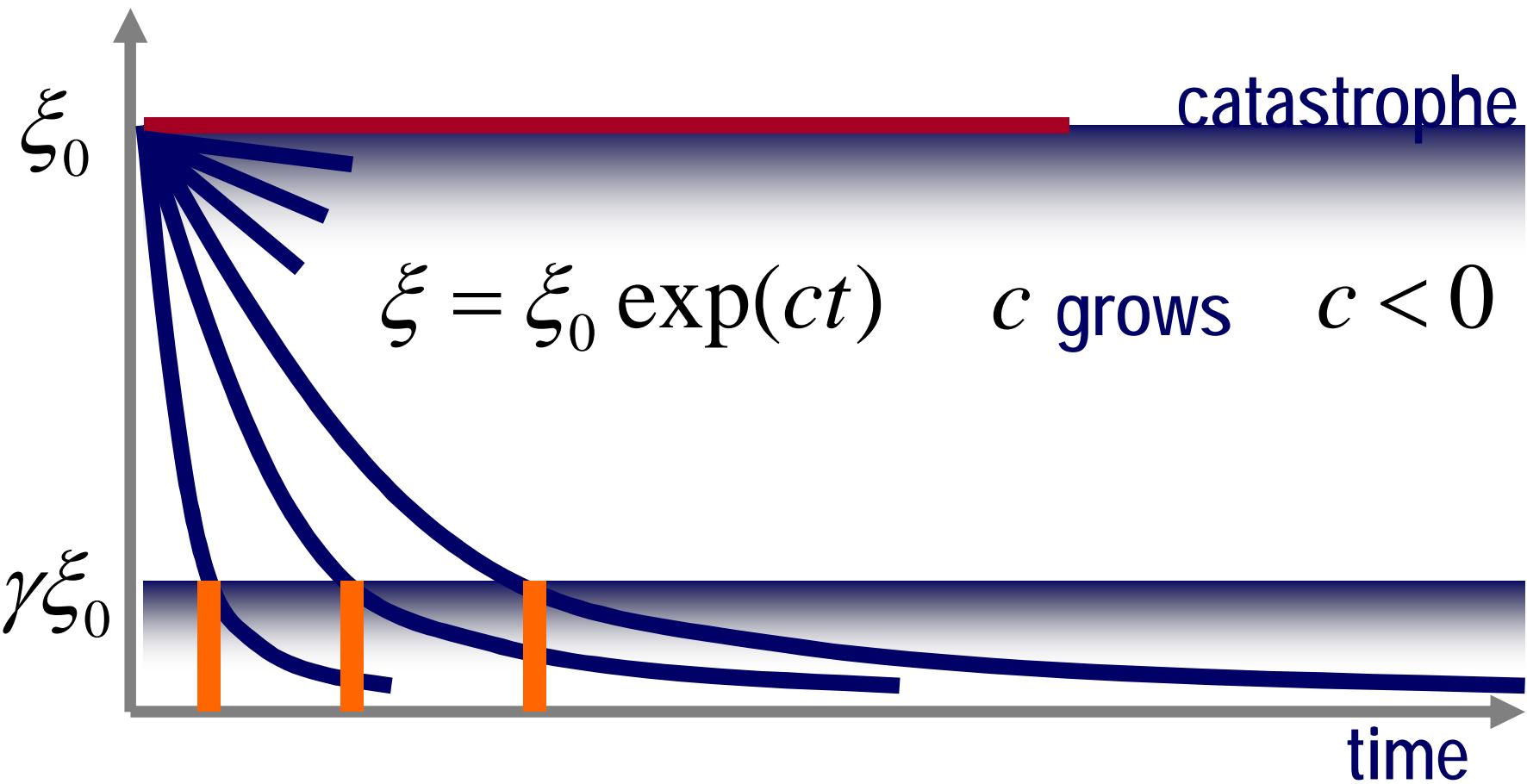
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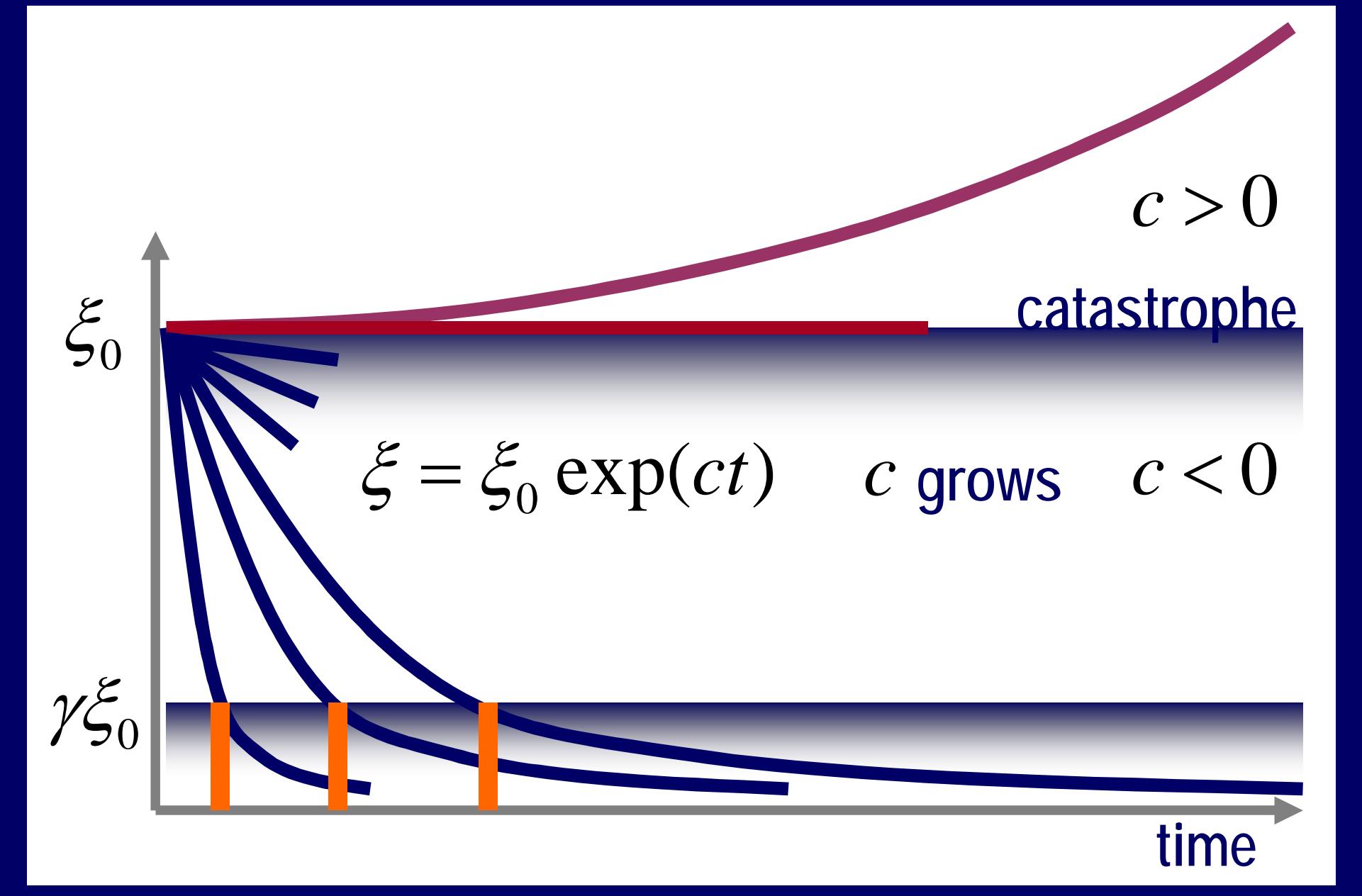
Mathematical interpretation

$$\dot{\xi} = c \xi$$

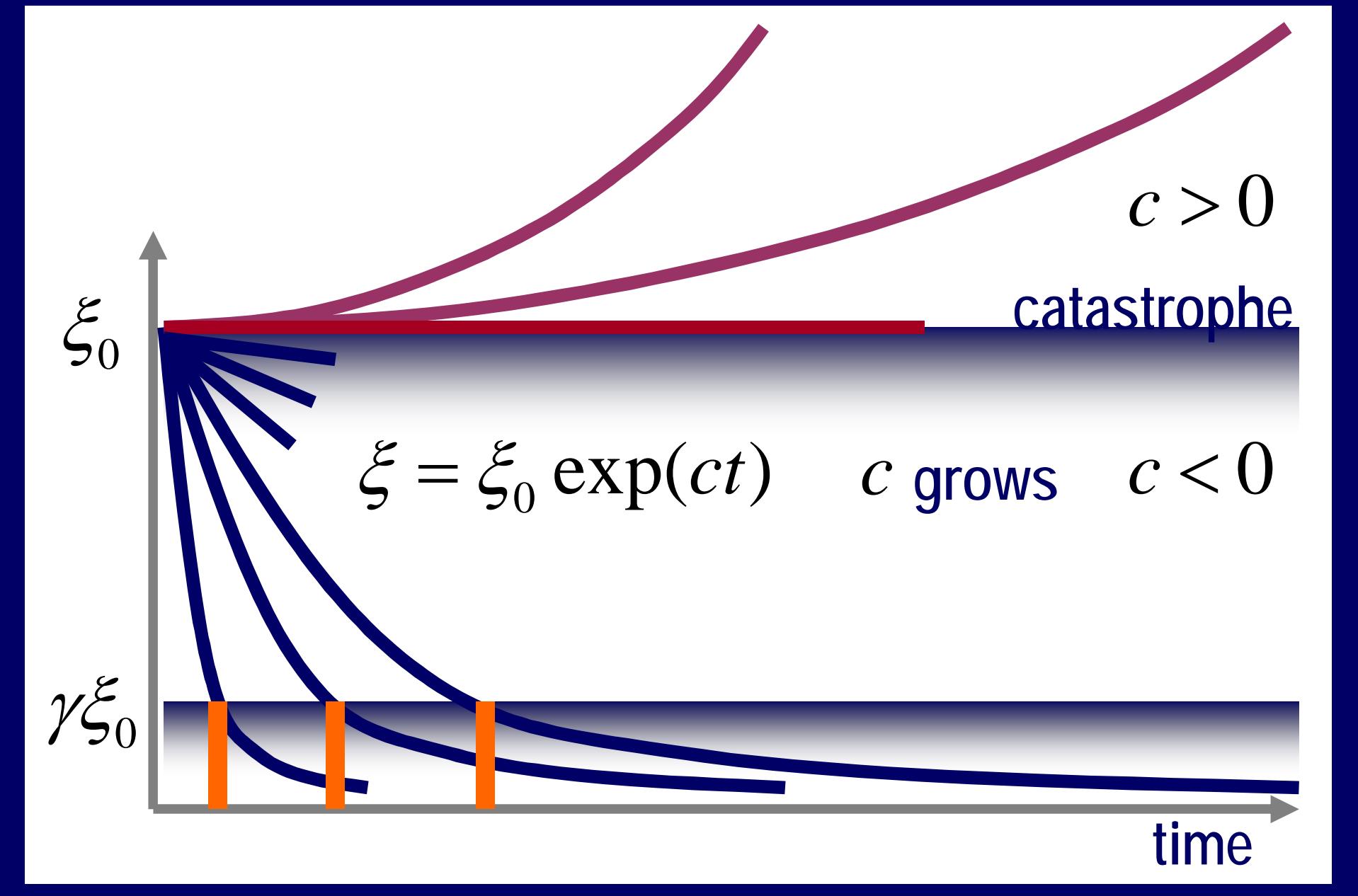
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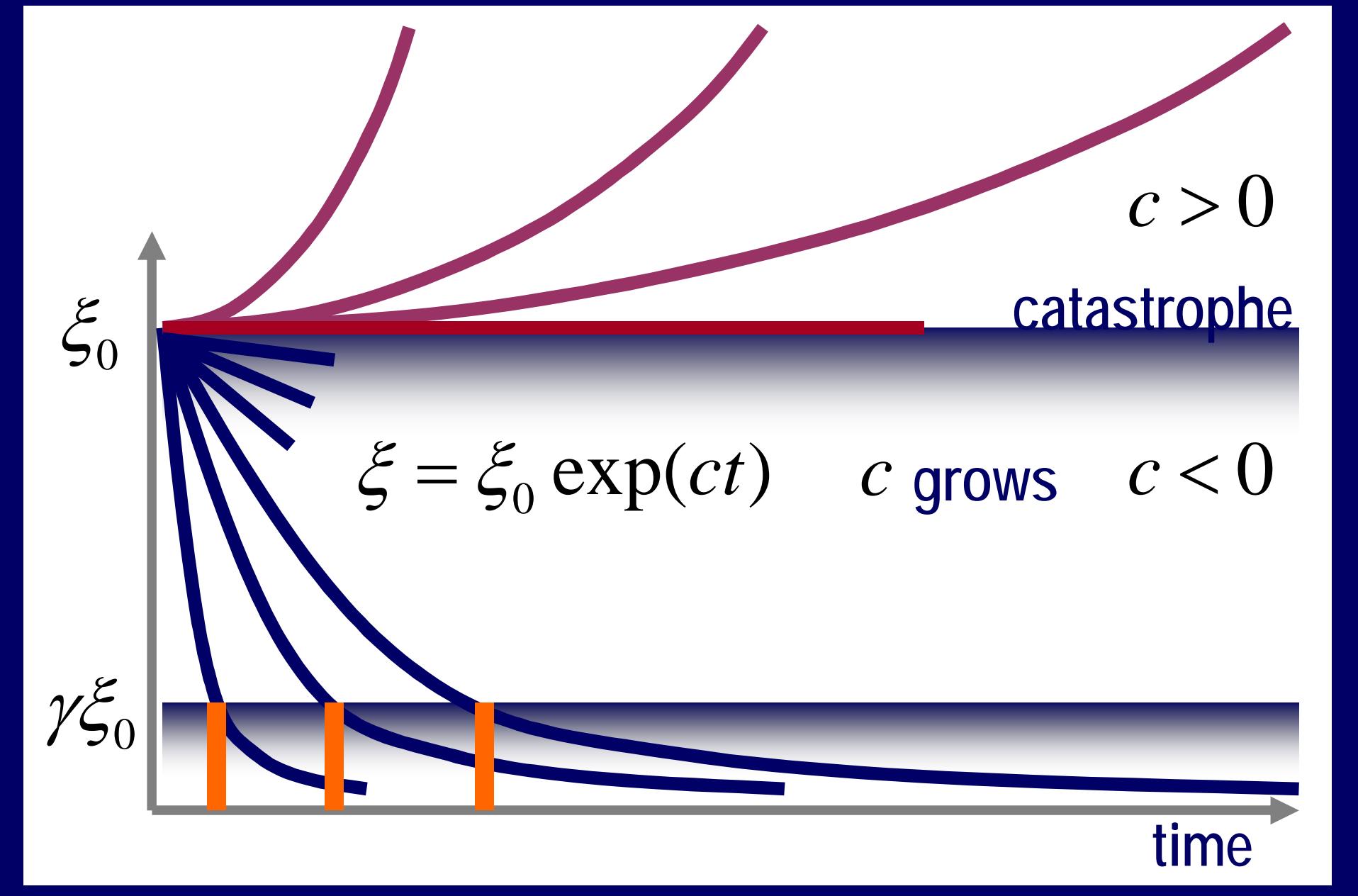
Mathematical interpretation



Mathematical interpretation



Mathematical interpretation



Multi-dimensional system

Multi-dimensional system

$$\dot{\xi} = C\xi \quad C = (c_{ik})$$

$$\dot{\xi}_i = \sum_{k=1}^n \frac{\partial f_i(x^0)}{\partial x_k} \xi_k \quad c_{ik}$$

$$\xi = x - x^0 = (\xi_1, \dots, \xi_n)^T$$

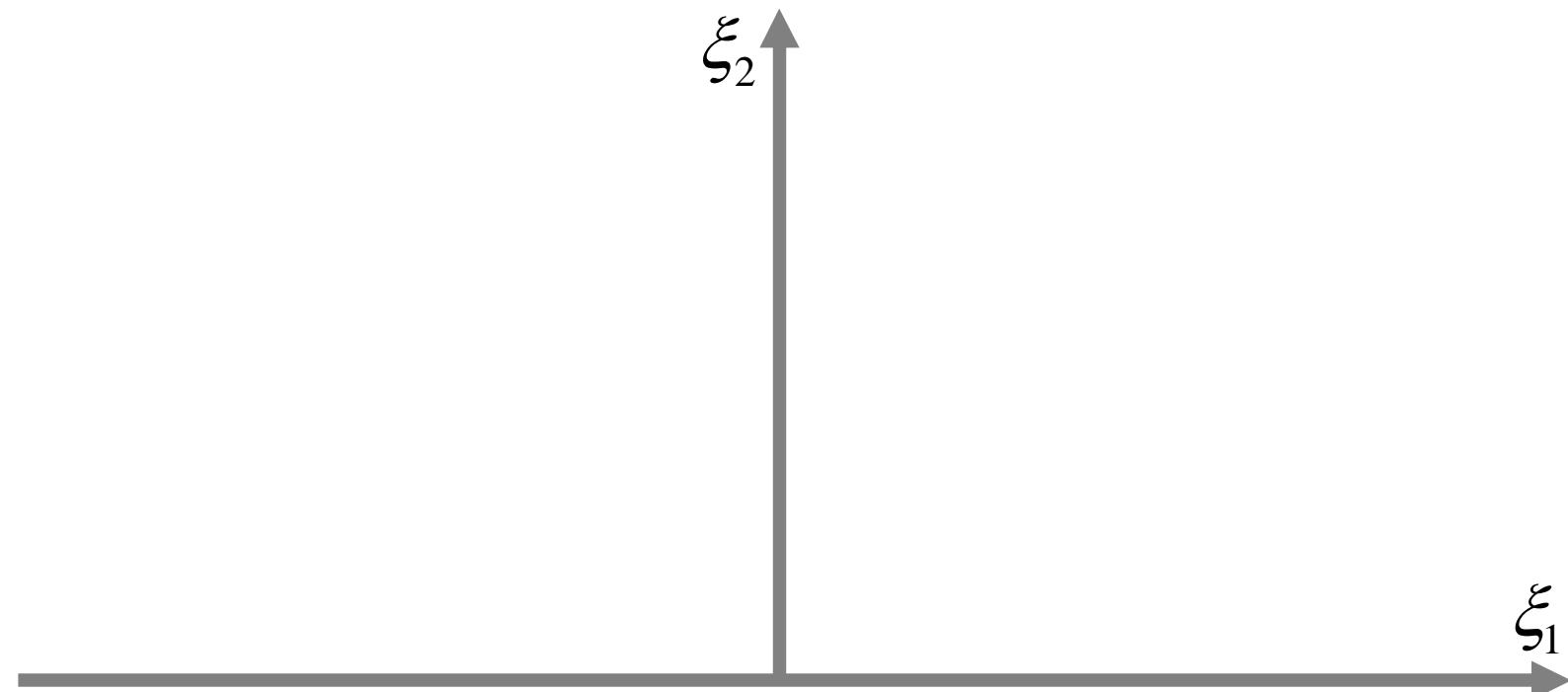
$$\dot{x}_i = f_i(x) \quad x = (x_1, \dots, x_n)^T$$

Multi-dimensional system

$$\dot{\xi} = C\xi$$

$$Cy = \lambda y$$

$$\xi = \alpha_1 \exp(\lambda_1 t) y_1 + \alpha_2 \exp(\lambda_2 t) y_2$$



$$y_1, \dots, y_n$$

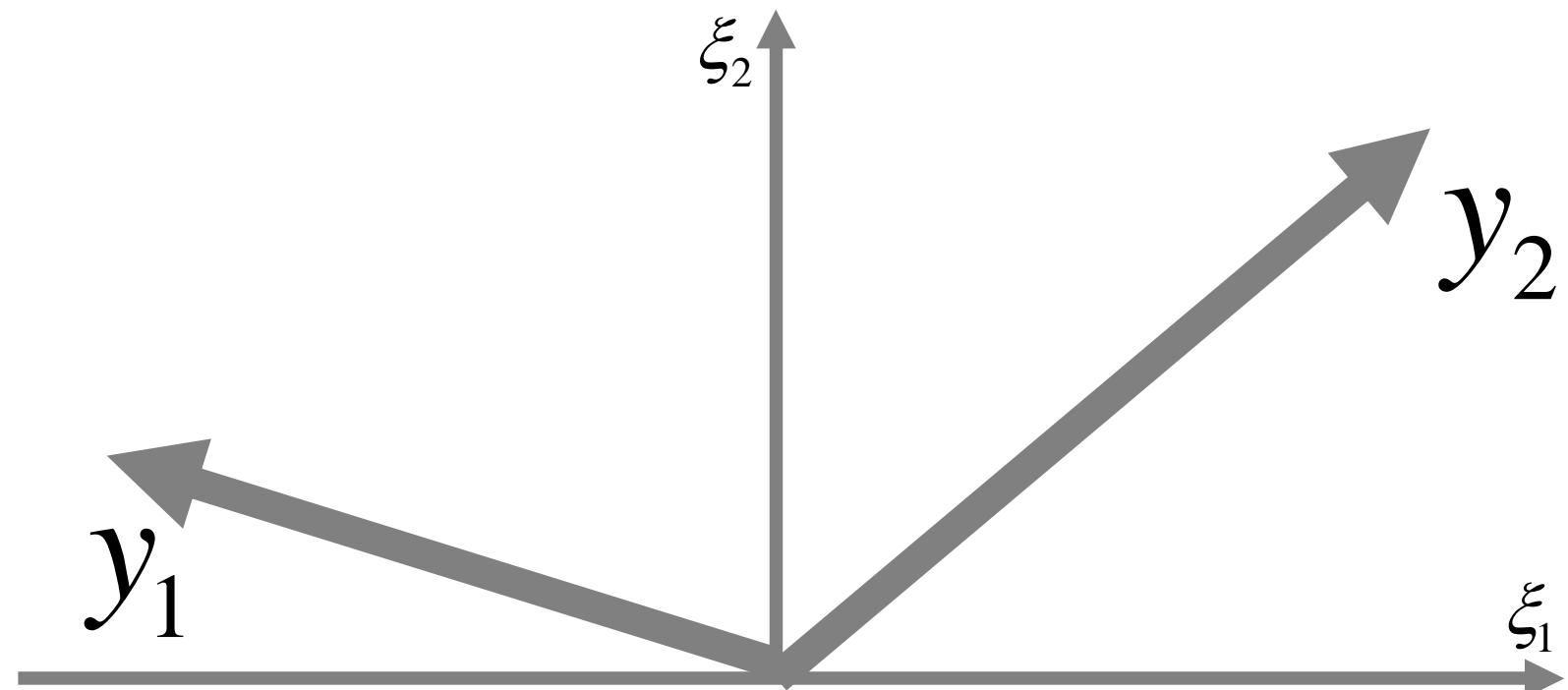
$$0 > \lambda_1 > \dots > \lambda_n$$

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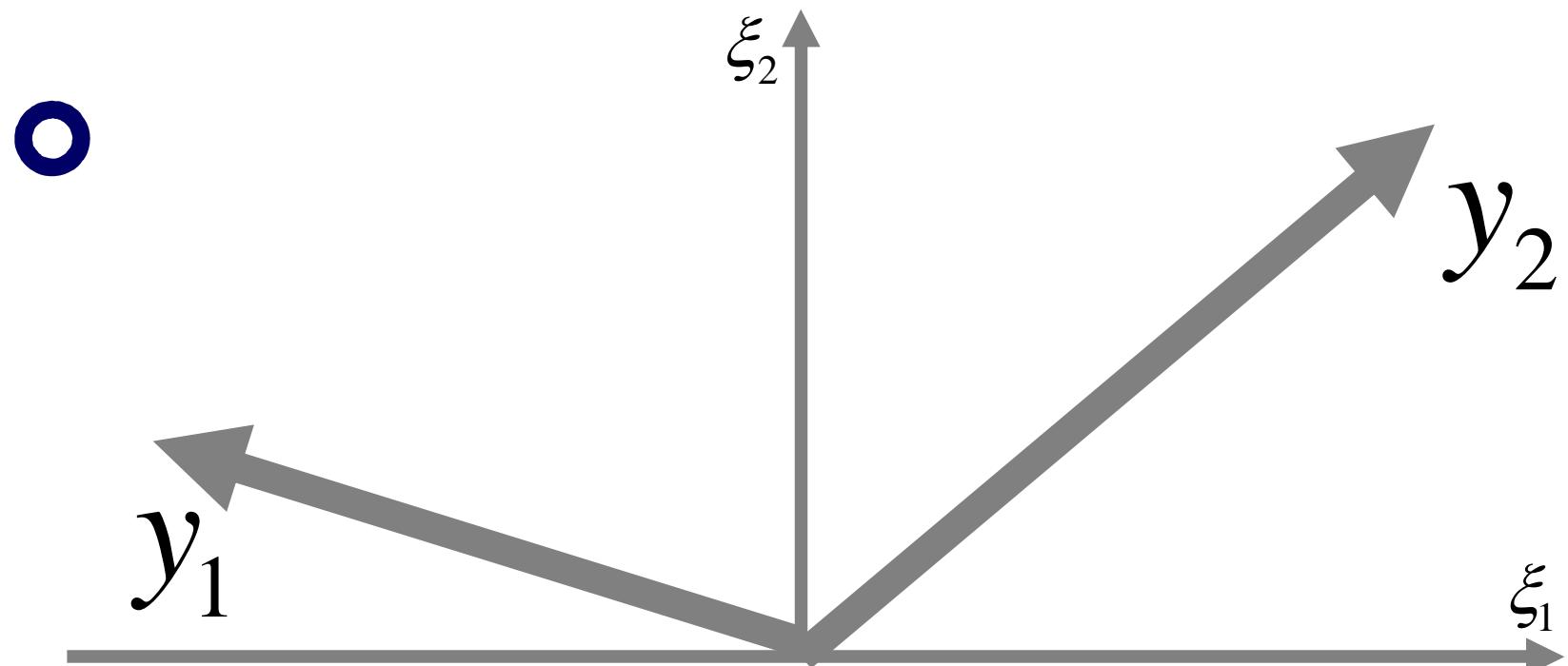
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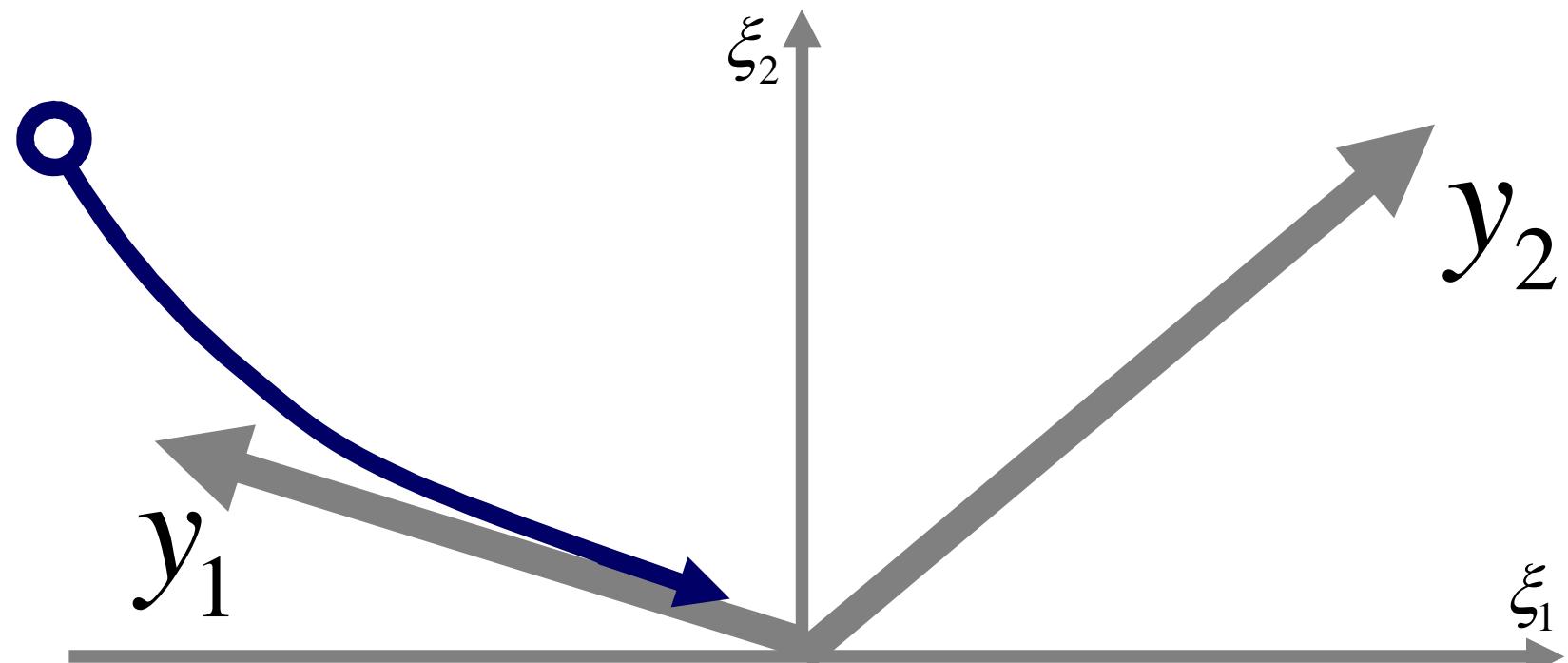
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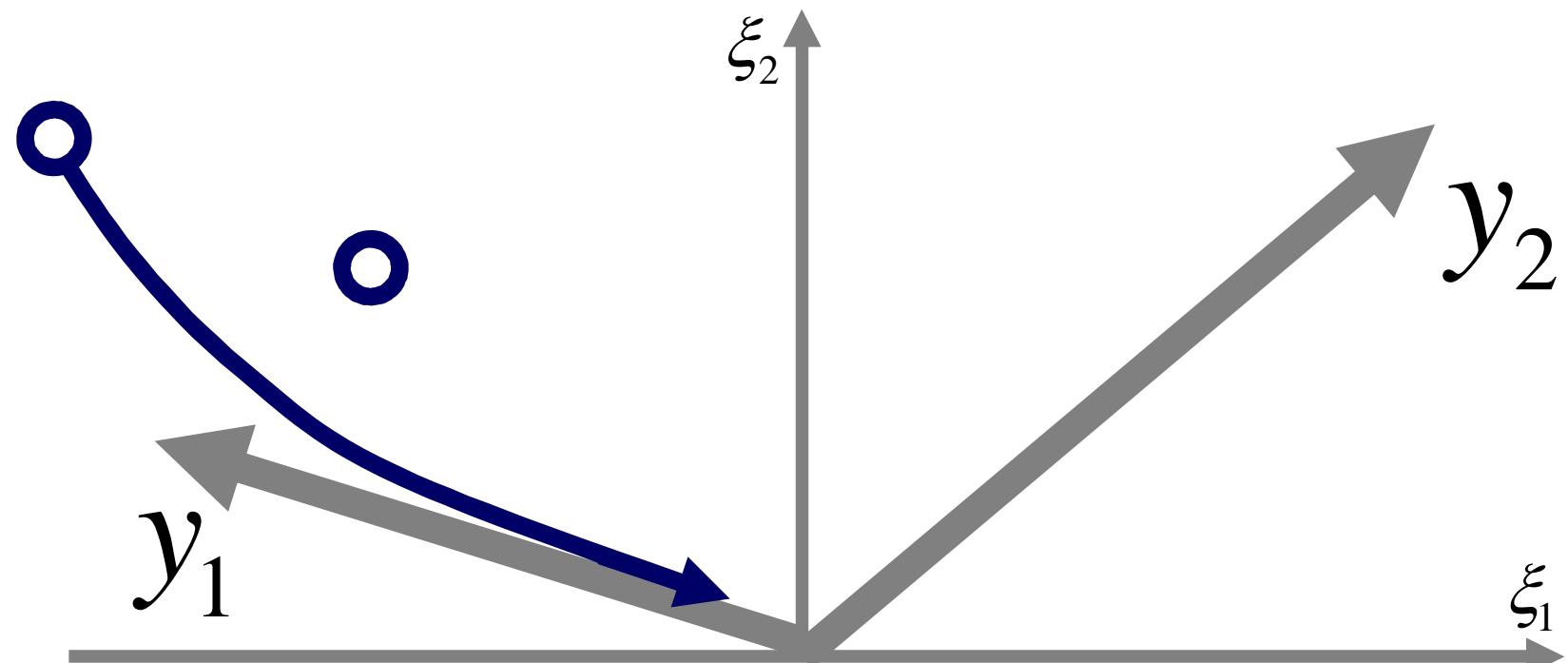
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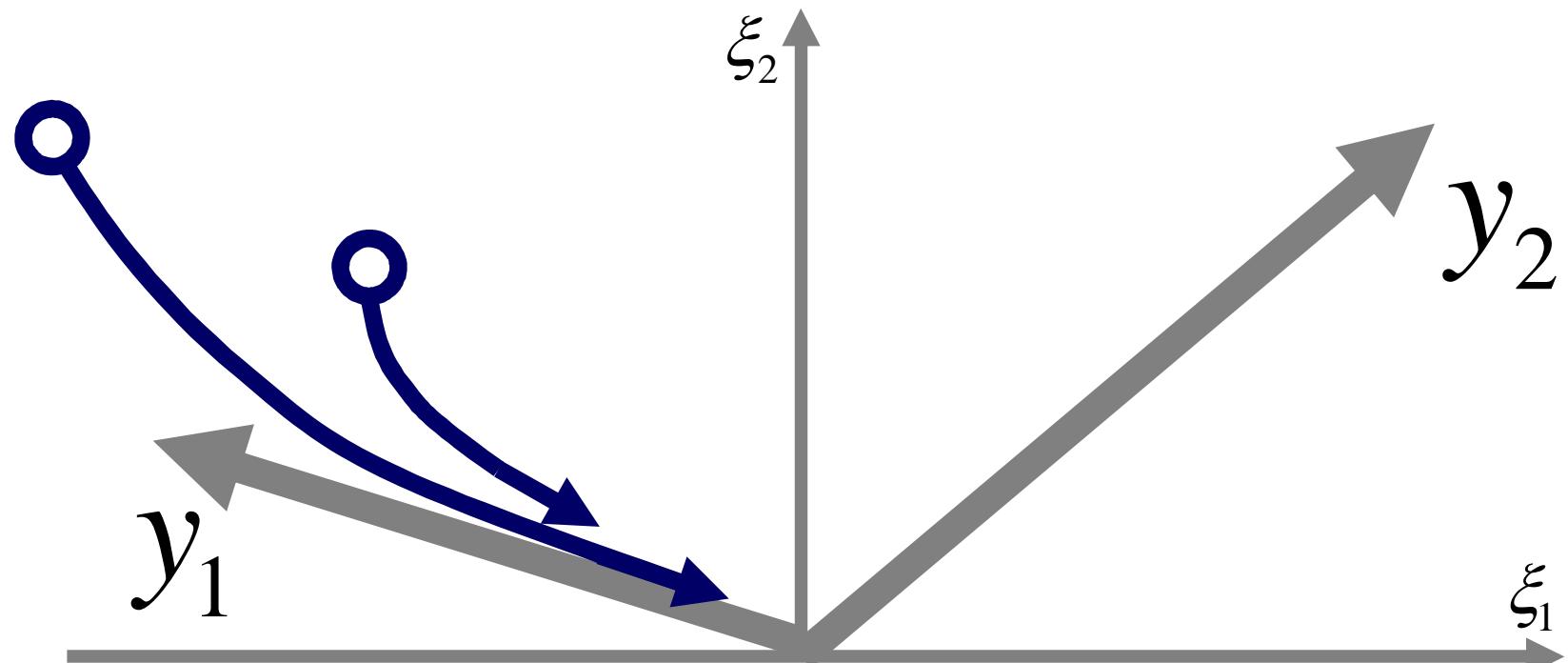
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$$y_1, \dots, y_n$$

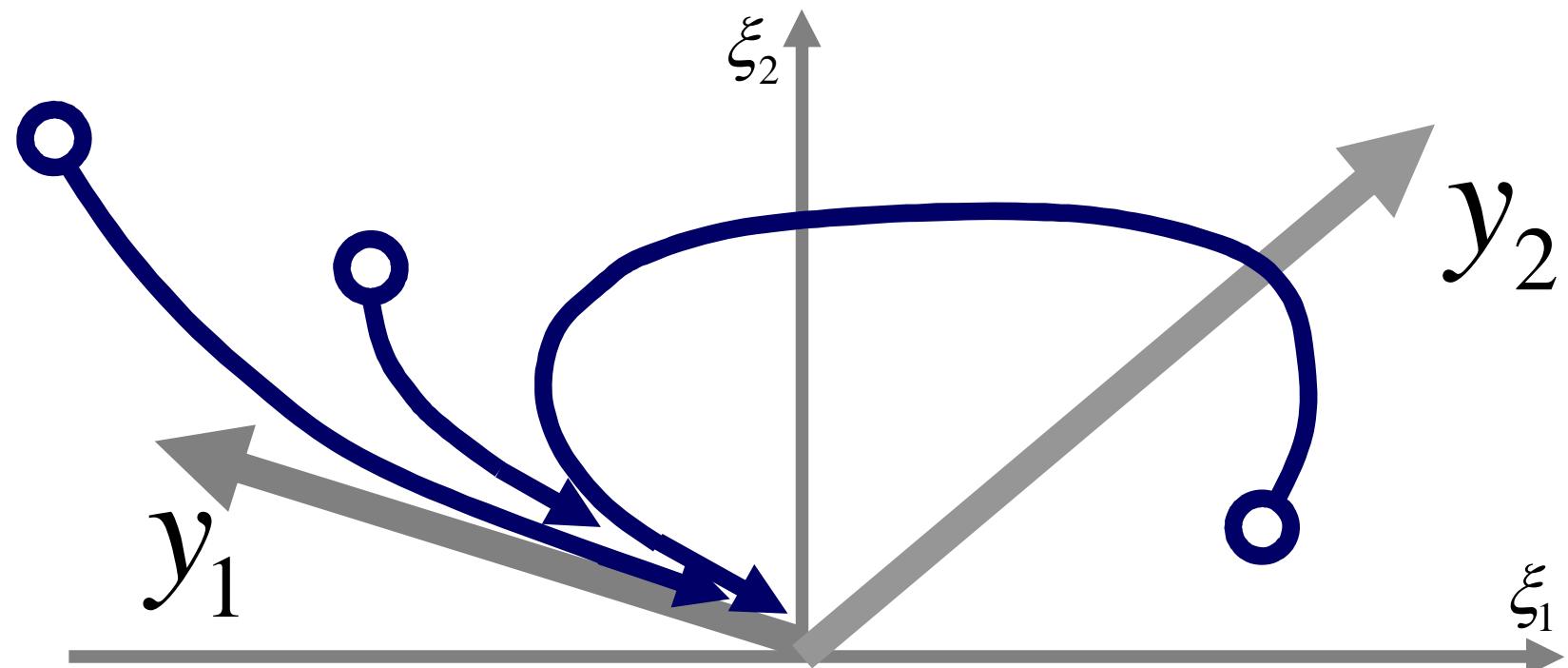
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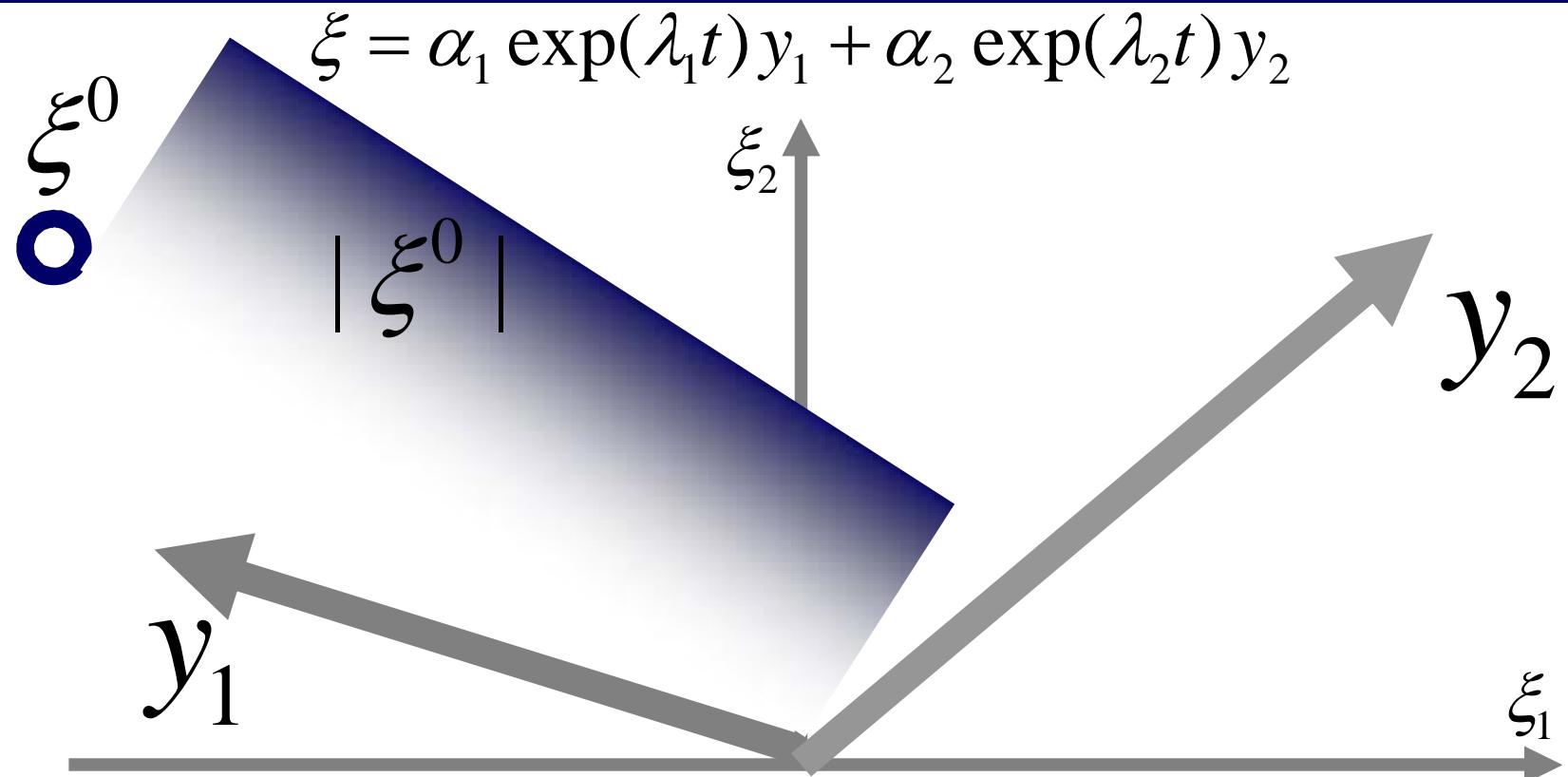
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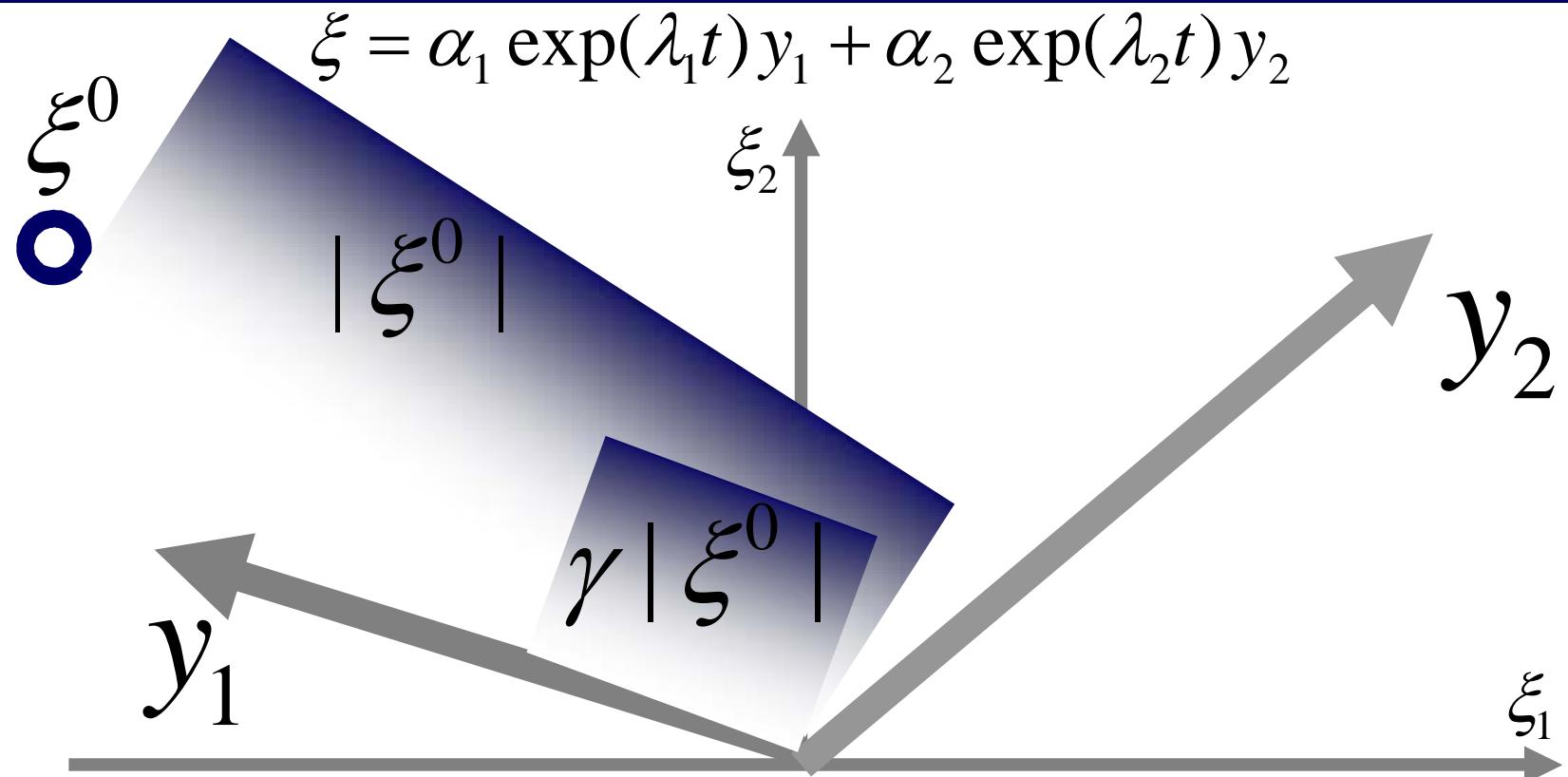
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Multi-dimensional system

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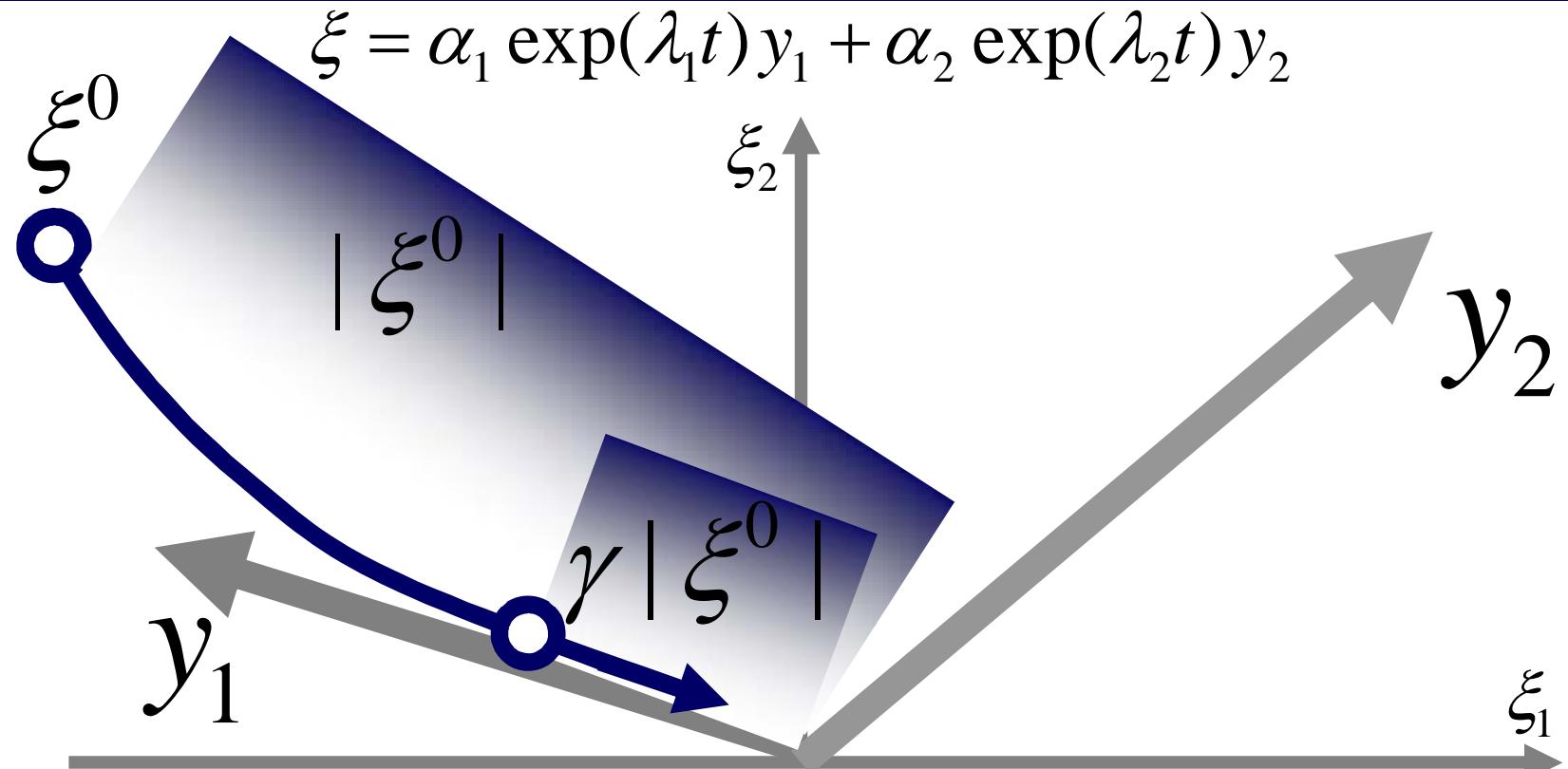
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Multi-dimensional system

$$\dot{\xi} = C\xi$$

$$Cy = \lambda y$$



γ -recovery time

$\tau(\xi^0, \gamma)$

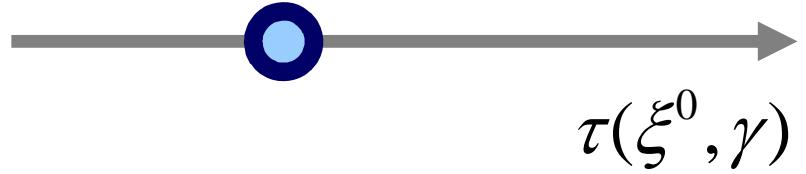
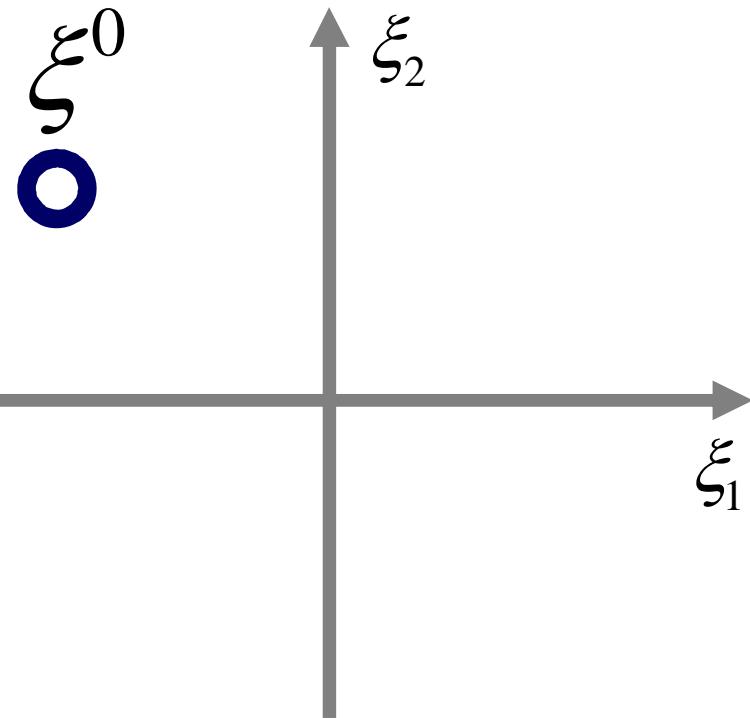
Multi-dimensional system

γ -recovery time $\tau(\xi^0, \gamma)$

Expected recovery time

γ -recovery time $\tau(\xi^0, \gamma)$

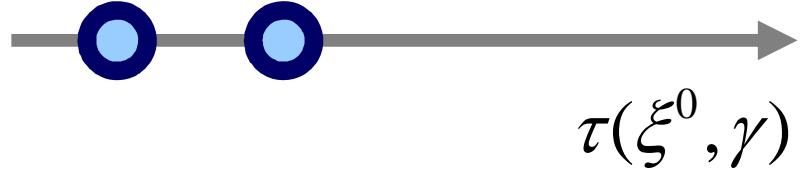
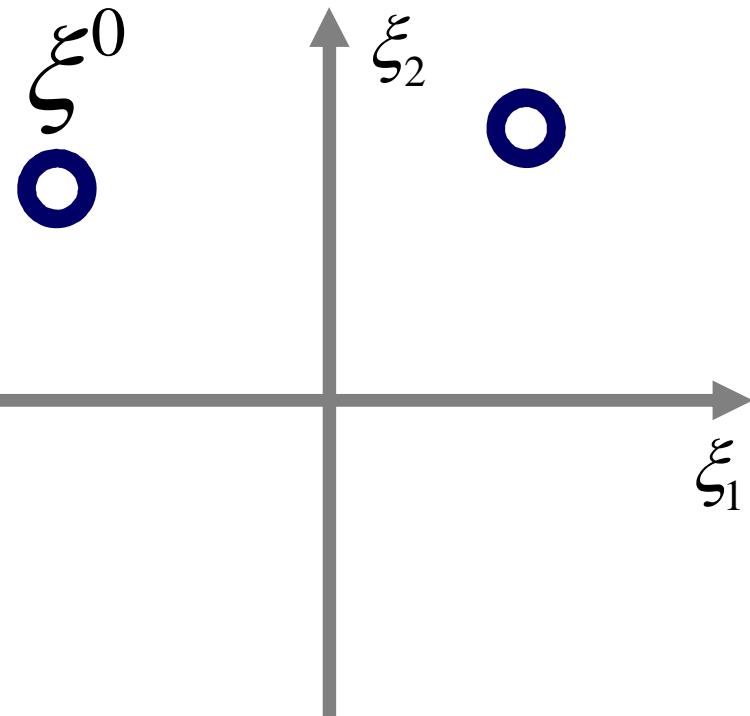
Expected recovery time



γ -recovery time

$\tau(\xi^0, \gamma)$

Expected recovery time



γ -recovery time

$\tau(\xi^0, \gamma)$

Expected recovery time

ξ^0

ξ_2

ξ_1



$\tau(\xi^0, \gamma)$

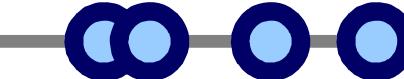
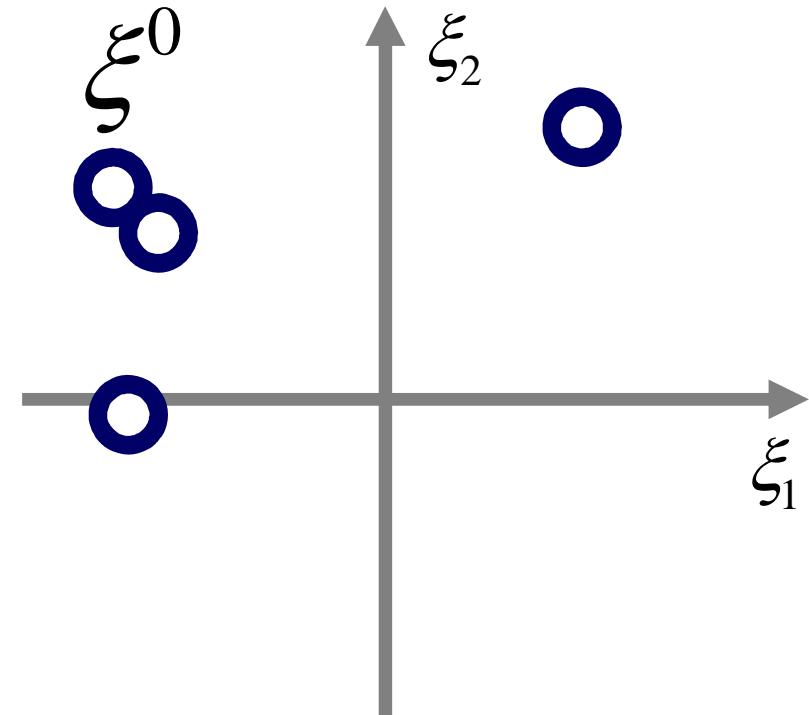
γ -recovery time

$\tau(\xi^0, \gamma)$

Expected recovery time

$$E\tau(\cdot, \gamma) \approx [\tau(\xi^{01}, \gamma) + \dots + \tau(\xi^{0k}, \gamma)]/k$$

ξ^0



$\tau(\xi^0, \gamma)$

γ -recovery time

$\tau(\xi^0, \gamma)$

Slow drift to catastrophe

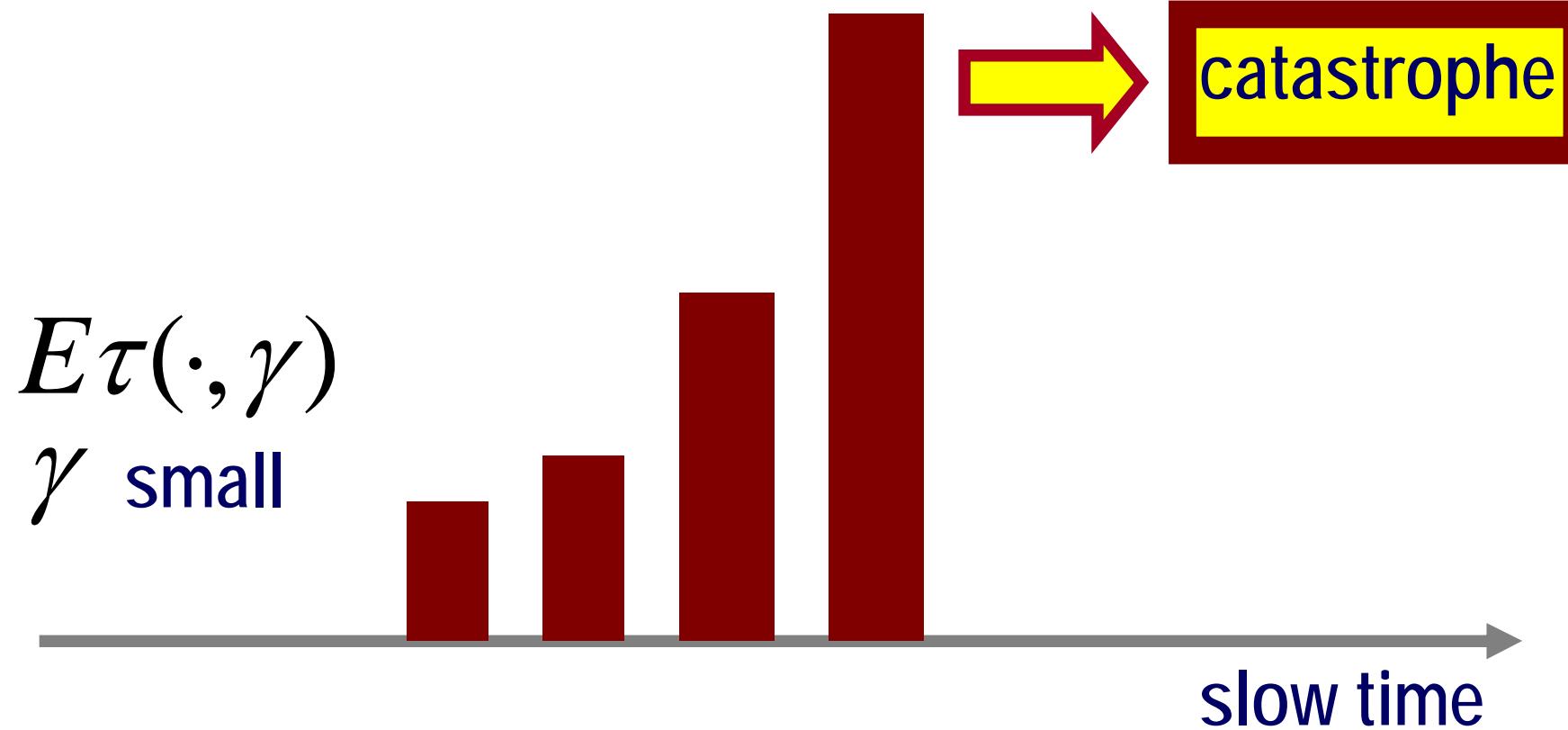
Slow drift to catastrophe

$$\dot{x} = f_p(x)$$

$$\dot{\xi} = C_p \xi$$

$$\lambda_{1p} \rightarrow 0$$

random shock-recovery tests



Most vulnerable coordinate

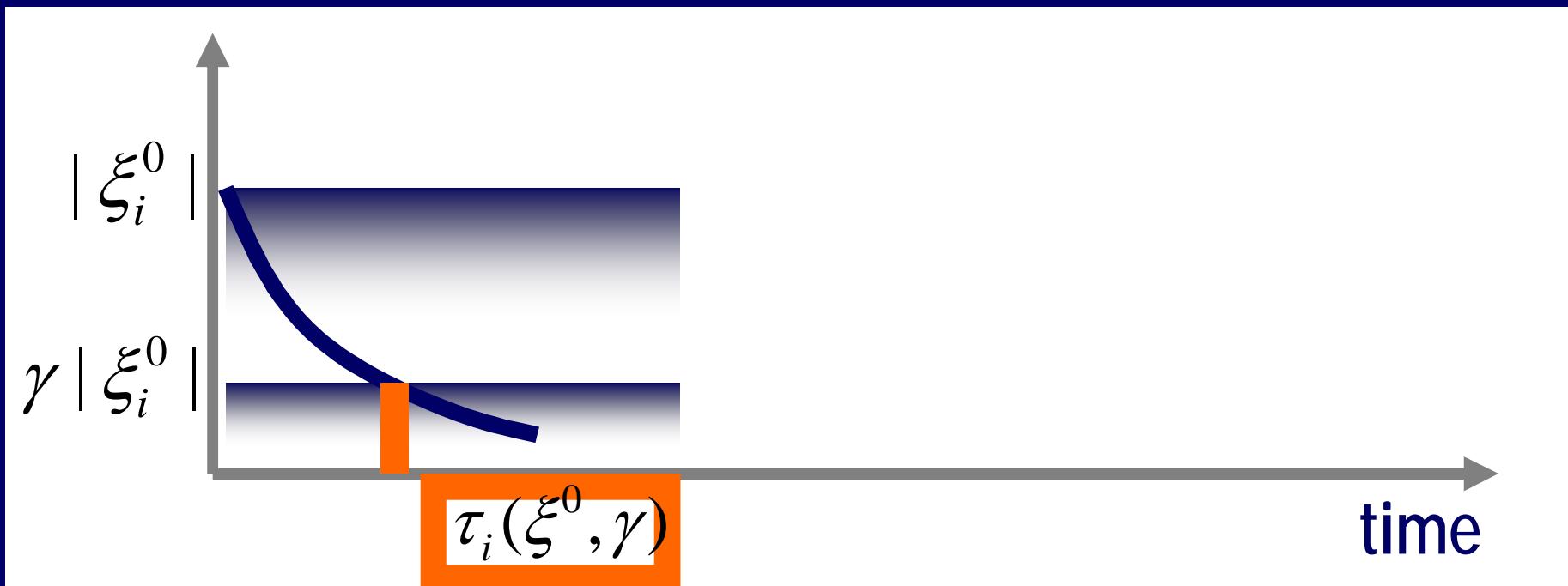
Most vulnerable coordinate



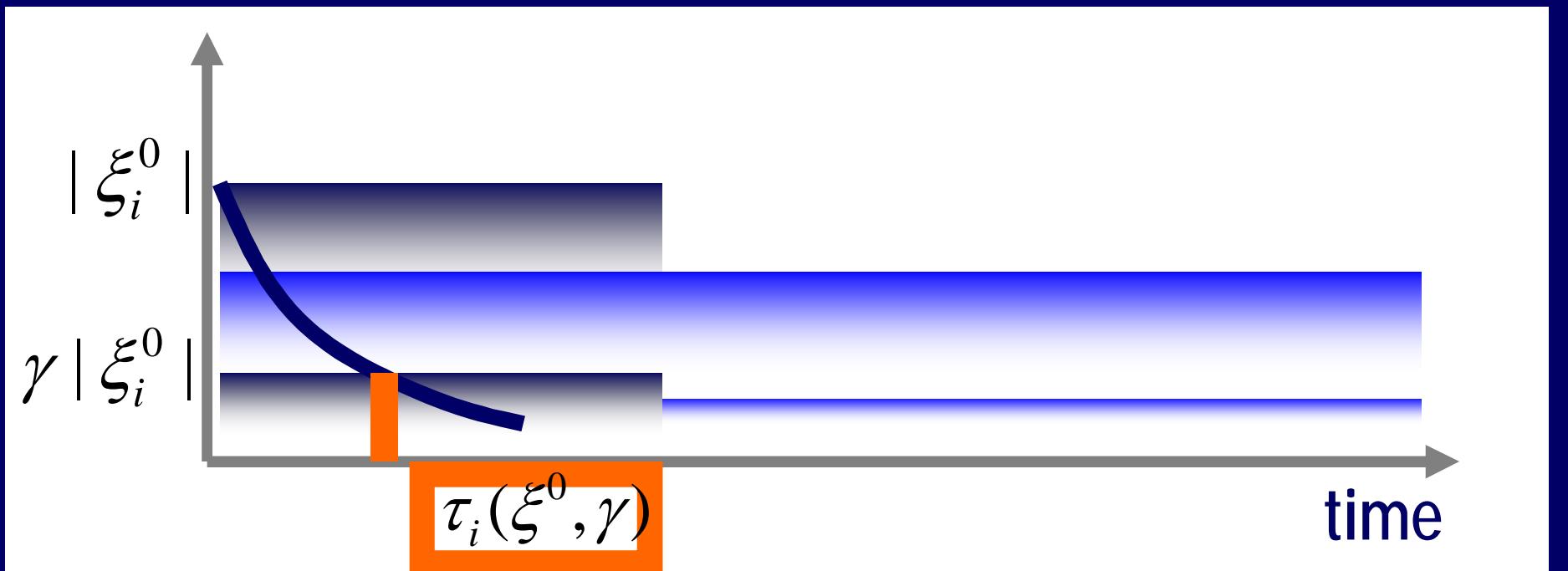
Most vulnerable coordinate



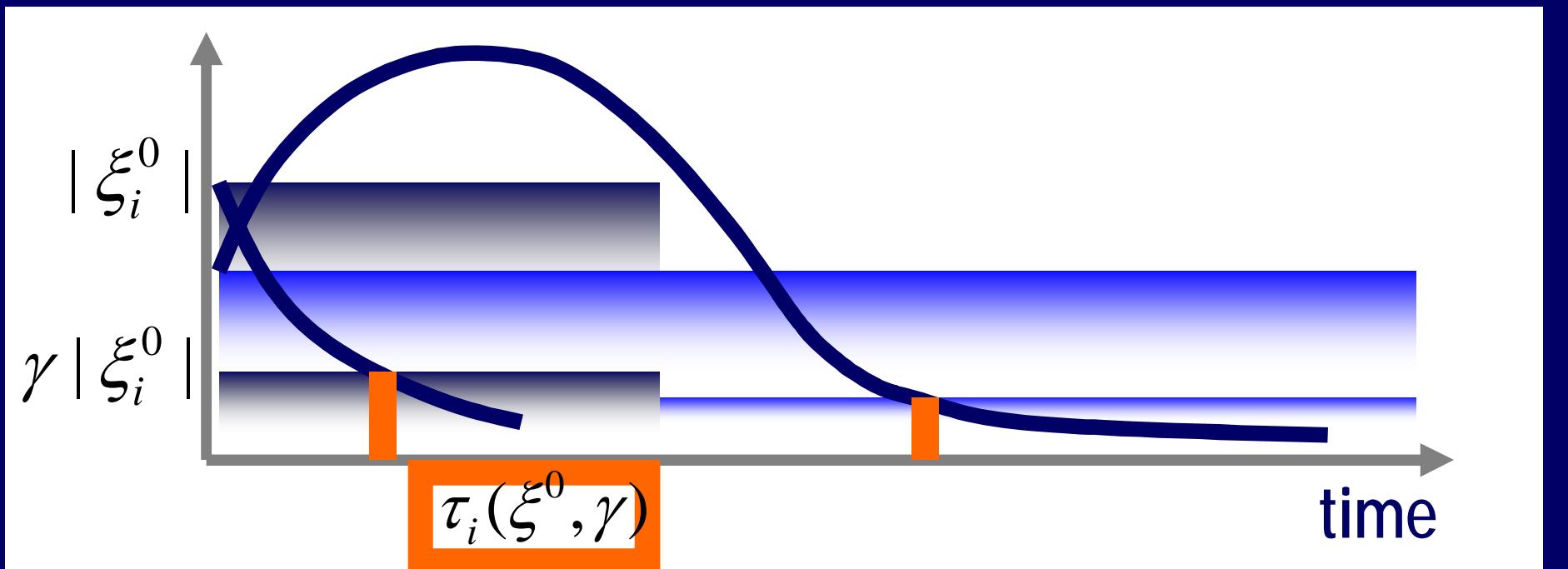
Most vulnerable coordinate



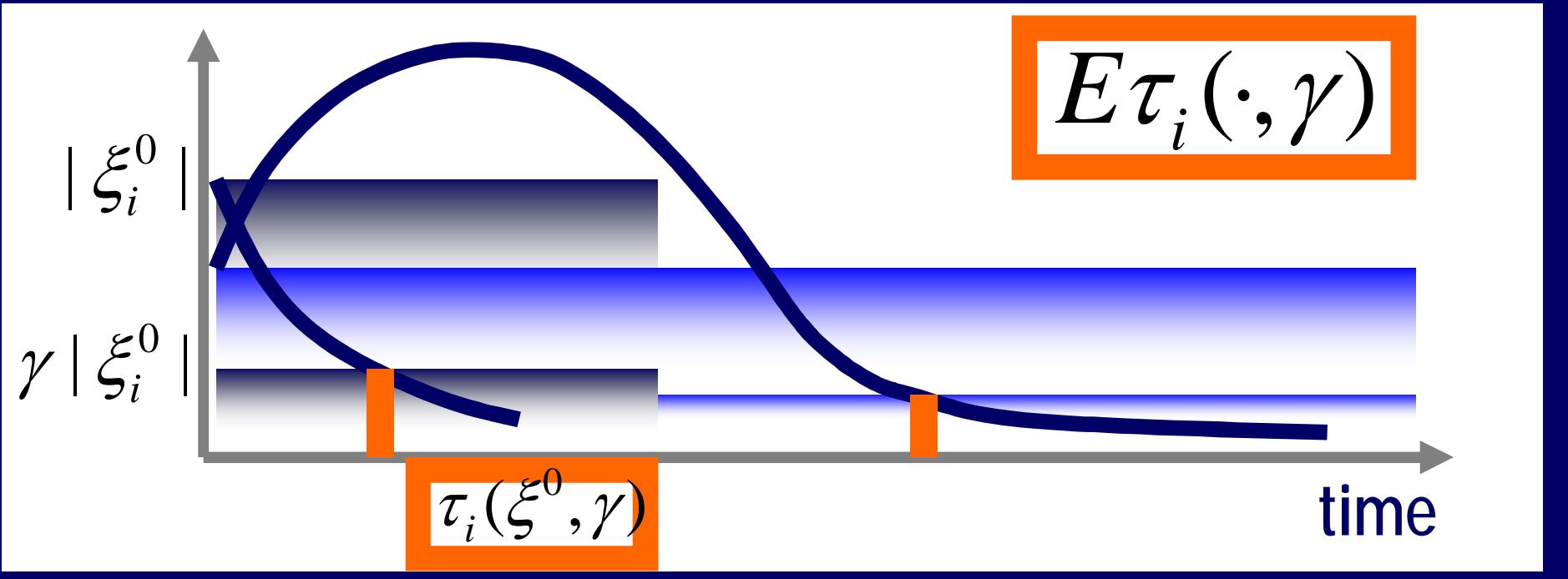
Most vulnerable coordinate



Most vulnerable coordinate



Most vulnerable coordinate



$$E\tau_i(\cdot, \gamma) = \frac{\log[\gamma |\xi_i^0| \alpha_i(\xi^0)] - \log |y_{1i}|}{\lambda_1} + o(\gamma)$$

$$\xi^0 = \alpha_1(\xi^0)y_1 + \dots + \alpha_n(\xi^0)y_n$$

Most vulnerable coordinate

Assumption: ξ_i^0 identically distributed

$$|y_{1k}| = \max_i |y_{1i}| \rightarrow k \text{ most vulnerable}$$

$$E\tau_i(\cdot, \gamma) - E\tau_j(\cdot, \gamma) = \frac{\log |y_{1j}| - \log |y_{1i}|}{\lambda_1} + o(\gamma)$$

$$\tau_i(\xi^0, \gamma) = \frac{\log[\gamma | \xi_i^0 | \alpha_i(\xi^0)] - \log |y_{1i}|}{\lambda_1} + o(\gamma)$$

$$\xi^0 = \alpha_1(\xi^0)y_1 + \dots + \alpha_n(\xi^0)y_n$$

Slow drift to catastrophe

$$\dot{x} = f_p(x)$$

$$\dot{\xi} = C_p \xi$$

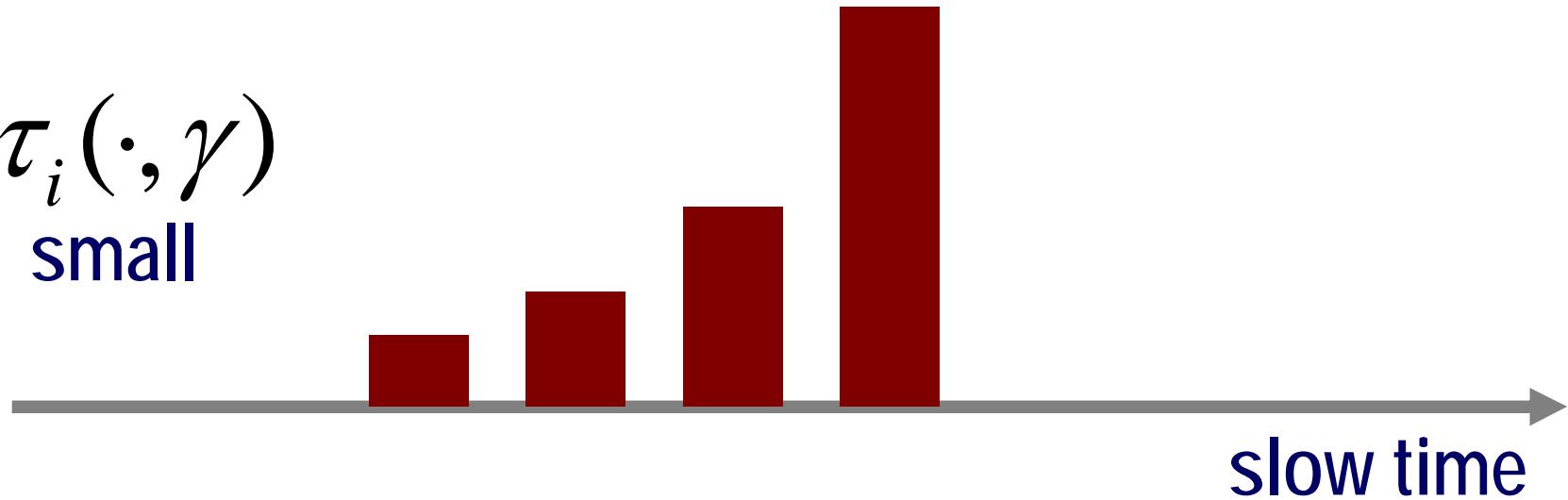
$$\lambda_{1p} \rightarrow 0$$

random shock-recovery tests

$$\tau_i(\xi^0, \gamma) = \frac{\log[\gamma |\xi_i^0| \alpha_i(\xi^0)] - \log |y_{1i}|}{\lambda_1} + o(\gamma)$$

$$E\tau_i(\cdot, \gamma)$$

γ small



Slow drift to catastrophe

$$\dot{x} = f_p(x)$$

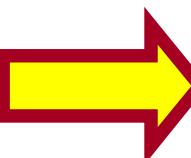
$$\dot{\xi} = C_p \xi$$

$$\lambda_{1p} \rightarrow 0$$

random shock-recovery tests

$E\tau_k(\cdot, \gamma)$
 k most vulnerable

$E\tau_i(\cdot, \gamma)$
 γ small



catastrophe

slow time

Thank you