Kinetics of Bose Condensation

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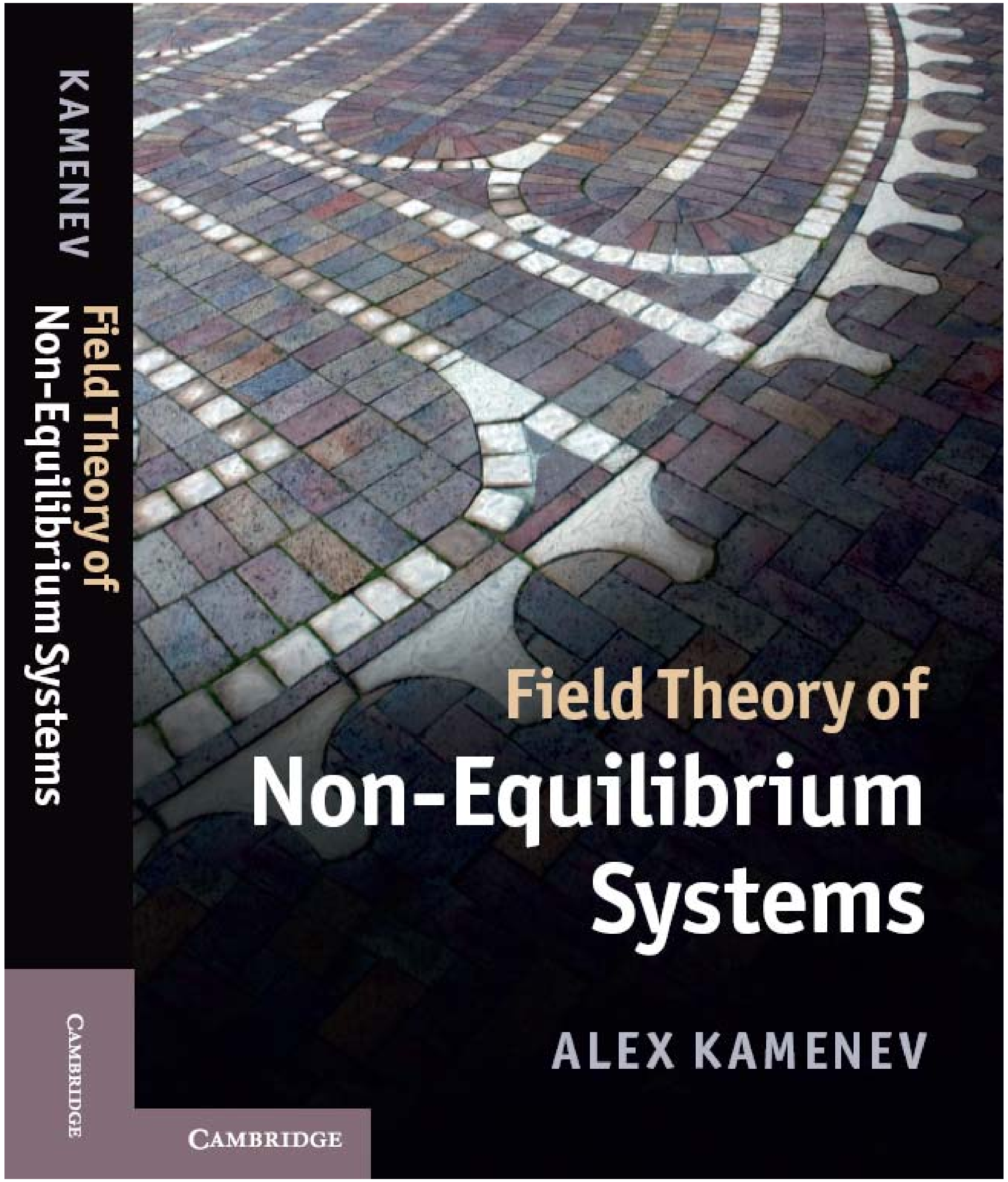
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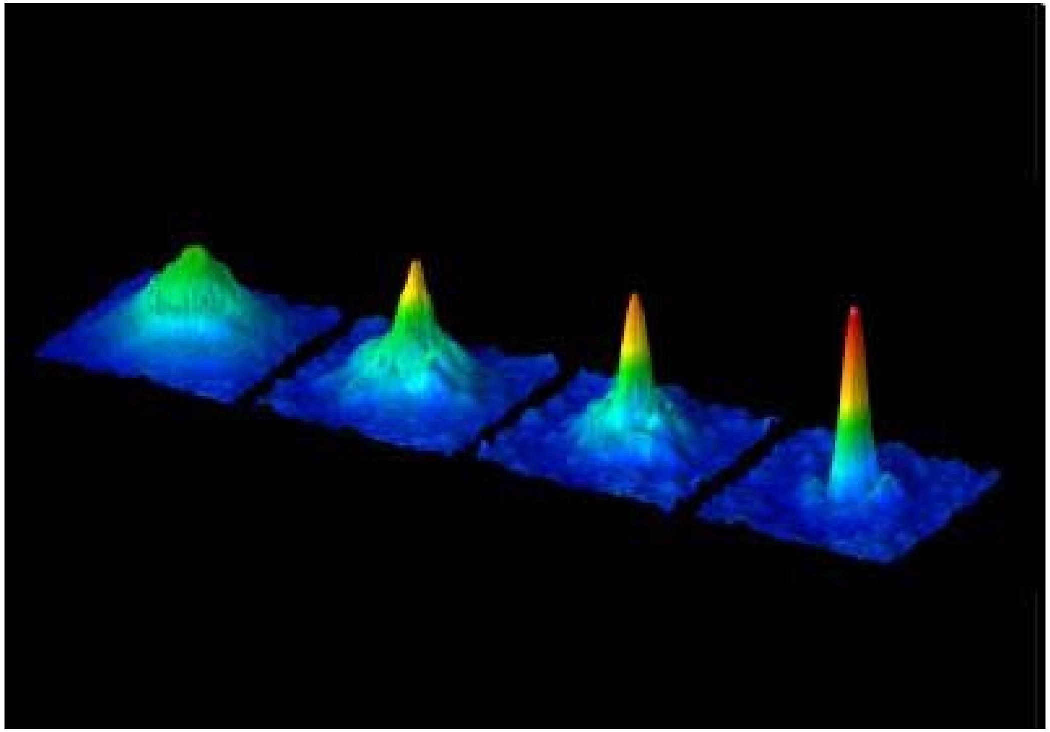
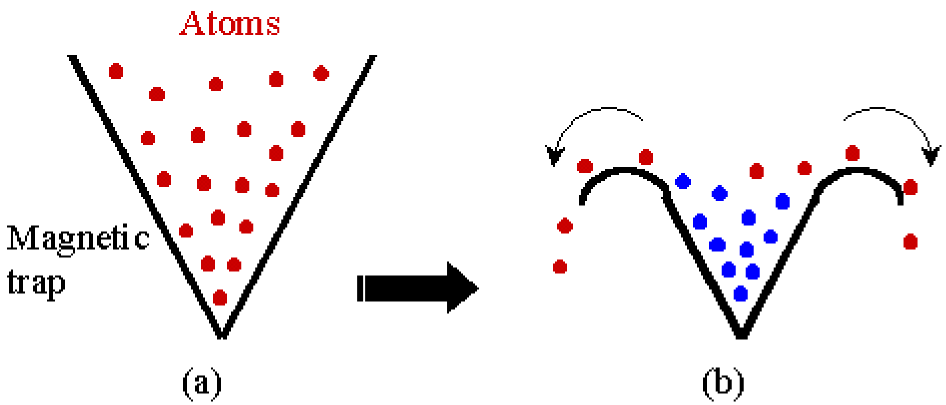
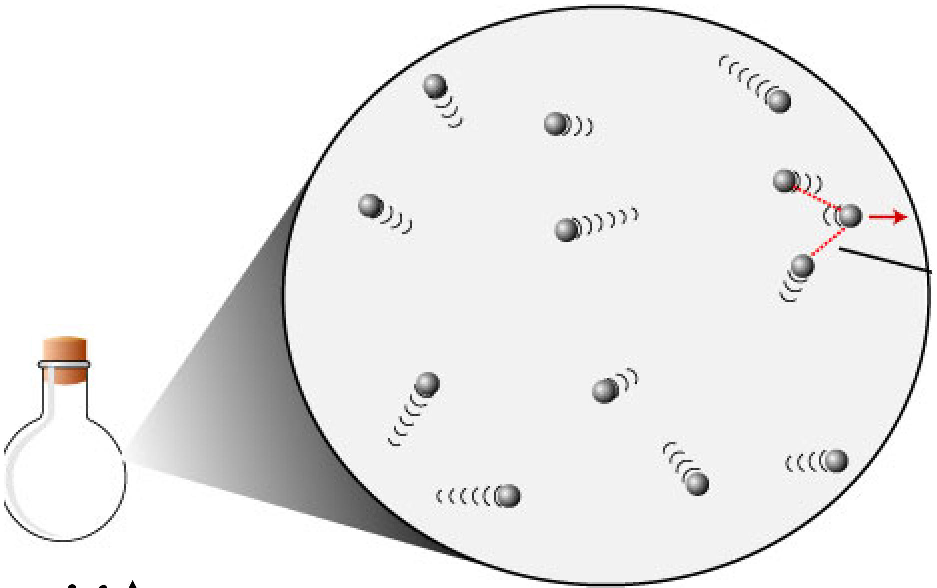
Bose Condensation



Evaporative cooling

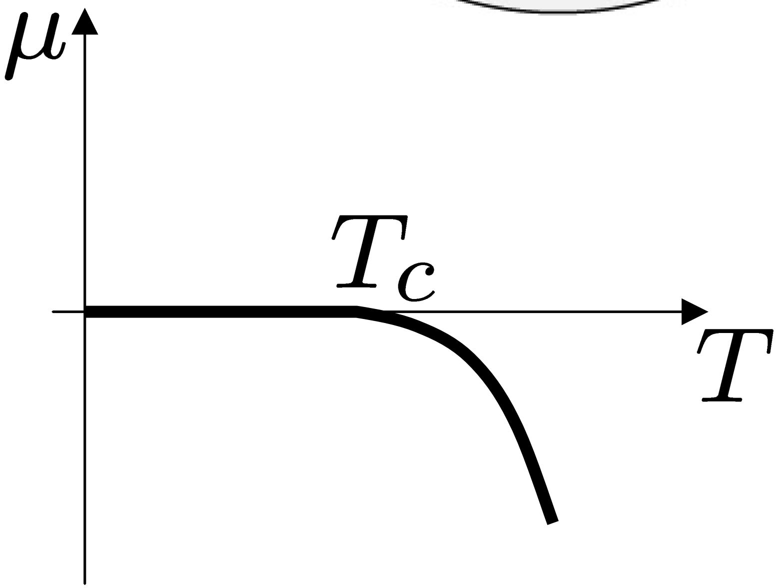


Non-Interacting Bosons



$$N = \sum_k \frac{1}{e^{\frac{\epsilon_k - \mu}{T}} - 1}$$

$$T_c \propto \frac{\rho^{2/3}}{m}$$



condenses into the lowest non-interacting eigenstate

Repulsive Interactions

$$\hat{H} = \int d\mathbf{r} \left[\phi^\dagger \frac{-\nabla^2_{\mathbf{r}}}{2m} \phi + V(\mathbf{r}) \phi^\dagger \phi + \frac{g}{2} \phi^\dagger \phi^\dagger \phi \phi \right]$$

Kinetic energy

External potential

interactions

$$g = \frac{4\pi a_s}{m}$$

s-wave scattering length

$$a_s \rho^{1/3} \ll 1$$

dilute limit

gas parameter

Condensate

Macroscopic eigenvalue of the density matrix operator:

$$\widehat{\rho}(\mathbf{r},\mathbf{r}') = \langle \phi^\dagger(\mathbf{r})\phi(\mathbf{r}')\rangle$$

$$\int d\mathbf{r}'\widehat{\rho}(\mathbf{r},\mathbf{r}')\Phi_0(\mathbf{r}') = \lambda_0\Phi_0(\mathbf{r})$$

$$\lambda_0 \propto N$$

condensate wavefunction

Gross-Pitaevskii Equation

Dilute limit: variational many-body ground-state

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) \propto \Phi_0(\mathbf{r}_1)\Phi_0(\mathbf{r}_2)\ldots\Phi_0(\mathbf{r}_N)$$

Classical approximation for the creation/annihilation operators

$$\left(-\frac{1}{2m}\nabla^2_{\mathbf{r}} + V(\mathbf{r}) + g\,|\Phi_0|^2\right)\Phi_0 = \mu\Phi_0$$

$$\int \mathrm{d}\mathbf{r}\,|\Phi_0(\mathbf{r},t)|^2 = N$$

Lagrange multiplier

Uniform gas at T=0

$$\mu = \partial\langle\hat{H}\rangle/\partial N$$

chemical potential

$$\mu = g|\Phi_0|^2 = g\rho$$

non-zero **positive** chemical potential

$$\frac{\mu}{T_c} \propto \frac{a_s\rho/m}{\rho^{2/3}/m} \propto a_s\rho^{1/3} \ll 1$$

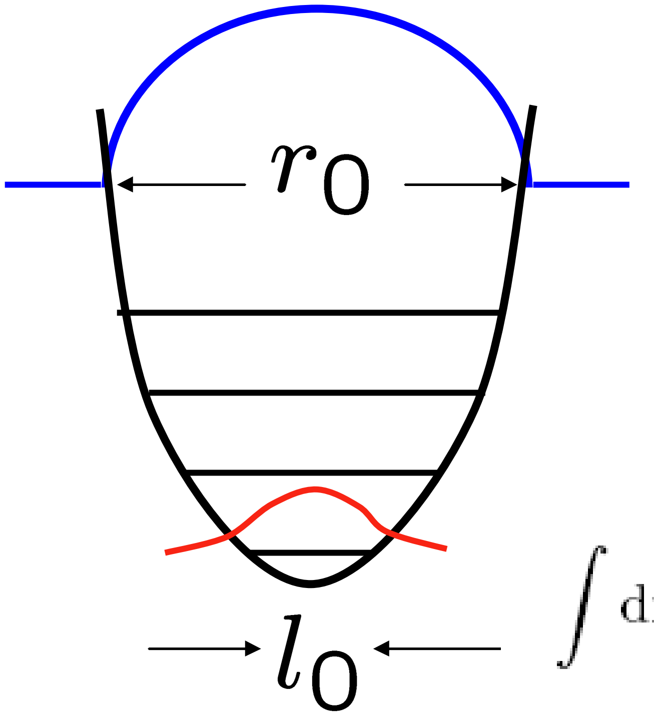
dilute limit

$$\mu \ll T < T_c$$

Thomas-Fermi Approximation

$$\left(\cancel{-\frac{1}{2m}\nabla_{\mathbf{r}}^2} + V(\mathbf{r}) + g\,|\Phi_0|^2\right)\Phi_0 = \mu\Phi_0$$

$$V(\mathbf{r}) + g\,|\Phi_0|^2 = \mu$$



$$|\Phi_0(\mathbf{r})| = \sqrt{(\mu - V(\mathbf{r}))/g}$$

semicircle

$$\int \mathrm{d}\mathbf{r}\, |\Phi_0(\mathbf{r},t)|^2 = N$$

$$r_0 \propto l_0 \left(\frac{N a_s}{l_0}\right)^{\frac{1}{5}}$$

Interactions are important even in the **dilute** limit

Thomas-Fermi approximation

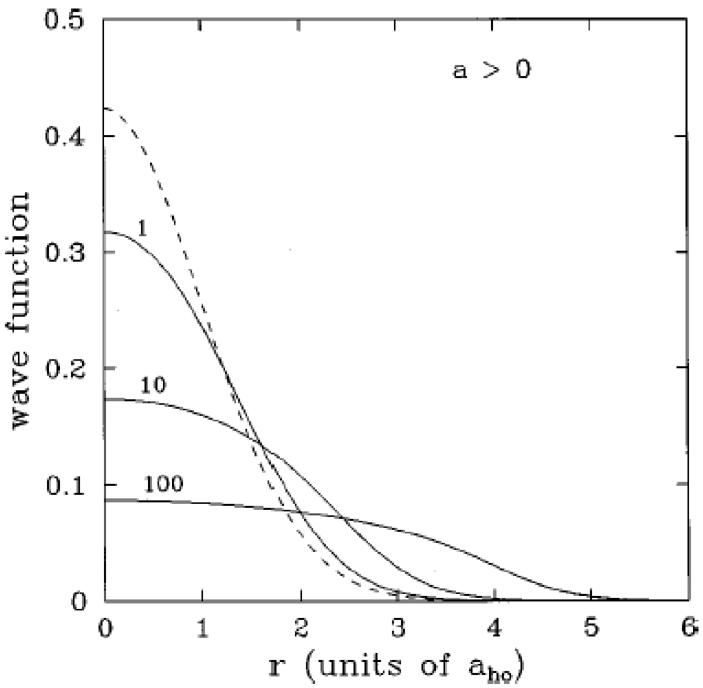


FIG. 9. Same as in Fig. 8, but for repulsive interaction ($a > 0$) and $Na/a_{\rm ho}=1,10,100$.

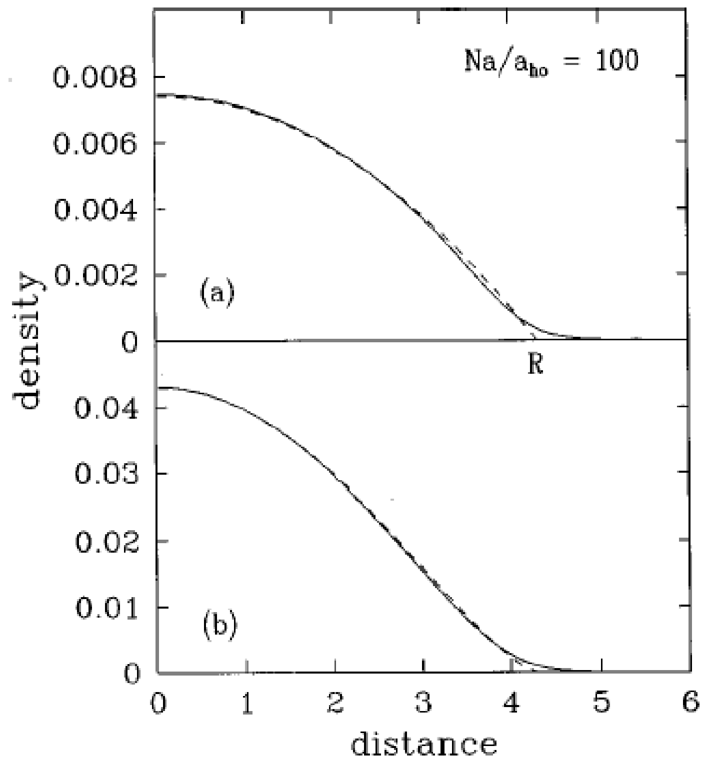


FIG. 13. Density profile for atoms interacting with repulsiv forces in a spherical trap, with $Na/a_{\rm ho}=100$. Solid line: solution of the stationary GP Eq. (39). Dashed line: Thomas-Ferm approximation (50). In the upper part, the atom density is plotted in arbitrary units, while the distance from the center of th trap is in units of $a_{\rm ho}$. The classical turning point is at $R \simeq 4.31 a_{\rm ho}$. In the lower part, the column density for the sam system is reported.

Thomas-Fermi approximation

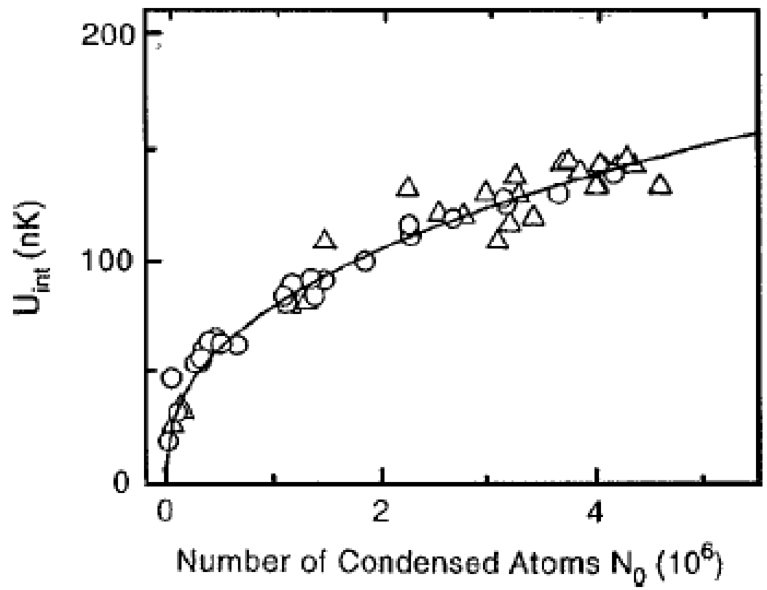


FIG. 11. Release energy of the condensate as a function of the number of condensed atoms in the MIT trap with sodium atoms. For these condensates the initial kinetic energy is negligible and the release energy coincides with the mean-field energy. The symbol $U_{\rm int}$ is here used for the mean-field energy per particle. Triangles: clouds with no visible thermal component. Circles: clouds with both thermal and condensed fractions visible. The solid line is a fit proportional to $N_0^{2/5}$ (see discussion in Sec. III.D). From Mewes *et al.* (1996a).

Time-dependent Gross-Pitaevskii

$$\hat{H} = \int d\mathbf{r} \left[\phi^\dagger \frac{-\nabla_\mathbf{r}^2}{2m} \phi + V(\mathbf{r}) \phi^\dagger \phi + \frac{g}{2} \phi^\dagger \phi^\dagger \phi \phi \right]$$

$$[\phi, \phi^\dagger] = 1$$

treat them as a classical **canonical** pair

$$\mathcal{L} = \bar{\phi} \partial_t \phi - H(\bar{\phi}, \phi)$$

$$\frac{\delta \mathcal{L}}{\delta \bar{\phi}} = 0 \qquad \partial_t \phi = \partial_{\bar{\phi}} H$$

$$\frac{\delta \mathcal{L}}{\delta \phi} = 0 \qquad \partial_t \bar{\phi} = -\partial_\phi H$$

Time-dependent Gross-Pitaevskii

$$\left(i\partial_t + \frac{1}{2m}\nabla^2_{\mathbf{r}} - V(\mathbf{r},t) - g\,|\Phi_0|^2\right)\Phi_0 = 0\,;$$
$$\left(-i\partial_t + \frac{1}{2m}\nabla^2_{\mathbf{r}} - V(\mathbf{r},t) - g\,|\Phi_0|^2\right)\bar{\Phi}_0 = 0\,.$$

normalization: $\int \mathrm{d}\mathbf{r}\,|\Phi_0(\mathbf{r},t)|^2 = N$

static limit: $\Phi_0(\mathbf{r},t) = e^{-i\mu t}\Phi_0(\mathbf{r})$

Small fluctuations

$$\Phi_0(\mathbf{r},t) \;=\; e^{-i\mu t}\big(\Phi_0 + \varphi(\mathbf{r},t)\big) \qquad\qquad \mu = |\Phi_0|^2 g$$

$$\begin{aligned} i\partial_t\varphi + \frac{1}{2m}\nabla^2_{\mathbf{r}}\varphi - g|\Phi_0|^2\varphi - g\Phi_0^2\bar{\varphi} &= 0\,; \\ -i\partial_t\bar{\varphi} + \frac{1}{2m}\nabla^2_{\mathbf{r}}\bar{\varphi} - g|\Phi_0|^2\bar{\varphi} - g\bar{\Phi}_0^2\varphi &= 0\,. \end{aligned}$$

$$\begin{pmatrix} \omega - \mathbf{q}^2/(2m) - g|\Phi_0|^2 & -g\Phi_0^2 \\ -g\bar{\Phi}_0^2 & -\omega - \mathbf{q}^2/(2m) - g|\Phi_0|^2 \end{pmatrix} \begin{pmatrix} \varphi(\mathbf{q},\omega) \\ \bar{\varphi}(\mathbf{q},\omega) \end{pmatrix} = 0$$

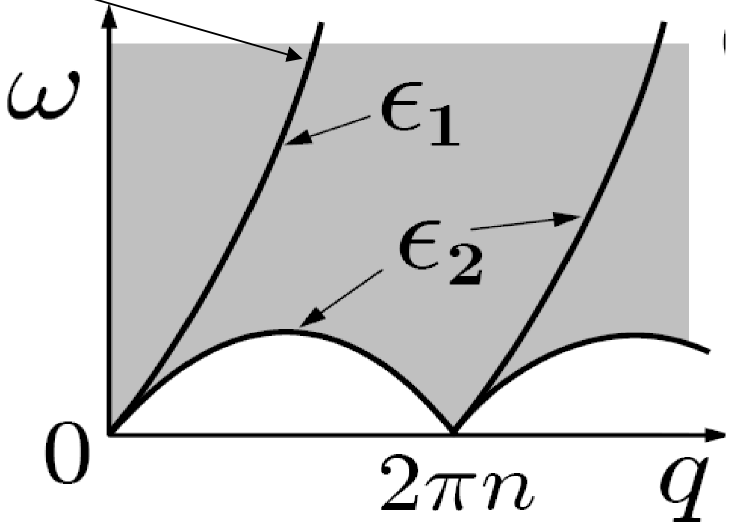
$$\omega^2 = \left(\frac{\mathbf{q}^2}{2m} + g|\Phi_0|^2\right)^2 - (g)^2\,|\Phi_0|^4 = \left(\frac{\mathbf{q}^2}{2m}\right)^2 + \frac{\mathbf{q}^2}{m}g|\Phi_0|^2$$

Bogoliubov mode

$$\omega_B(\mathbf{q}) = \sqrt{c^2\mathbf{q}^2 + \left(\frac{\mathbf{q}^2}{2m}\right)^2}$$

$$c^2 = g|\Phi_0|^2/m = g\rho/m$$

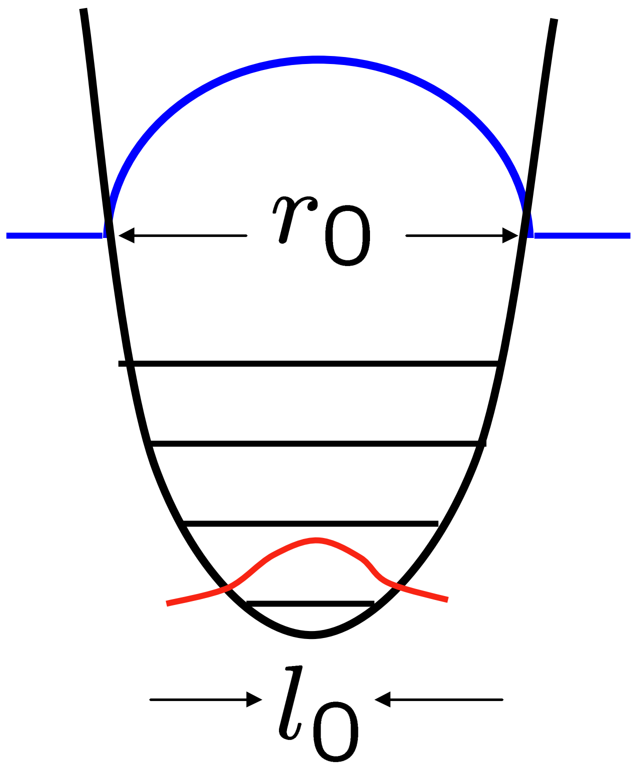
$$mc^2 = \rho\partial\mu/\partial\rho$$



sound-like for energy below the chemical potential

particle-like for energy above the chemical potential

Collective excitations in the trap



$$r_0 \propto l_0 \left(\frac{Na_s}{l_0}\right)^{\frac{1}{5}}$$

1. $\left(\frac{Na_s}{l_0}\right)^{\frac{1}{5}} \ll 1$

$$\omega(n_r, l) = \omega_{\mathrm{ho}}(2n_r + l)$$

2. $\left(\frac{Na_s}{l_0}\right)^{\frac{1}{5}} \gg 1$

surface modes: $n_r = 0$

$$\omega(0, l) = \omega_{ho}\sqrt{l}$$

$$\omega(n_r, l) = \omega_{\mathrm{ho}}(2n_r^2 + 2n_r l + 3n_r + l)^{1/2}$$

Stringari, 1996

Hydrodynamics

$$\Phi_0(\mathbf{r},t) = \sqrt{\rho_0(\mathbf{r},t)}\, e^{-i\theta(\mathbf{r},t)}$$

current

$$\partial_t \rho_0 + \nabla_{\mathbf{r}}\big(\rho_0 \nabla_{\mathbf{r}}\theta/m\big) = 0\,;$$

continuity equation

$$\partial_t \theta = \frac{\nabla^2_{\mathbf{r}}\sqrt{\rho_0}}{m\sqrt{\rho_0}} - \frac{(\nabla_{\mathbf{r}}\theta)^2}{2m} - V + g\rho_0$$

$$\mathbf{v}_{\mathrm{sf}} = \nabla_{\mathbf{r}}\theta/m$$

$$\partial_t(m\mathbf{v}_{\mathrm{sf}}) = -\nabla_{\mathbf{r}}\left(\frac{m\mathbf{v}^2_{\mathrm{sf}}}{2} + V - \mu\right)$$

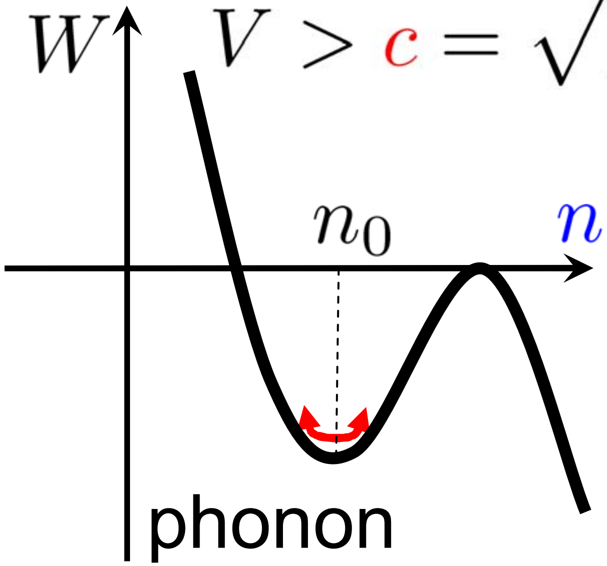
Euler (Newton) equation

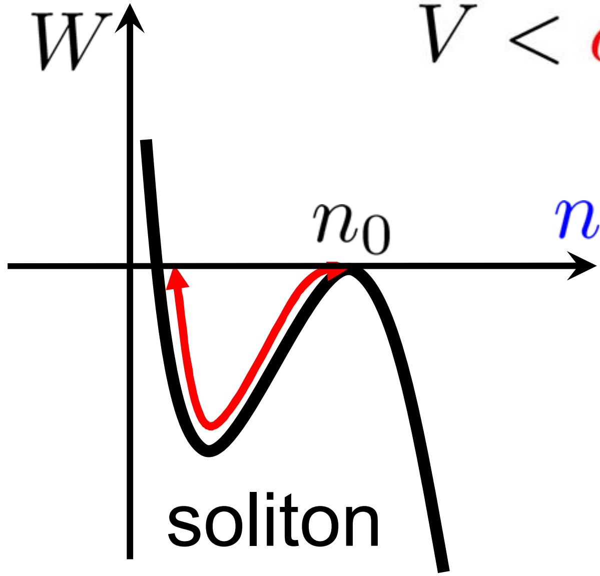
Dark Solitons

$$\psi(x - Vt) = \sqrt{n(x - Vt)}\, e^{i\vartheta(x - Vt)}$$

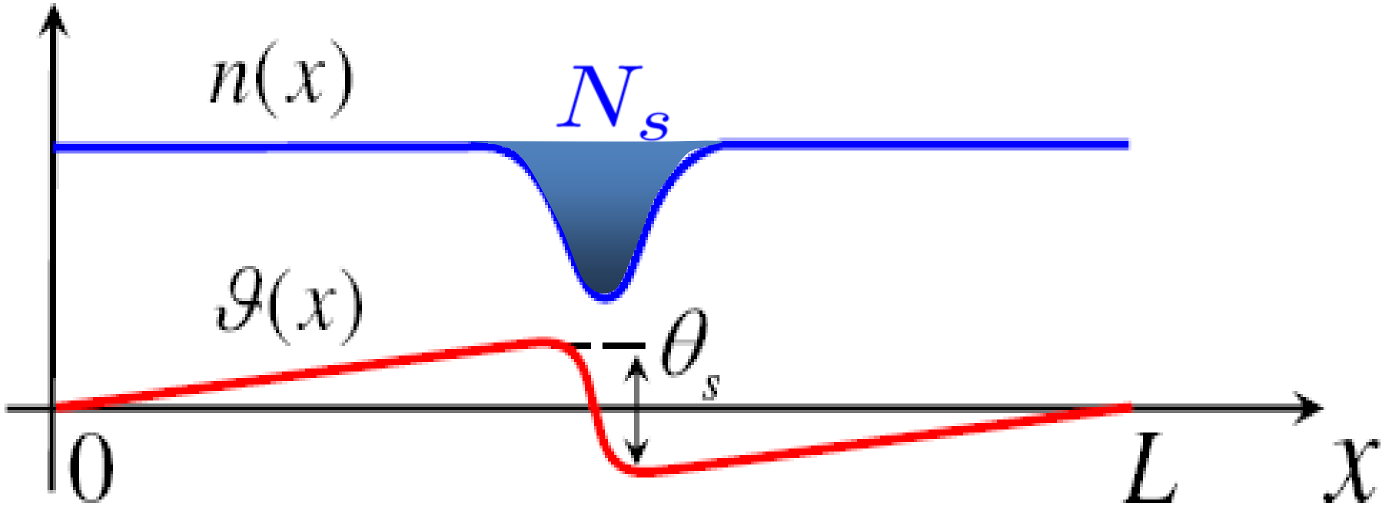
$$\Big(n[\vartheta' - mV]\Big) = const \qquad\qquad (n)'' = -\frac{dW(n)}{dn}$$

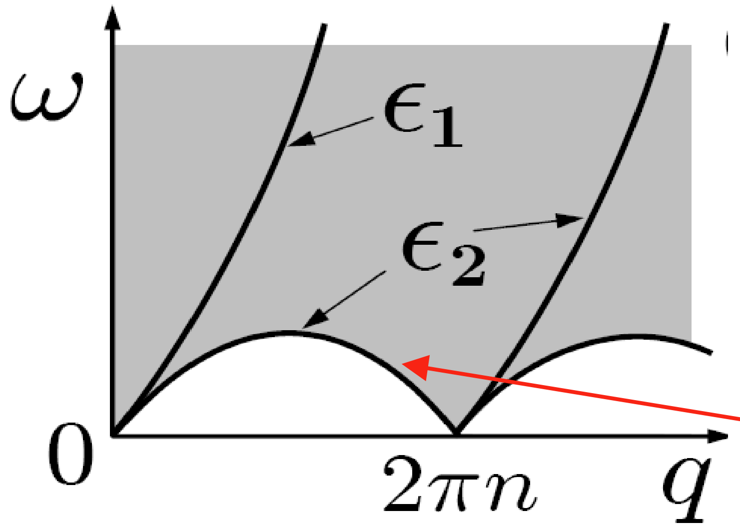
$$V > c = \sqrt{gn_0/m} \qquad\qquad V < c$$





Dark Solitons





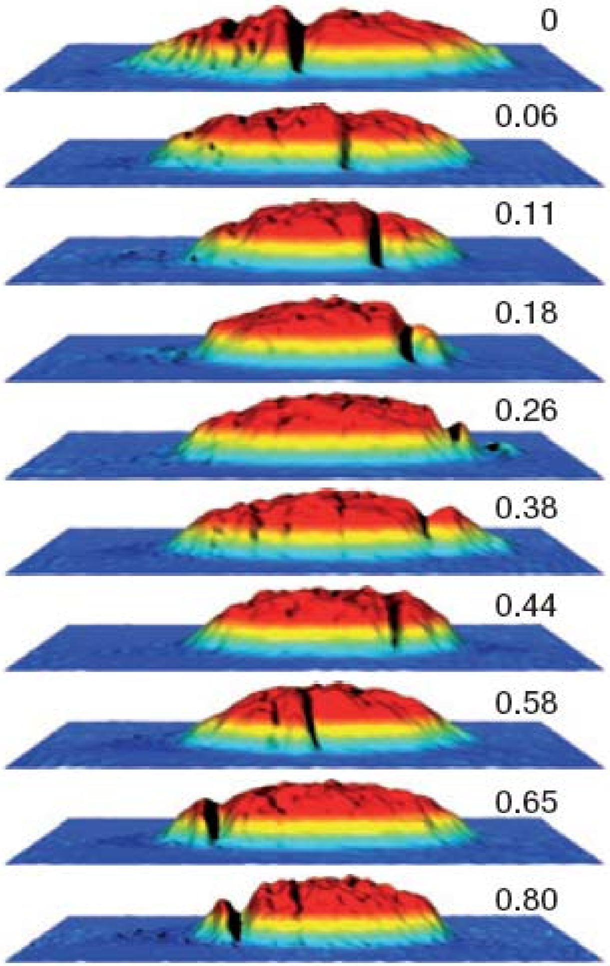
$$V = c\cos\theta_s/2$$

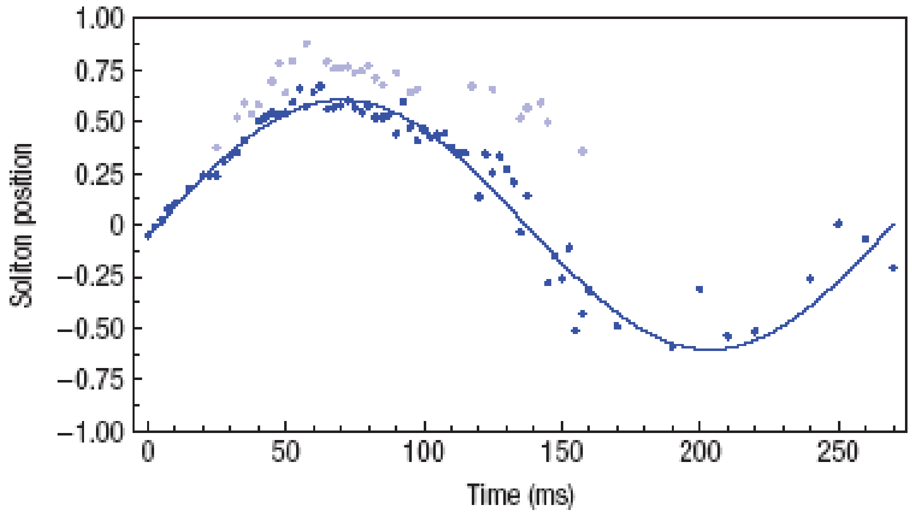
$$N_s = \frac{2}{\sqrt{\gamma}}\sin\theta_s/2 \gg 1$$

$$p_s = n_0\left(\theta_s - \sin\theta_s\right)$$

$$E_s = \frac{4}{3}n_0 c\sin^3(\theta_s/2)$$

Dark solitons in Bose gases





Lifetime ~300ms

C. Becker *et al.*, Nature Physics **4**, 496 (2008)

Quasiparticles

Energy scales separation: $\mu \ll T < T_c$

$$\phi(\mathbf{r},t) = e^{-i\mu t}[\Phi_0(\mathbf{r},t) + \varphi(\mathbf{r},t)]$$

slow condensate fast quasiparticles

$\langle \varphi^\dagger \varphi \rangle = \rho_{qp}(\mathbf{r},t)$ neglect: $\langle \varphi\varphi \rangle$ Popov approximation

Hartree-Fock-but-not-Bogoliubov theory

$$[i\partial_t + \frac{\nabla_{\mathbf{r}}^2}{2m} - V + \mu - g(|\Phi_0|^2 + 2\rho_{qp})]\Phi_0 = 0$$

Quasiparticles Distribution Function

occupation number of state **k** at point (**r**,t): $n(\mathbf{r}, t, \mathbf{k})$

$$\rho_{qp}(\mathbf{r}, t) = \sum_{\mathbf{k}} n(\mathbf{r}, t, \mathbf{k})$$

Kinetic equation:

$$\partial_t n + \nabla_{\mathbf{k}} \epsilon_{\mathbf{k}} \nabla_{\mathbf{r}} n - \nabla_{\mathbf{r}} \epsilon_{\mathbf{k}} \nabla_{\mathbf{k}} n = I^{\mathrm{coll}}[n, \Phi_0]$$

$$\frac{dn}{dt} = \partial_t n - \{\epsilon_{\mathbf{k}}, n\} = I^{\mathrm{coll}}$$

Poisson brackets

$$\epsilon_{\mathbf{k}}(\mathbf{r}, t) = \frac{\mathbf{k}^2}{2m} + V - \mu + 2g(|\Phi_0|^2 + \rho_{qp})$$

Hartree+Fock

Collisionless Dynamics

Modified Gross-Pitaevskii equation coupled to collisionless kinetic equation

$$\left[i\partial_t + \frac{\nabla_{\mathbf{r}}^2}{2m} - V + \mu - g(|\Phi_0|^2 + 2\rho_{qp})\right]\Phi_0 = 0$$

$$\partial_t n + \nabla_{\mathbf{k}}\epsilon_{\mathbf{k}}\nabla_{\mathbf{r}} n - \nabla_{\mathbf{r}}\epsilon_{\mathbf{k}}\nabla_{\mathbf{k}} n = 0$$

$$\rho_{qp}(\mathbf{r},t) = \sum_{\mathbf{k}} n(\mathbf{r},t,\mathbf{k})$$

$$\epsilon_{\mathbf{k}}(\mathbf{r},t) = \frac{\mathbf{k}^2}{2m} + V - \mu + 2g(|\Phi_0|^2 + \rho_{qp})$$

Linearized Collisionless Dynamics

Linearization: $\Phi_0(x) = \Phi_0 + \phi(x)$ $\delta\rho_0 = \bar{\Phi}_0\phi + \Phi_0\bar{\phi}$

$n = n_B(\epsilon_{\mathbf{k}}) + n^{(1)}(x, \mathbf{k})$ $\delta\rho_{qp} = \sum_{\mathbf{k}} n^{(1)}$

$$i\partial_t\phi(x) = -\frac{\nabla^2_{\mathbf{r}}\phi(x)}{2m} + g\,|\Phi_0|^2\phi(x) + g\,\Phi_0^2\,\bar{\phi}(x) + 2g\delta\rho_{qp}(x)\Phi_0$$

$$\left[\partial_t + \frac{\mathbf{k}}{m}\nabla_{\mathbf{r}}\right] n^{(1)}(x, \mathbf{k}) = 2g\nabla_{\mathbf{r}}\left[\delta\rho_0 + \delta\rho_{qp}\right]\nabla_{\mathbf{k}} n_B(\epsilon_{\mathbf{k}})$$

Linear differential equations ⇒ Fourier transform

$i\partial_t \to \omega$ $\nabla_{\mathbf{r}} \to i\mathbf{q}$

Three linear algebraic homogeneous equations

Dispersion of collective modes

$$c^2 = g|\Phi_0|^2/m = g\rho/m$$

$$\omega^2 - \omega_B^2(\mathbf{q}) = -c^2\mathbf{q}^2 4g\Pi^R(\mathbf{q},\omega)$$

$$\Pi^R(\mathbf{q},\omega) = \sum_{\mathbf{k}} \frac{\mathbf{q}\nabla_{\mathbf{k}} n_B(\epsilon_{\mathbf{k}})}{\omega + i0 - \mathbf{v}_{\mathbf{k}}\mathbf{q}}$$

Real part – renormalization of the speed of sound.

$$\mathbf{v}_{\mathbf{k}} = \mathbf{k}/m$$

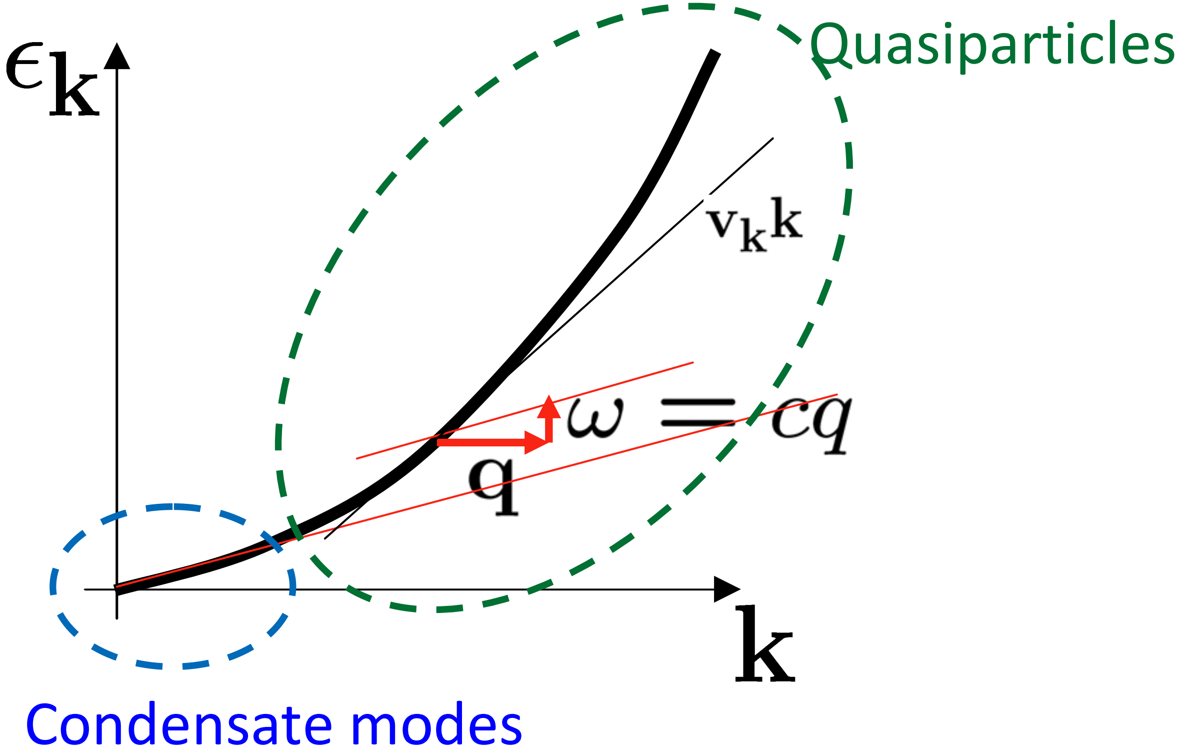
$$\mathrm{Im}\frac{1}{\omega+i0-\mathbf{v}_{\mathbf{k}}\mathbf{q}} = -i\pi\delta(\omega - \mathbf{v}_{\mathbf{k}}\mathbf{q})$$

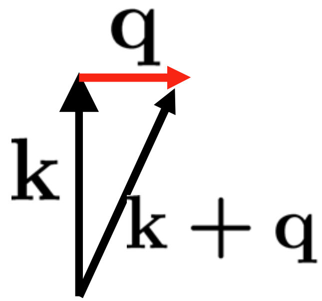
$$\tilde{\omega}_B(\mathbf{q}) = \omega_B(\mathbf{q}) - i\Gamma_2(\mathbf{q})$$

Damping

Landau Damping

$$\Gamma_2(\mathbf{q}) = \frac{c^2\mathbf{q}^2}{\omega_B(\mathbf{q})}\, 2g\sum_{\mathbf{k}} [n_B(\epsilon_{\mathbf{k}+\mathbf{q}}) - n_B(\epsilon_{\mathbf{k}})]\delta(\omega + \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}})$$





Landau vs. Beliaev damping

$$\Gamma_2(\mathbf{q}) = 4T\, q a_s$$

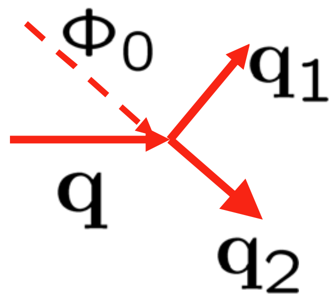
needs thermally excited quasiparticles

$$\Gamma_2(\mathbf{q}) \ll T; cq$$

condensate modes are well-defined

Beliaev damping:

One condensate mode decays on two lower energy modes



$$\Gamma_{Beliaev}(\mathbf{q}) \propto q^5/m\rho_0$$

Zero temperature effect

$$\Gamma_2(\mathbf{q}) \gg \Gamma_{Beliaev}$$

$$T > \mu$$

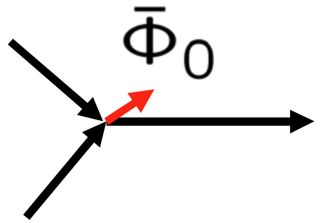
Condensate Growth and Collapse

So far number of particles in the condensate and
in the quasiparticle cloud are conserved **separately**

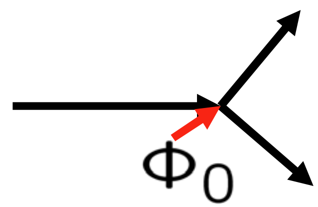
Because we have only been taking into account terms
like: $\bar{\Phi}_0 \Phi_0 \bar{\varphi} \varphi$

terms describing the exchange between them are:

$\bar{\Phi}_0 \bar{\varphi} \varphi \varphi$ and $\bar{\varphi} \bar{\varphi} \varphi \Phi_0$



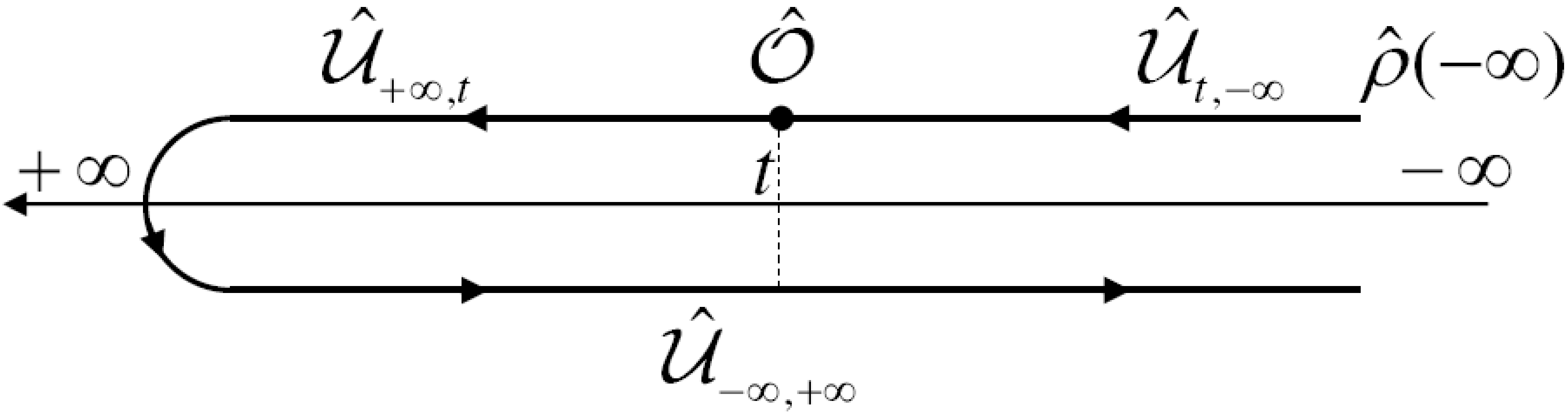
condensate growths



condensate collapses

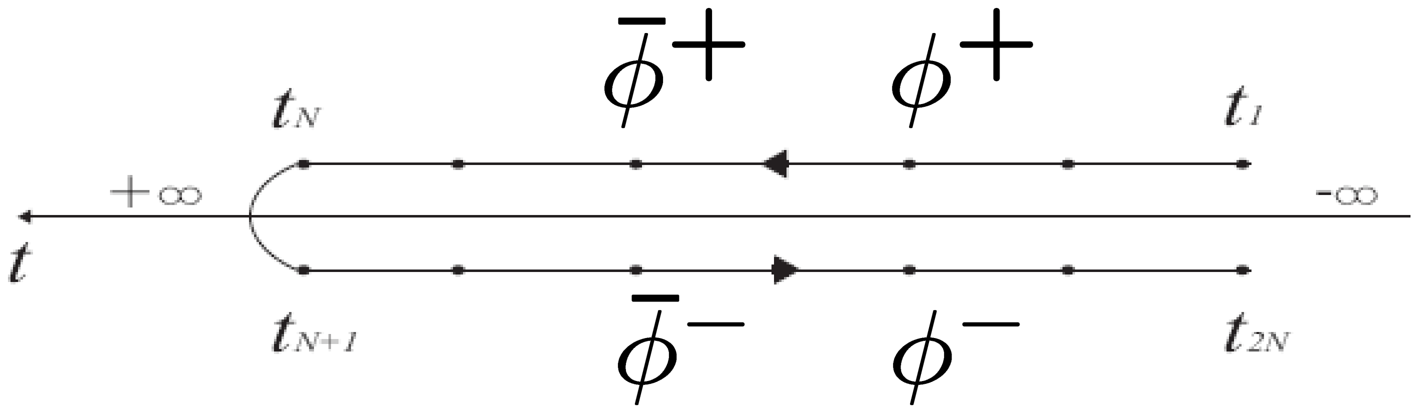
Keldysh Technique

$$\langle \hat{\mathcal{O}} \rangle (t) \equiv \frac{\mathrm{Tr}\{\hat{\mathcal{O}}\hat{\rho}(t)\}}{\mathrm{Tr}\{\hat{\rho}(t)\}} = \frac{1}{\mathrm{Tr}\{\hat{\rho}(t)\}} \ \mathrm{Tr}\{\hat{\mathcal{U}}_{-\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\}$$



$$S = \int_{\mathcal{C}} dt \, \mathcal{L} = \int_{\mathcal{C}} dt \, [\bar{\phi}\partial_t\phi - H(\bar{\phi},\phi)]$$

Keldysh Rotation



$$\phi^{cl}(t) = \frac{1}{\sqrt{2}}\left(\phi^{+}(t) + \phi^{-}(t)\right), \qquad \phi^{q}(t) = \frac{1}{\sqrt{2}}\left(\phi^{+}(t) - \phi^{-}(t)\right)$$

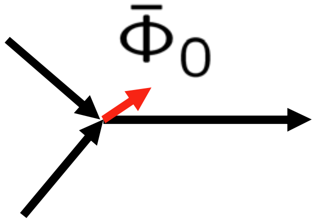
classical field quantum field

$$\langle \phi^{\alpha}(t)\, \bar{\phi}^{\beta}(t') \rangle \equiv iG^{\alpha\beta}(t,t') = \begin{pmatrix} iG^{K}(t,t') & iG^{R}(t,t') \\ iG^{A}(t,t') & 0 \end{pmatrix}$$

$$\alpha, \beta = (cl, q)$$

Condensate Growth and Collapse

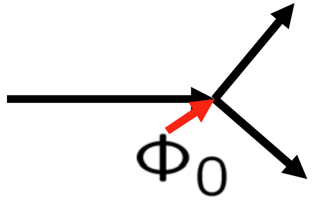
$$\bar{\Phi}_0\bar{\varphi}\varphi\varphi$$



condensate growths

and

$$\bar{\varphi}\bar{\varphi}\varphi\Phi_0$$



condensate collapses

$$\phi = \Phi_0 + \varphi$$

$$\Phi_0^{\pm}; \qquad \varphi^{\pm}$$

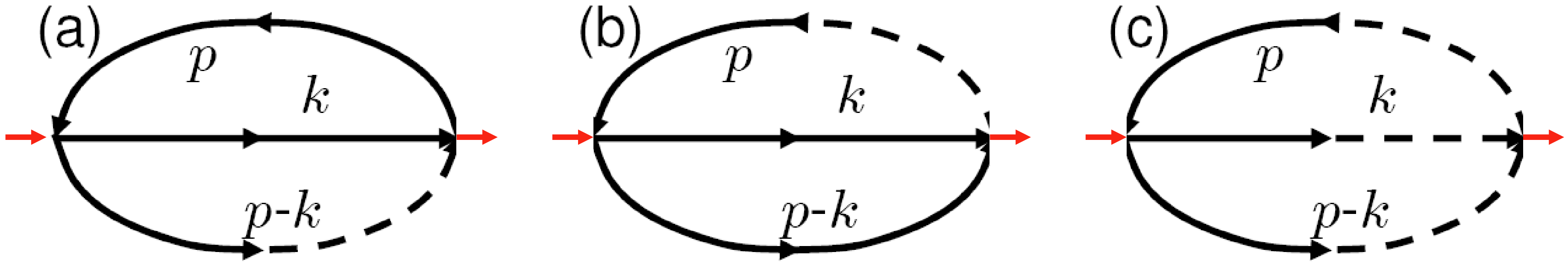
$$\bar{\Phi}^q\left(\bar{\varphi}\varphi\varphi + 2\bar{\varphi}^q\varphi^q\varphi + \bar{\varphi}\varphi^q\varphi^q\right) + c.c.$$

$$+\left(2\bar{\varphi}^q\bar{\varphi}\varphi + \bar{\varphi}\bar{\varphi}\varphi^q + \bar{\varphi}^q\bar{\varphi}^q\varphi^q\right)\Phi_0 + c.c.$$

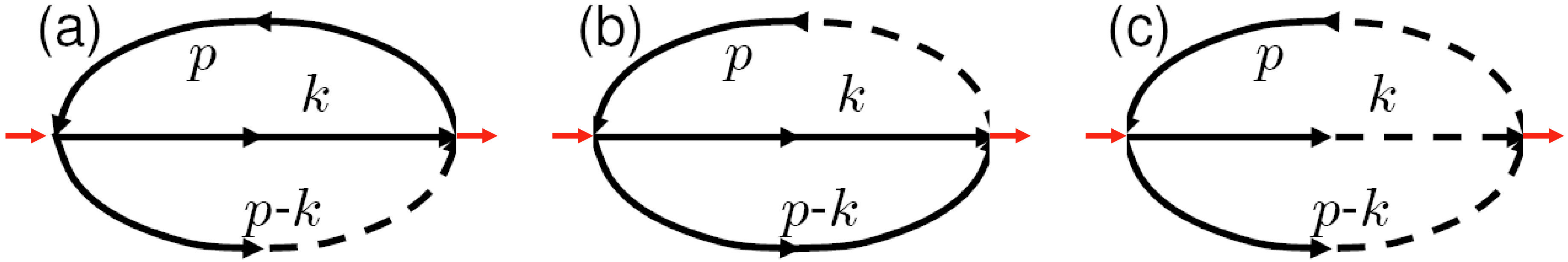
Condensate Growth and Collapse

$$S_3^{\mathrm{coll}} = -g\int \mathrm{d}\mathbf{r}\,\mathrm{d}t\left[\bar{\Phi}^q\left(\bar{\varphi}\varphi\varphi + 2\bar{\varphi}^q\varphi^q\varphi + \bar{\varphi}\varphi^q\varphi^q\right) + c.c.\right.$$
$$\left. + \left(2\bar{\varphi}^q\bar{\varphi}\varphi + \bar{\varphi}\bar{\varphi}\varphi^q + \bar{\varphi}^q\bar{\varphi}^q\varphi^q\right)\Phi_0 + c.c.\right]$$

$$\left\langle e^{iS^{\mathrm{coll}}}\right\rangle \approx 1 + i\left\langle S^{\mathrm{coll}}\right\rangle - \frac{1}{2}\left\langle\left(S^{\mathrm{coll}}\right)^2\right\rangle \approx e^{-\frac{1}{2}\left\langle\left(S^{\mathrm{coll}}\right)^2\right\rangle}\;; \qquad \delta S = \frac{i}{2}\left\langle\left(S^{\mathrm{coll}}\right)^2\right\rangle$$

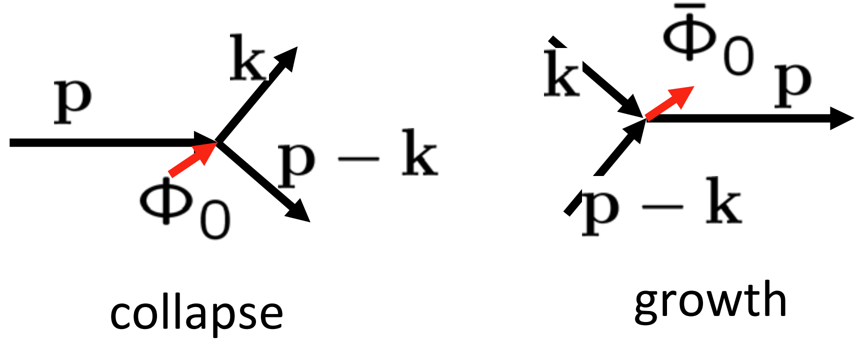


Condensate Growth and Collapse

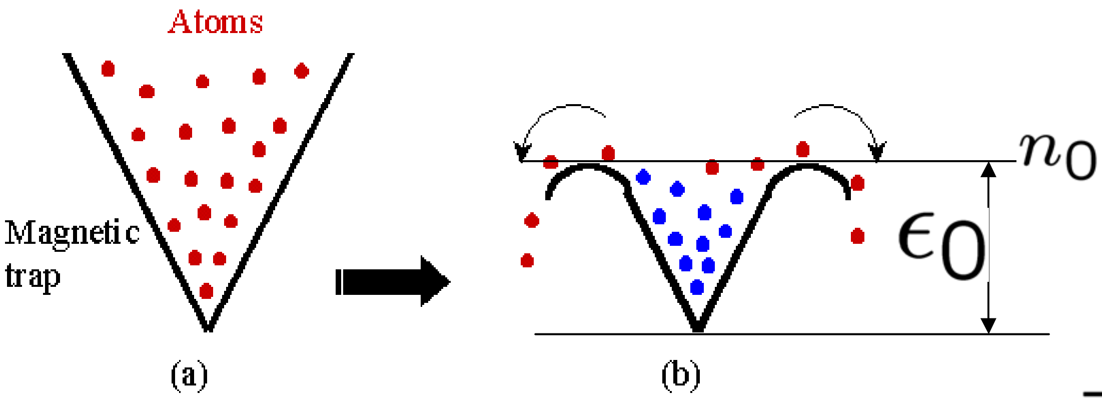


$$[i\partial_t - H_{GP}]\Phi_0 = -i\Gamma_3 \Phi_0$$

$$\Gamma_3 = 2\pi g^2 \sum_{\mathbf{p},\mathbf{k}} \delta(\epsilon_{\mathbf{p}} - \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{p}-\mathbf{k}}) \left[n_{\mathbf{p}}(n_{\mathbf{k}} + n_{\mathbf{p}-\mathbf{k}} + 1) - n_{\mathbf{k}} n_{\mathbf{p}-\mathbf{k}} \right]$$



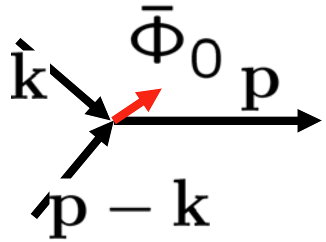
Evaporative Cooling



$$n_{\mathbf{k}} = \begin{cases} n_B(\epsilon_{\mathbf{k}}) & \epsilon_{\mathbf{k}} < \epsilon_0 \\ 0 & \epsilon_{\mathbf{k}} > \epsilon_0 \end{cases}$$

$$\epsilon_{\mathbf{k}} < \epsilon_0 \qquad \epsilon_{\mathbf{p}} > \epsilon_0$$

$$\epsilon_{\mathbf{p}-\mathbf{k}} = \epsilon_{\mathbf{p}} - \epsilon_{\mathbf{k}} < \epsilon_0$$

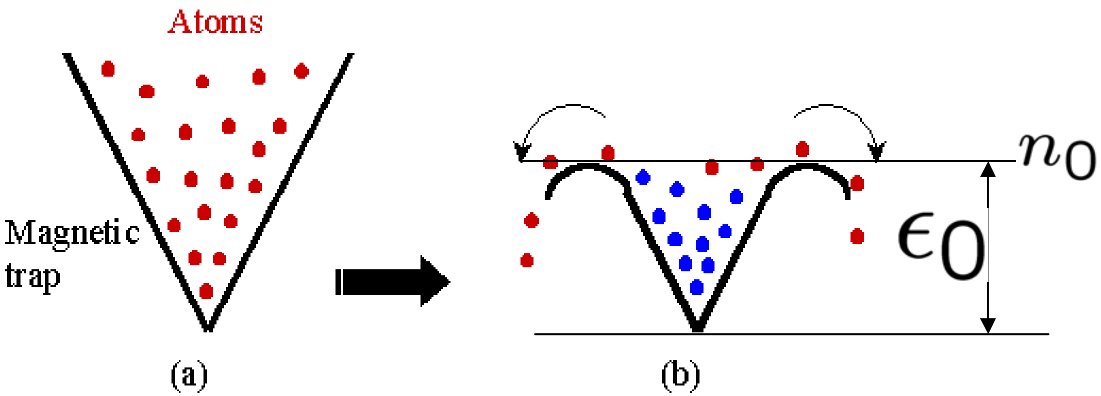


growth

$$-\Gamma_3 = \frac{2g^2m^3}{(2\pi)^3}\int_0^{\epsilon_0} d\epsilon_k\, n_B(\epsilon_k)\int_{\epsilon_0}^{\epsilon_0+\epsilon_k} d\epsilon_p\, n_B(\epsilon_p-\epsilon_k) = \frac{g^2m^3T^2}{4\pi^3}\,B(n_B(\epsilon_0))$$

$B = \pi^2/6$ if $n_0 \gg 1$ and $B \ll 1$ if $n_0 \ll 1$

Evaporative Cooling



$$n_{\mathbf{k}} = \begin{cases} n_B(\epsilon_{\mathbf{k}}) & \epsilon_{\mathbf{k}} < \epsilon_0 \\ 0 & \epsilon_{\mathbf{k}} > \epsilon_0 \end{cases}$$

Once $n_0 1$ condensate growth rate saturates to:

$$\Gamma_3^{\max} = \frac{2\pi}{3} m a_s^2 T^2$$

Nature of the Condensation

$$\Gamma_2(\mathrm{q}) = 4T\, q a_s$$

Landau damping

$$\Gamma_3^{\max} = \frac{2\pi}{3}\, m a_s^2\, T^2$$

Growth rate

$$q_c = m a_s T$$

box size

critical temperature

$$q_c \approx L^{-1}; \qquad T \approx T_c \propto \rho^{2/3}/m$$

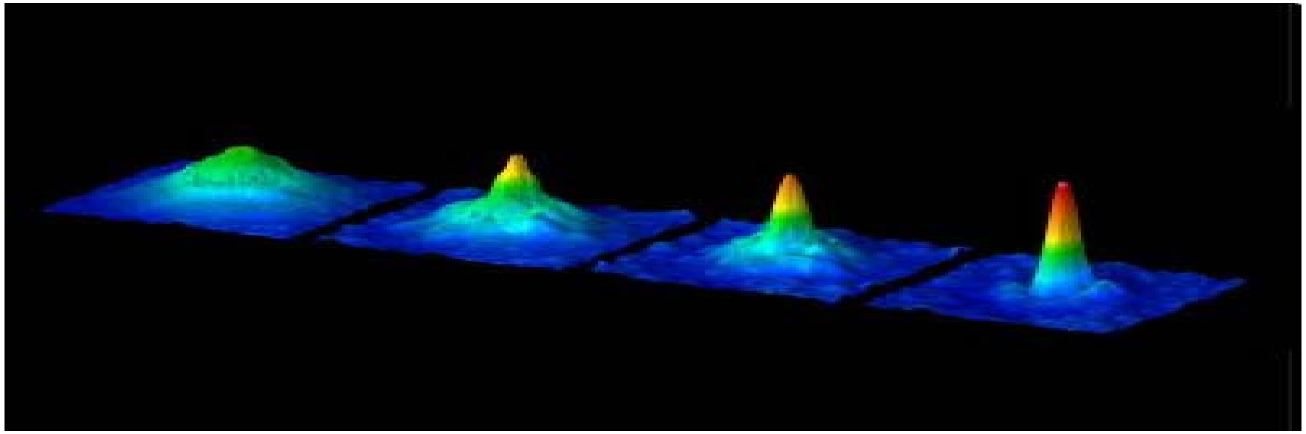
$$N_c = \frac{1}{a_s^3 \rho} \gg 1$$

Nature of the Condensation

$$N_c = \frac{1}{a_s^3 \rho} \gg 1 \qquad \partial_t \Phi_0 = \Gamma_3 \Phi_0$$

$N < N_c$ Landau damping >> Growth rate

Nice and smooth growth of the condensate wave function



$N > N_c$ Landau damping << Growth rate for $q > q_c$

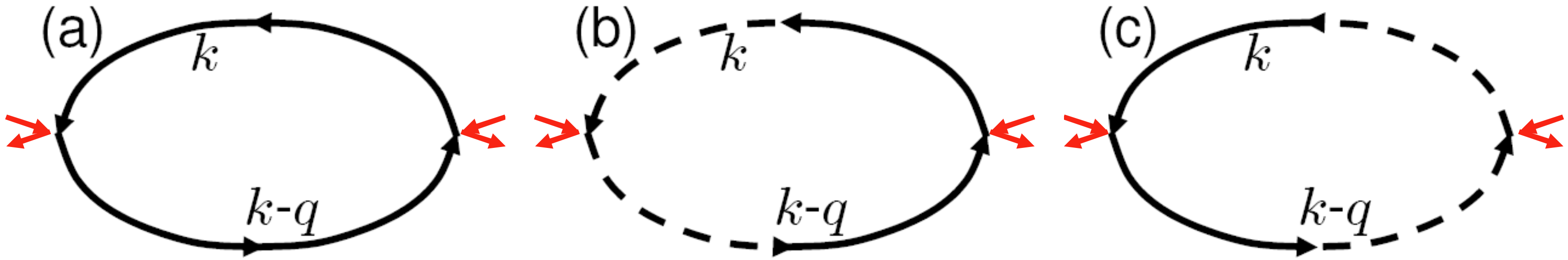
Local structures of size < q_c^{-1} grow instead of uniform condensate "Kimble-Zurek mechanism"

Fluctuations

Damping $\Longrightarrow$ Fluctuations

$$S_2^{\mathrm{coll}} = -\frac{g}{2}\int \mathrm{d}\mathbf{r}\,\mathrm{d}t\left[\bar{\Phi}^q\Phi_0\left(2\bar{\varphi}\varphi - 2\langle\bar{\varphi}\varphi\rangle + 2\bar{\varphi}^q\varphi^q\right) + \bar{\Phi}^q\bar{\Phi}_0\left(\varphi\varphi + \varphi^q\varphi^q\right) + c.c.\right]$$

$$\left\langle e^{iS^{\mathrm{coll}}}\right\rangle \approx 1 + i\left\langle S^{\mathrm{coll}}\right\rangle - \frac{1}{2}\left\langle\left(S^{\mathrm{coll}}\right)^2\right\rangle \approx e^{-\frac{1}{2}\left\langle\left(S^{\mathrm{coll}}\right)^2\right\rangle}\;; \qquad \delta S = \frac{i}{2}\left\langle\left(S^{\mathrm{coll}}\right)^2\right\rangle$$



$$\delta S_2 = g^2\int \mathrm{d}x\,\mathrm{d}x'\,\left(\bar{\Phi}^q\Phi_0 + \bar{\Phi}_0\Phi^q\right)_x \Pi^K(x,x')\left(\bar{\Phi}^q\Phi_0 + \bar{\Phi}_0\Phi^q\right)_{x'}$$

Hubbard Stratonovich Transform

$$\delta S_2 = g^2 \int \mathrm{d}x \, \mathrm{d}x' \left(\bar{\Phi}^q \Phi_0 + \bar{\Phi}_0 \Phi^q\right)_x \Pi^K(x, x') \left(\bar{\Phi}^q \Phi_0 + \bar{\Phi}_0 \Phi^q\right)_{x'}$$

$$e^{i\delta S_2} = \int \mathbf{D}[\xi] \; e^{-\frac{i}{4}\int \mathrm{d}x \mathrm{d}x' \, \xi(x) \Pi^{-1}(x,x') \xi(x') - ig \int \mathrm{d}x \, \xi(x) \left(\bar{\Phi}^q \Phi_0 + \bar{\Phi}_0 \Phi^q\right)_x}$$

$$\left[i\partial_t - H_{GP}\right]\Phi_0 = -i\Gamma_3 \Phi_0 + g\xi(x)\Phi_0$$

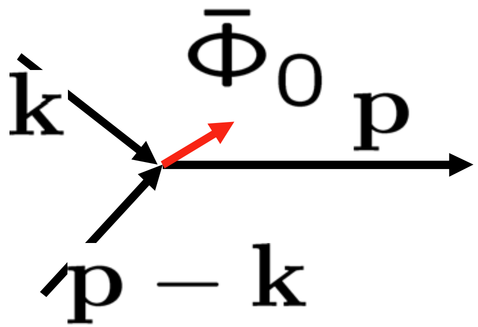
modified Gross-Pitaevskii　　growth/collapse　　fluctuations

$$\langle \xi\xi \rangle = \Pi^K[n] \qquad\qquad \Gamma_3 = \Gamma_3[n]$$

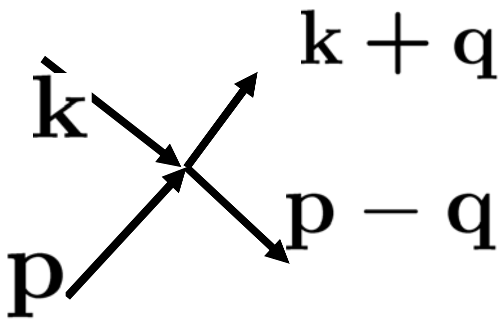
quasiparticles distribution function

Collision IntegralS

Three quasiparticles (+ condensate) collision



Four quasiparticles collisions



Three Particle Collisions

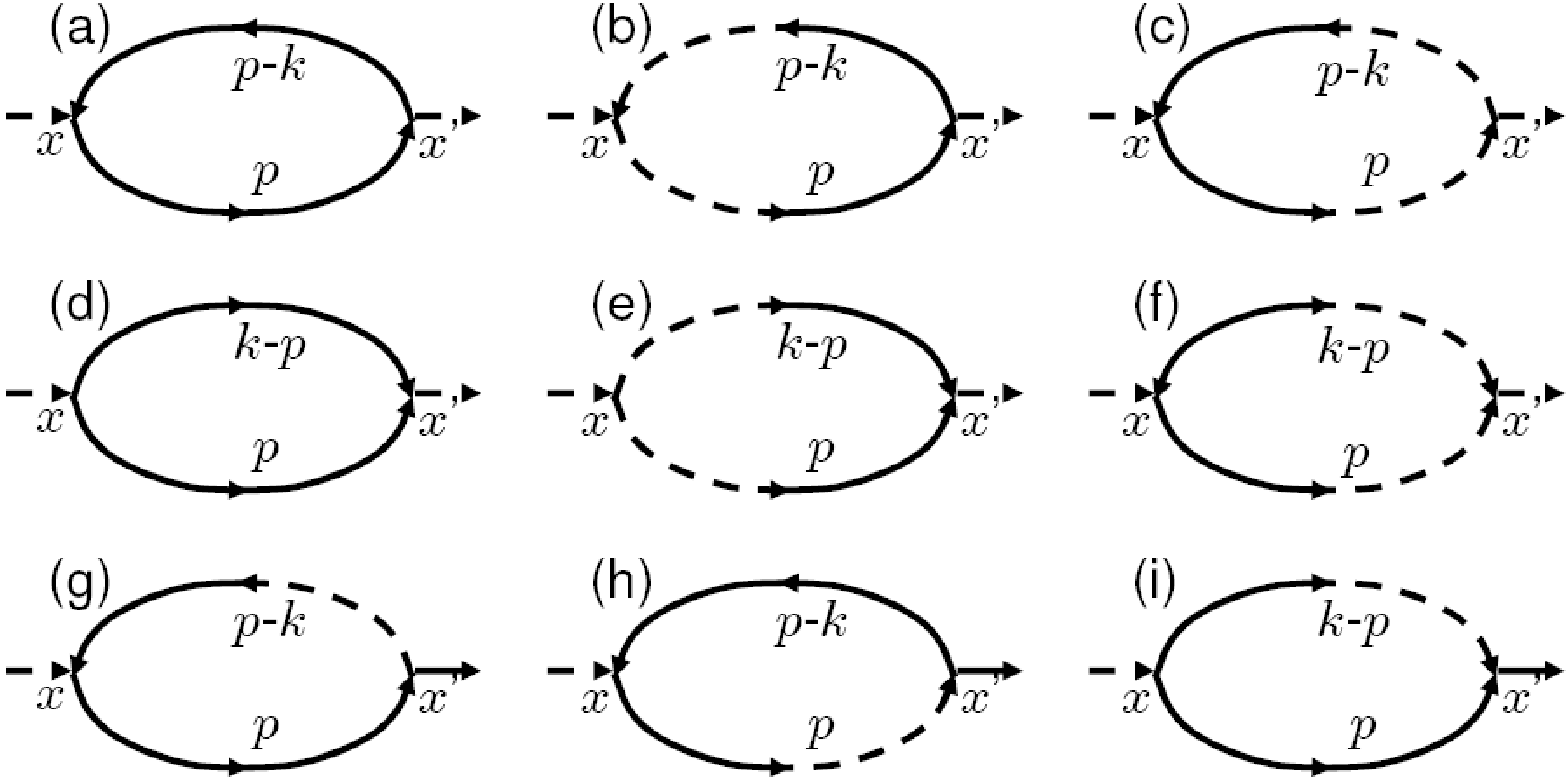
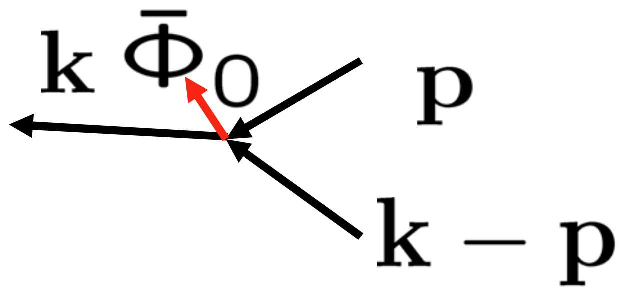


Fig. 7.4. (a)-(f) Six diagrams for $\Sigma_3^K(x,x')$. The normal diagrams (a)-(c) carry the combinatorial factor of 4, while the Bogoliubov ones (d)-(f) carry factor of 2. (g)-(i) three diagrams for $\Sigma_3^R(x,x')$, all carry the factor of 4.

Three Particle Collisions

$$I_3^{coll}[n_{\mathbf{k}}, \rho_0] = 2\pi g^2 \rho_0 \sum_{\mathbf{p}}$$
$$\Big\{2\delta(\epsilon_{\mathbf{p}} - \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{p}-\mathbf{k}}) \left[n_{\mathbf{p}-\mathbf{k}}(n_{\mathbf{p}} + n_{\mathbf{k}} + 1) - n_{\mathbf{p}} n_{\mathbf{k}}\right]$$
$$+ \delta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{p}} - \epsilon_{\mathbf{k}-\mathbf{p}}) \left[-n_{\mathbf{k}}(n_{\mathbf{p}} + n_{\mathbf{k}-\mathbf{p}} + 1) + n_{\mathbf{p}} n_{\mathbf{k}-\mathbf{p}}\right]\Big\}$$



$$\sum_{\mathbf{k}} I_3^{\mathrm{coll}}[\tilde{F}(x, \mathbf{k})] = 2\Gamma_3(x)\rho_0(x)$$

Particle conservation between condensate and quasiparticle cloud

Four Particle Collisions

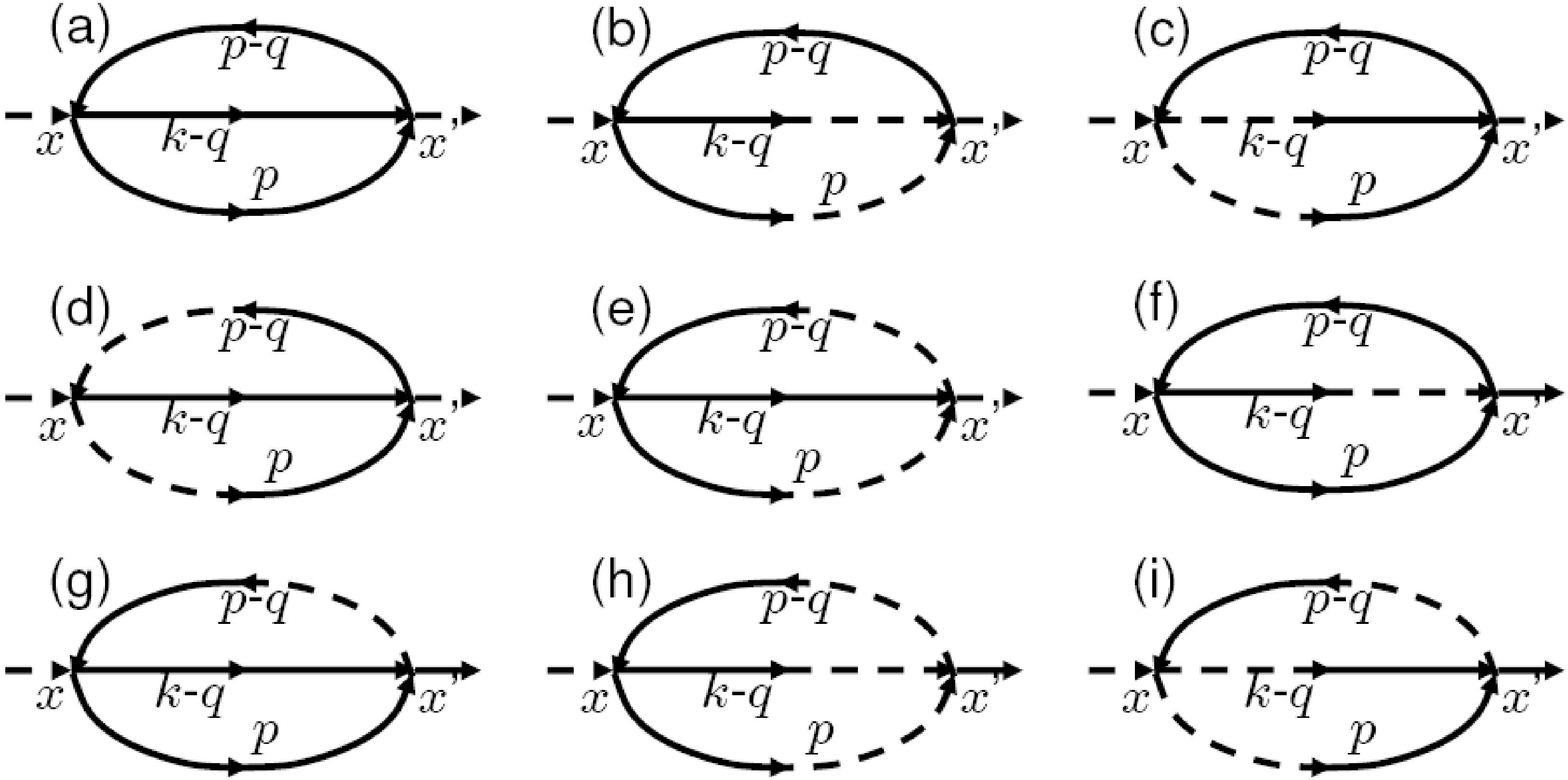
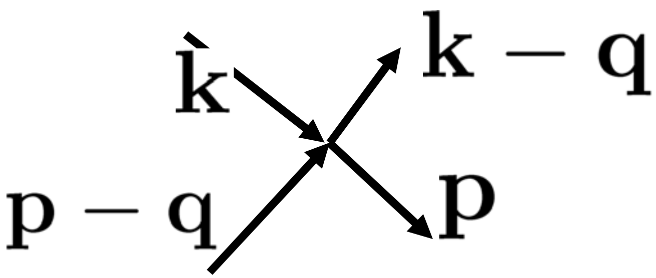


Fig. 7.5. (a)-(e) Five diagrams for $\Sigma_4^K(x, x')$. Diagrams (a)-(c) carry the combinatorial factor of 4, (d)-(e) carry factor of 8. (f)-(i) Four diagrams for $\Sigma_4^R(x, x')$; (f) carries the factor of 8 and (g)-(i) - 4.

Four Particle Collisions

$$I_4^{\text{coll}}[\tilde{F}(x,\mathbf{k}),\rho_0(x),\mathbf{v}_{\text{sf}}(x)] = \pi g^2 \sum_{\mathbf{pq}} \delta(\epsilon_{\mathbf{p}} + \epsilon_{\mathbf{k-q}} - \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{p-q}})$$

$$\{n_{\mathbf{p}} n_{\mathbf{k-q}} (n_{\mathbf{k}}+1)(n_{\mathbf{p-q}}+1) - n_{\mathbf{k}} n_{\mathbf{p-q}} (n_{\mathbf{p}}+1)(n_{\mathbf{k-q}}+1)\}$$



Kinetic Theory

occupation number of state **k** at point (**r**,t): $n(\mathbf{r}, t, \mathbf{k})$

Kinetic equation:

$$\partial_t n + \nabla_{\mathbf{k}} \epsilon_{\mathbf{k}} \nabla_{\mathbf{r}} n - \nabla_{\mathbf{r}} \epsilon_{\mathbf{k}} \nabla_{\mathbf{k}} n = I_3^{\mathrm{coll}}[n, \Phi_0] + I_4^{coll}[n]$$

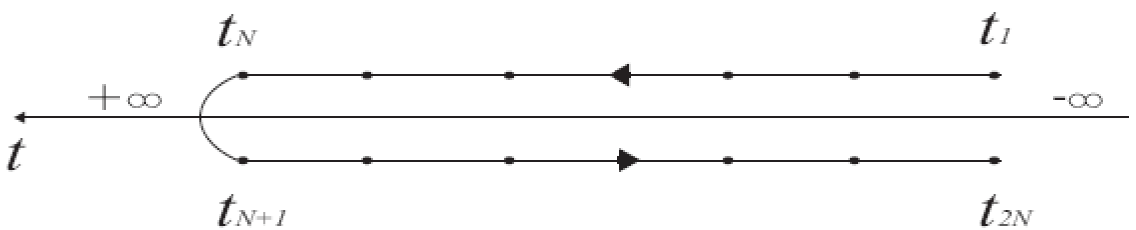
$$\epsilon_{\mathbf{k}}(\mathbf{r}, t) = \frac{\mathbf{k}^2}{2m} + V - \mu + 2g(|\Phi_0|^2 + \rho_{qp})$$

$$\left[i\partial_t - H_{GP}\right]\Phi_0 = -i\Gamma_3\Phi_0 + g\xi(x)\Phi_0$$

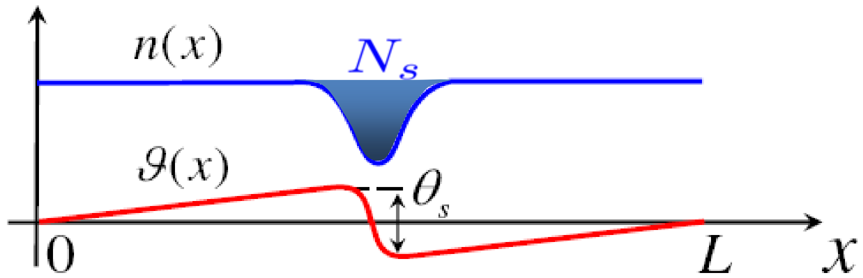
modified Gross-Pitaevskii growth/collapse fluctuations

Where do we go next ?

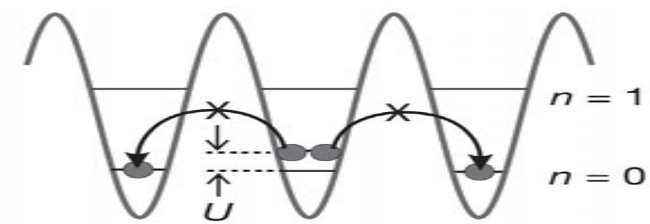
- Keldysh technique tutorial



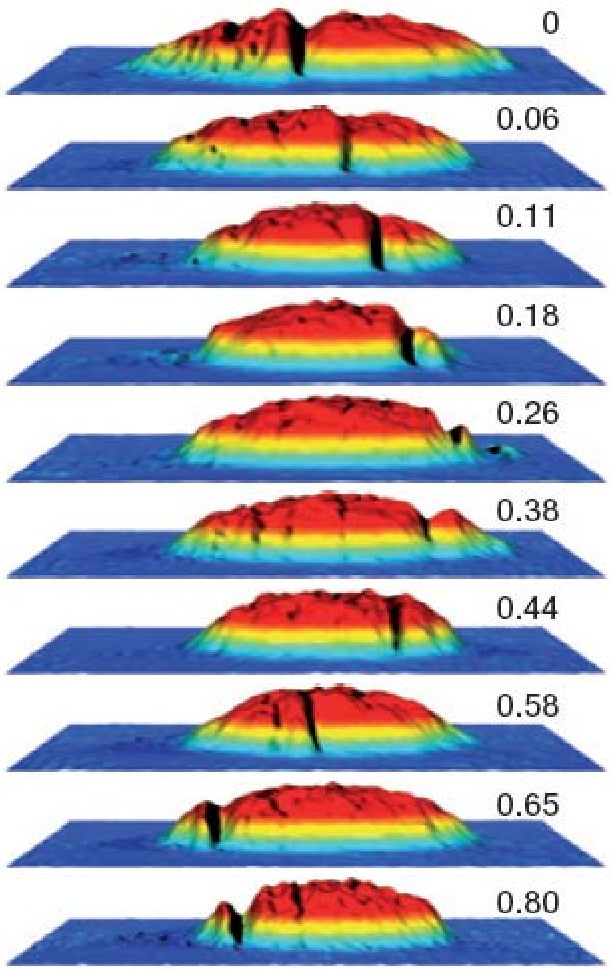
- Dynamics of dark solitons and impurities atoms in 1d Bose liquid



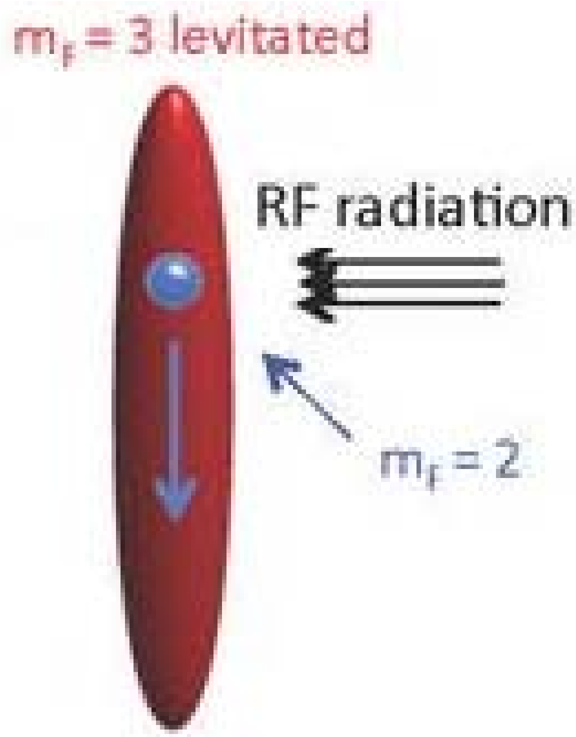
- Relaxation of dublons in optical lattices.



Dark Solitons and Impurities in 1d

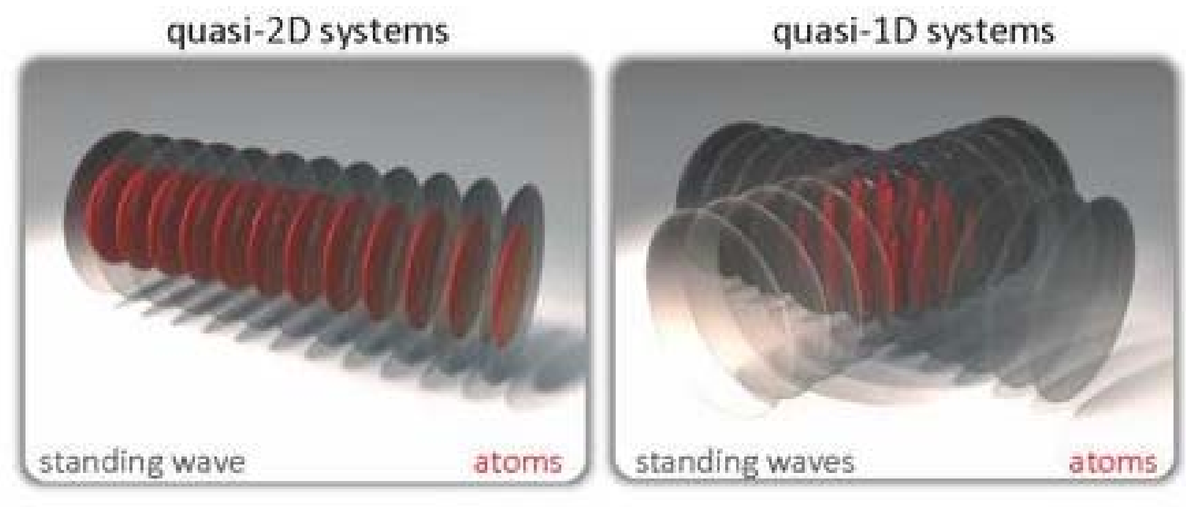


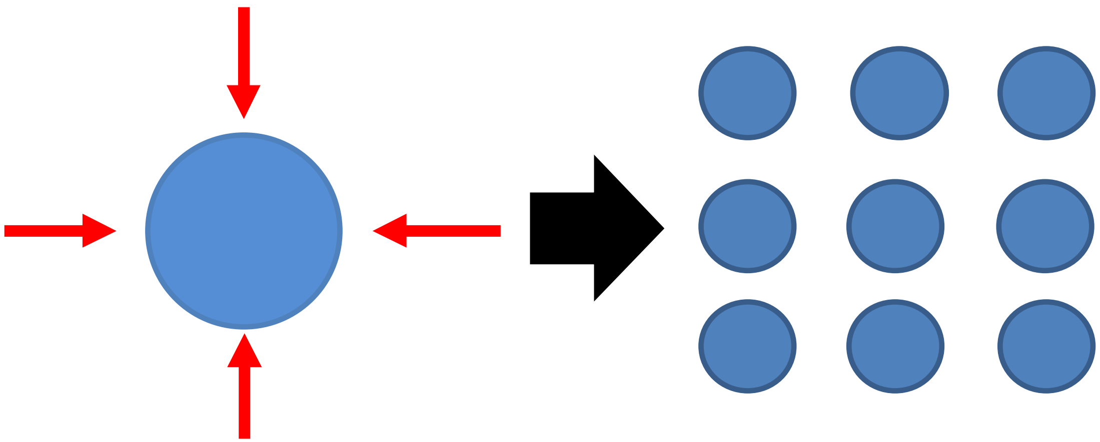
C. Becker *et al.*, Nature Physics 4, 496 (2008)

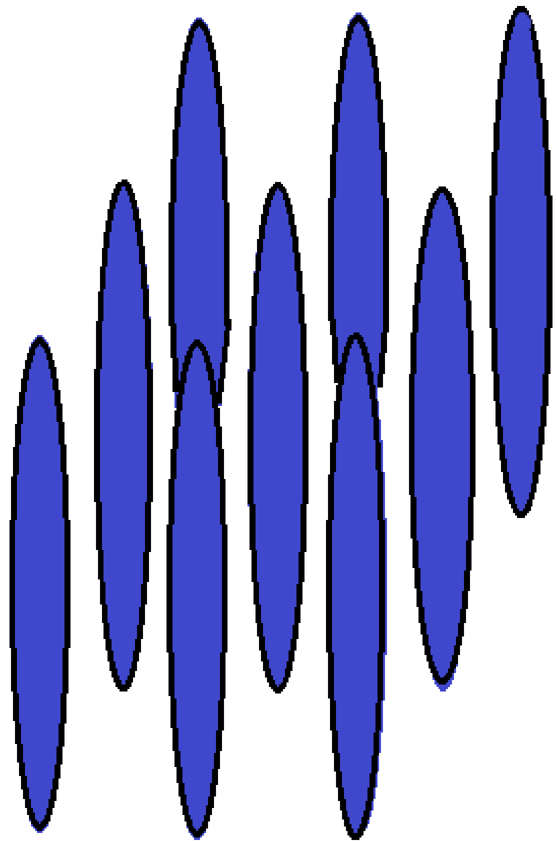


Kohl 2009, Nägerl 2010

1D Optical lattices





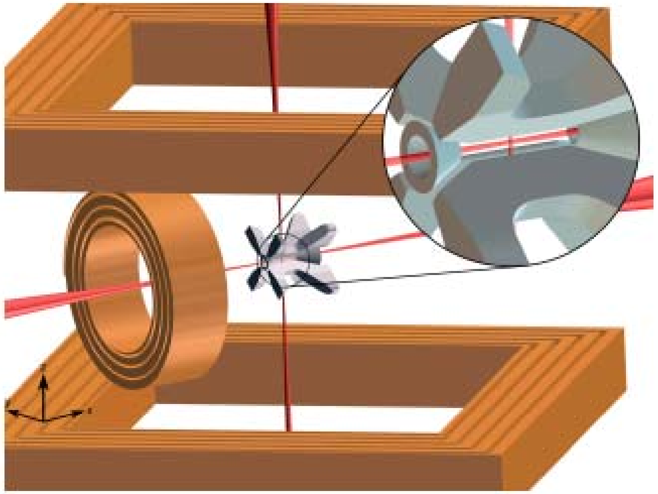


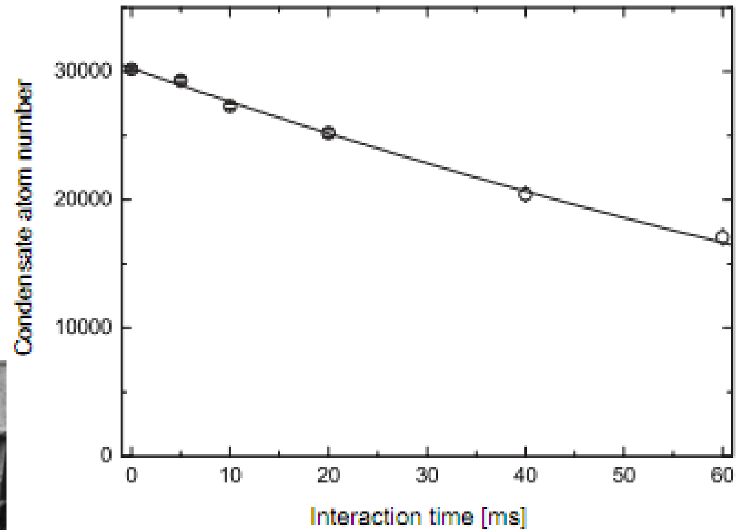
1D: $T,\ \mu \ll \omega_\perp$

Quantum impurity

A trapped single ion inside a Bose-Einstein condensate

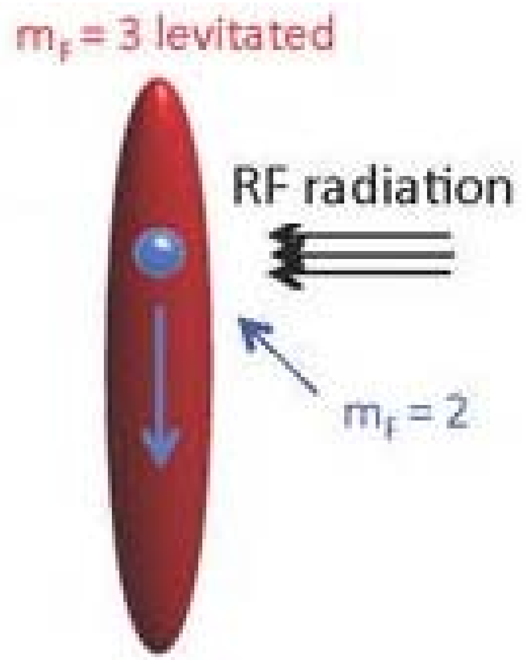
Christoph Zipkes, Stefan Palzer, Carlo Sias, and Michael Köhl
Cavendish Laboratory, University of Cambridge, JJ Thomson Avenue, Cambridge CB3 0HE, United Kingdom

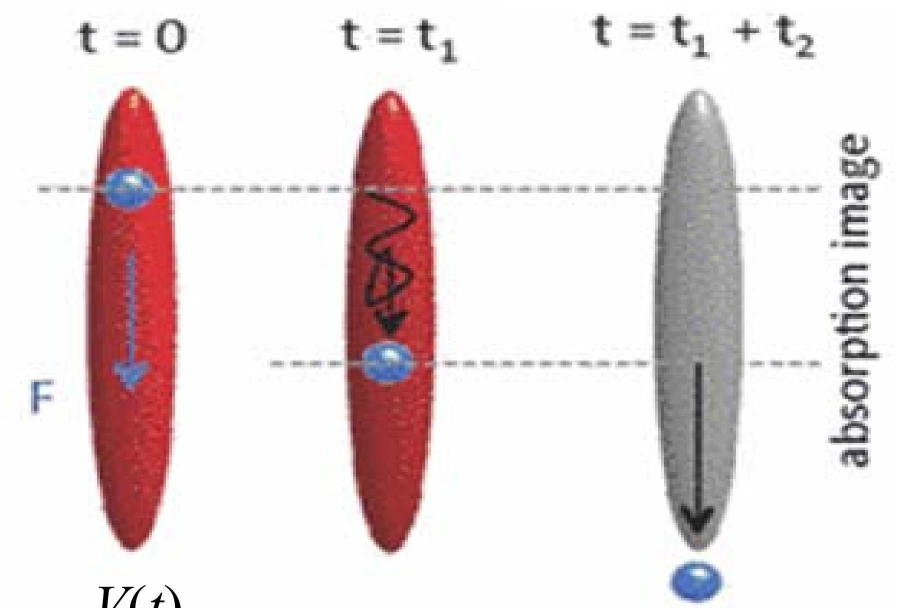




- ✓ Optical lattice + magnetic trap and cold neutral atoms: **Rb-87**
- ✓ Linear Paul trap controls ion: **Yb^+-174**

Spin flipped impurity

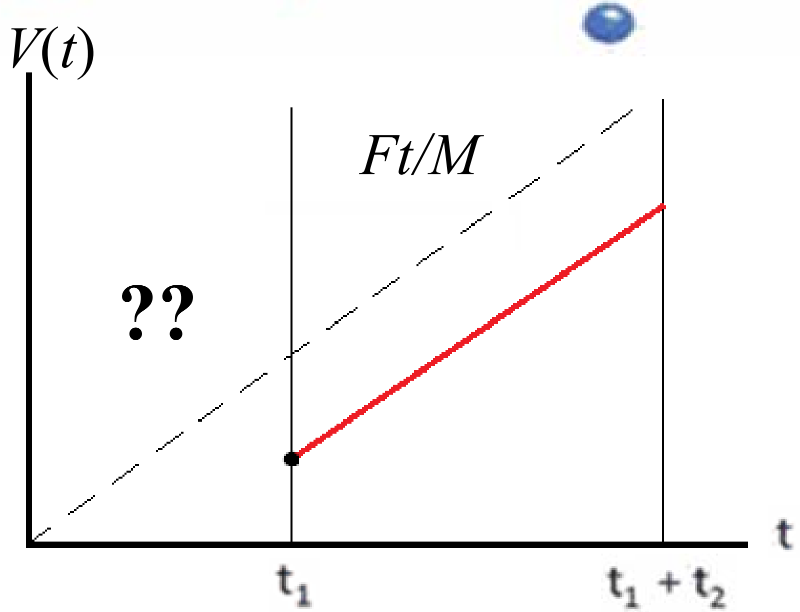




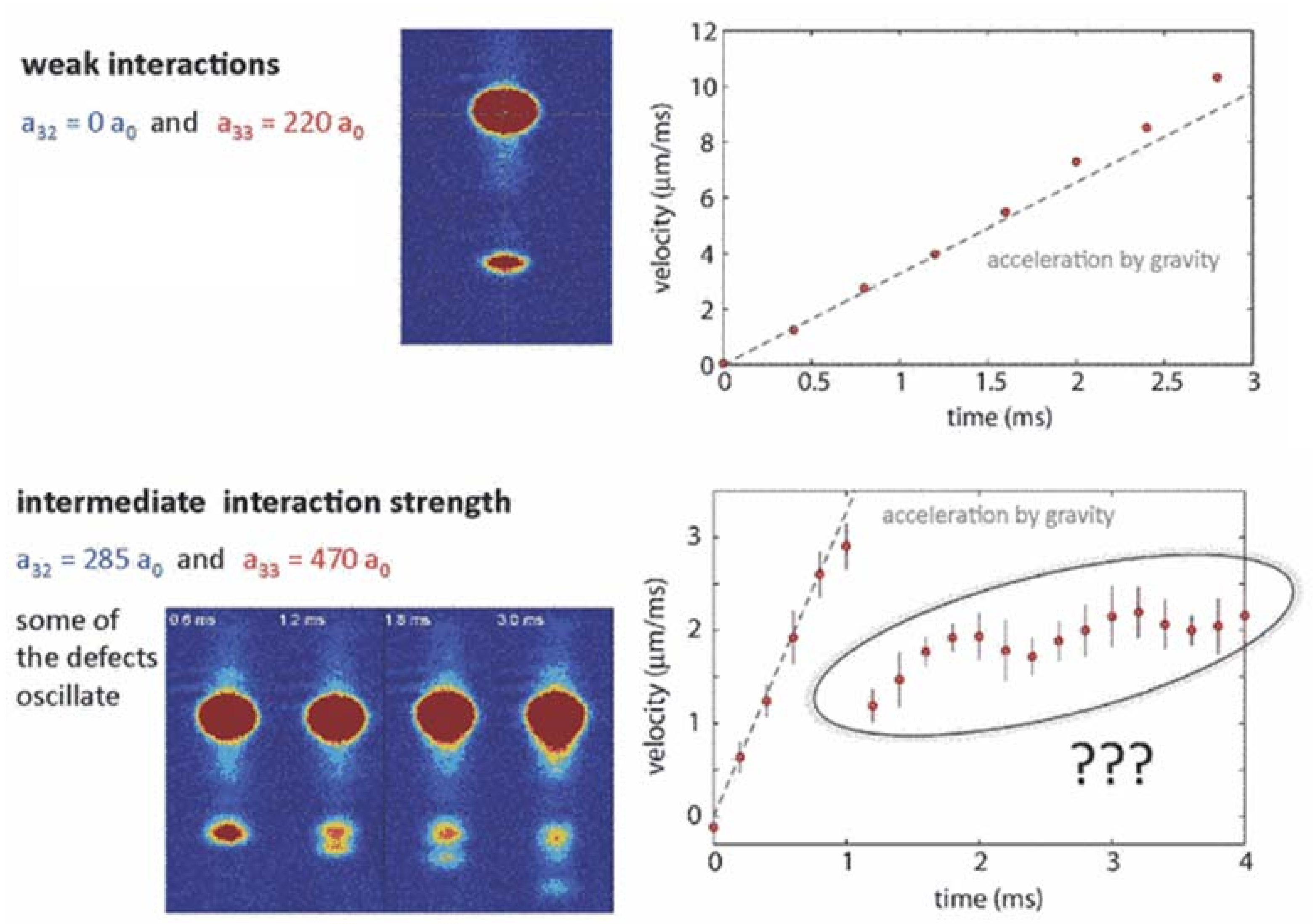
Repeat for different t_1

reconstruct $V(t)$

Kohl 2009, Nägerl 2010



Impurity Velocity



Hanns-Christoph Nägerl (2010)

Hydrodynamics in 1d

$$\phi(x,t) = \sqrt{\rho(x,t)}\, e^{i\theta(x,t)}$$

hydrodynamic parameterization

$$\begin{aligned}\mathcal{L} &= \bar{\phi}\partial_t\phi - H(\bar{\phi},\phi)\\ &= \bar{\phi}\partial_t\phi - \frac{1}{2m}\left|\partial_x\phi\right|^2 - \frac{g}{2}\left|\phi\right|^4\\ &= i\rho\partial_t\theta - \frac{\rho}{2m}(\partial_x\theta)^2 - \frac{1}{2m}\left(\frac{\partial_x\rho}{2\sqrt{\rho}}\right)^2 - \frac{g}{2}\rho^2\end{aligned}$$

$$\rho(x,t) = \rho_0 + \partial_x\varphi(x,t)/\pi$$

small density fluctuations

$$\mathcal{L} = \frac{1}{\pi}\left[i\varphi_x\theta_t - \frac{\pi\rho_0}{2m}(\theta_x)^2 - \frac{g}{2\pi}(\varphi_x)^2 - \frac{(\varphi_{xx})^2}{8\pi m\rho_0}\right]$$

+ nonlinear terms $\varphi_x(\theta_x)^2 + (\varphi_x)^3 + \ldots$

Luttinger Liquid

$$\mathcal{L} = \frac{1}{\pi}\left[\overbrace{i\varphi_x\theta_t - \frac{cK}{2}(\theta_x)^2 - \frac{c}{2K}(\varphi_x)^2}^{\text{Luttinger Liquid,} \quad \text{Popov 1973}} - \overbrace{\frac{(\varphi_{xx})^2}{8\pi m\rho_0}}^{\text{Bogoliubov dispersion}}\right]$$

$$c^2 = \rho_0 g/m\,, \qquad K = \pi\rho_0/(mc)$$

Equations of motion

$$i\theta_t = \frac{c}{K}\varphi_x + \frac{1}{4\pi m\rho_0}\varphi_{xxx}$$

$$i\varphi_t = cK\,\theta_x \qquad\qquad \varphi_{tt} + c^2\varphi_{xx} = 0$$

Dark Soliton

$$\phi(x,t) = \sqrt{\rho(x-Vt)}\, e^{-i\theta(x-Vt)}$$

$$\rho_t + \partial_x(\rho\theta_x/m) = 0$$

continuity equation

$$\partial_x[-V\rho + \rho\theta_x/m] = 0$$

$$\theta_x = \frac{mV}{\rho}[\rho - \rho_0]$$

$$\theta_t = \frac{\partial_x^2\sqrt{\rho}}{m\sqrt{\rho}} - \frac{(\theta_x)^2}{2m} + g\rho$$

Euler equation

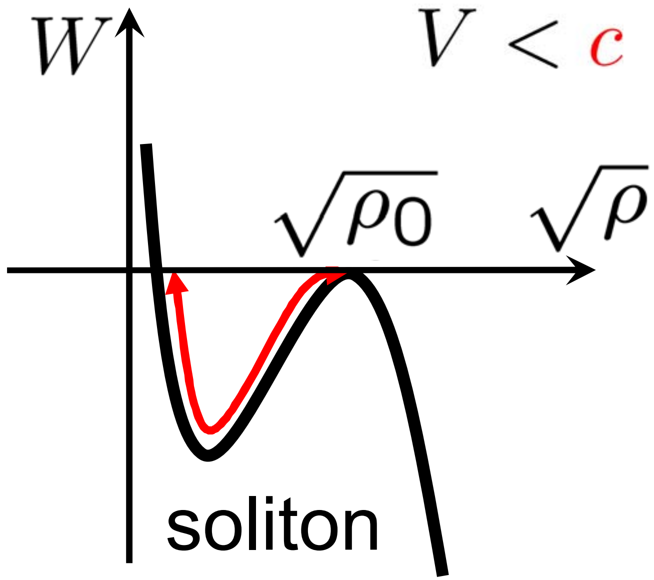
$$\partial_x^2\sqrt{\rho} = F(\sqrt{\rho})$$

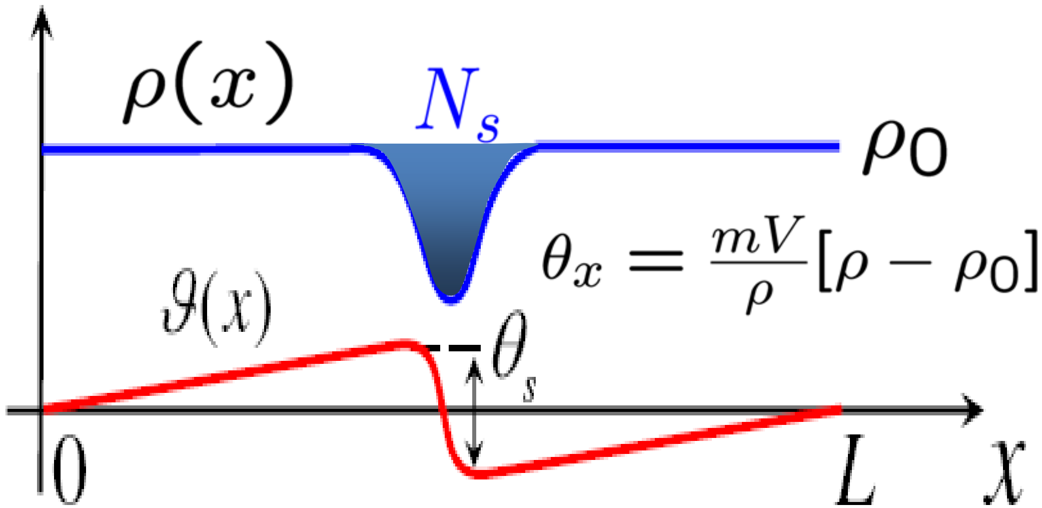
"Newtonian" mechanics for the amplitude

Dark Soliton

$$\partial_x^2 \sqrt{\rho} = -\frac{dW(\sqrt{\rho})}{d\sqrt{\rho}}$$

"Newtonian" mechanics for the amplitude

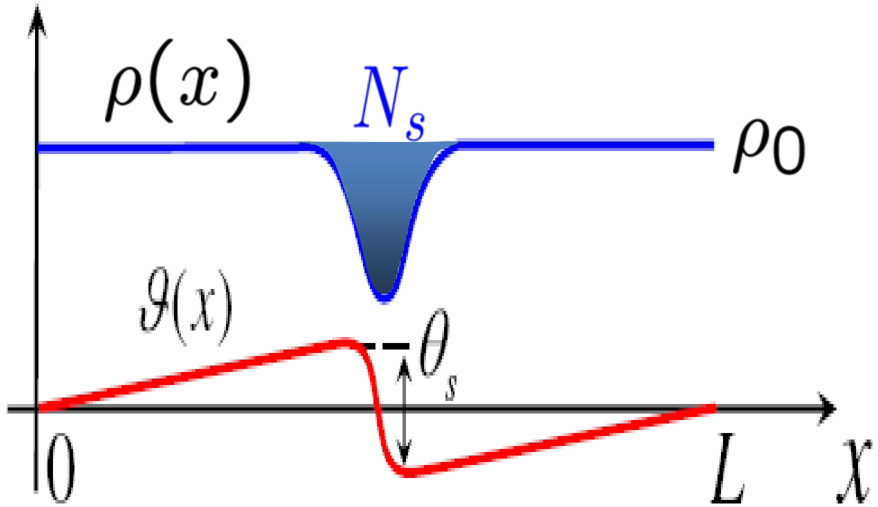


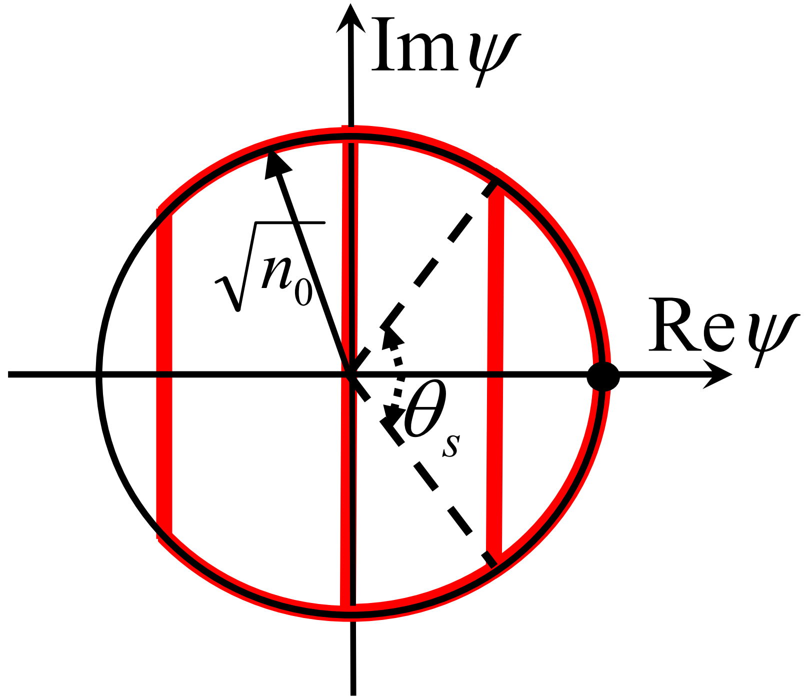


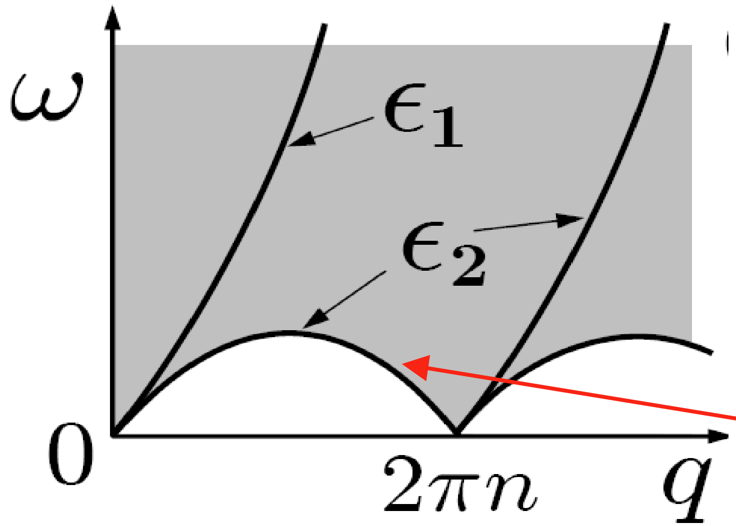
$$V = c\cos\theta_s/2$$

$$N_s = \frac{2K}{\pi}\sin\theta_s/2 \gg 1$$

Dark Solitons







$$p_s = n_0\left(\theta_s - \sin\theta_s\right)$$

$$E_s = \frac{4}{3} n_0 c \sin^3(\theta_s/2)$$

PHYSICAL REVIEW VOLUME 130, NUMBER 4 15 MAY 1963

Exact Analysis of an Interacting Bose Gas. I. The General Solution and the Ground State

Elliott H. Lieb and Werner Liniger
Thomas J. Watson Research Center, International Business Machines Corporation, Yorktown Heights, New York
(Received 7 January 1963)

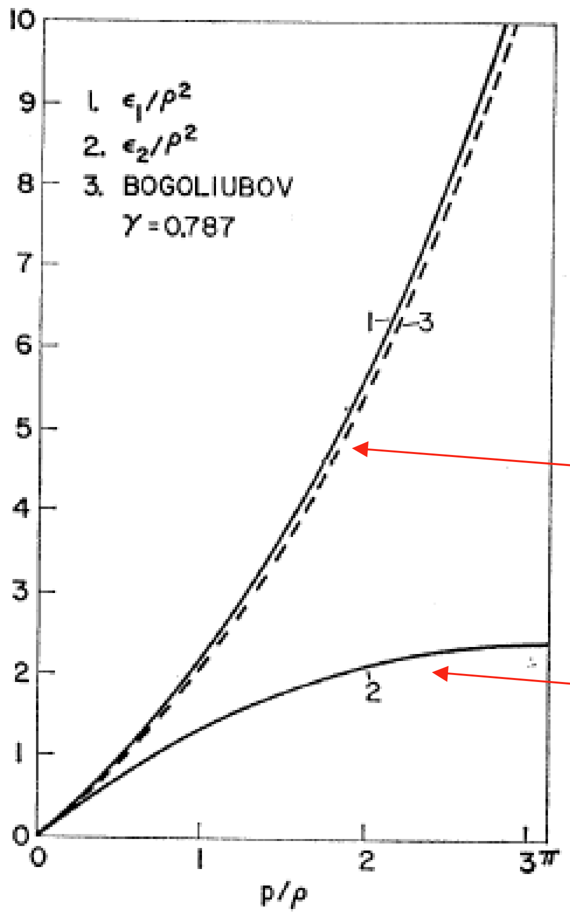
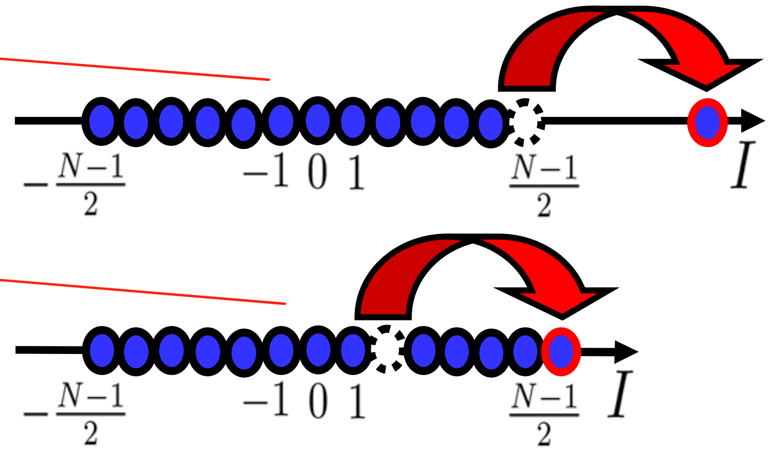
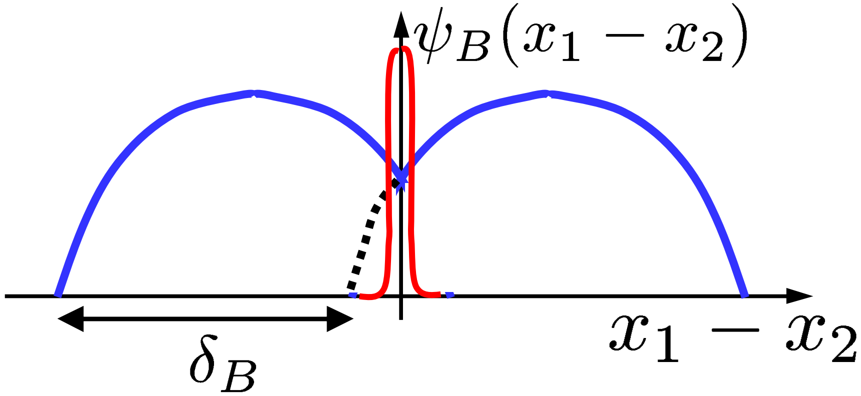


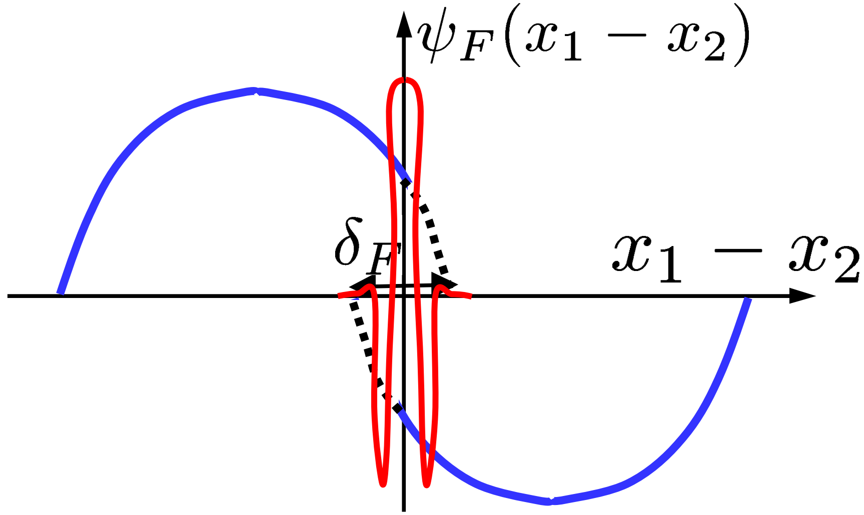
FIG. 4. A comparison plot of the two types of excitations, ϵ_1 and ϵ_2, for $\gamma = 0.787$. The dashed curve is Bogoliubov's spectrum which is quite close to the type I spectrum. The type II spectrum does not exist in Bogoliubov's theory.



Fermion-Boson Correspondence

Impenetrable bosons = non-interacting fermions.
Tonks-Girardeau limit





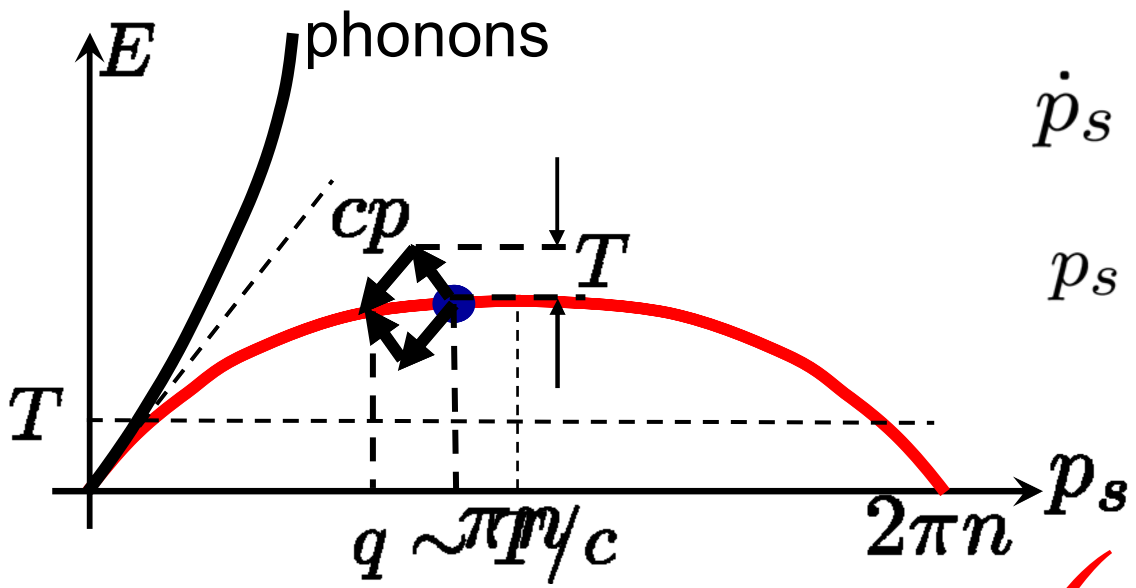
$$V_B = g\delta(x_1 - x_2)$$

$$\delta_F = \pi - \delta_B$$

$$V_F = -\frac{2}{m^2 g}\delta''(x_1 - x_2)$$

Girardeau, Olshanii, 2004

Finite Temperature Dynamics



$$\dot{p}_s = -\kappa(T)V + \xi(t)$$

$$p_s = \pi n - M^* V$$

$$\tau = M^*/\kappa$$

For Hamburg experiment

$$\tau \approx 200ms \times (\mu/T)^4$$

In a single Raman process the DS momentum change is small

$$q \sim T/c \ll n \sim p_s$$

One can thus develop semiclassical dynamics of DS

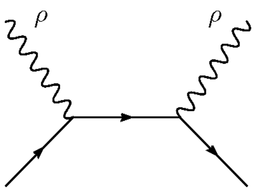
Soliton-Phonon Interactions

$$E_s(V,\rho_0) \to E_s(V-u,\rho_0+\delta\rho) + u p_s(V-u,\rho_0+\delta\rho)$$

$$u = \theta_x/m\,; \qquad \delta\rho = \varphi_x/\pi$$

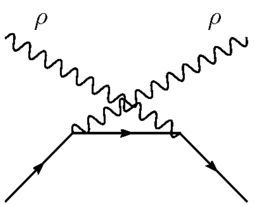
DS-phonon interactions are completely fixed by Galilean invariance and $E_s(V,\rho_0)$ dependence

a

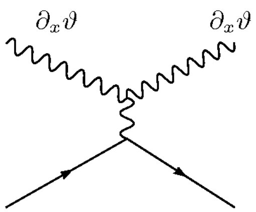


$$\boxed{L_{\mathrm{s-ph}} = -\frac{\Gamma_\rho}{2}\,\rho^2(X,t) - \frac{\Gamma_u}{2}\,u^2(X,t)}$$

b



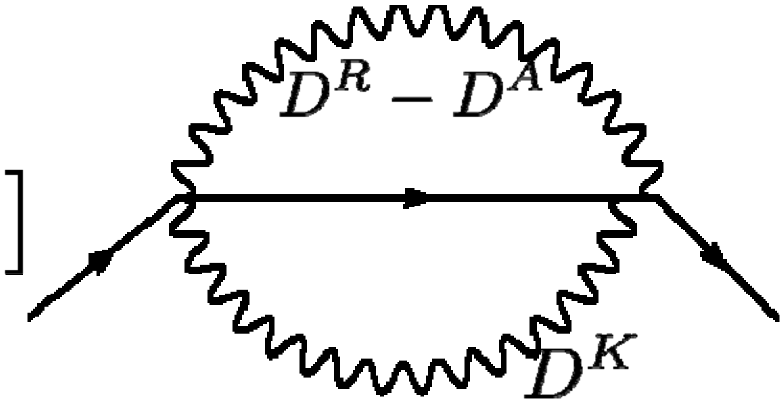
d



$$\Gamma_\rho = \frac{\partial\mu}{\partial\rho_0}\,\frac{\partial N_{\mathrm{s}}}{\partial\rho_0}\,; \qquad \Gamma_u = mN_{\mathrm{s}}\left(1+\frac{mN_{\mathrm{s}}}{M^*}\right)$$

Collision Integral

$$\dot{P} = \mathrm{Tr}\left[\Gamma\left(D^A - D^R\right)\Gamma D^K\right]$$



$$= -\frac{1}{4}\left(\Gamma_\rho - \Gamma_u \frac{c^2}{\rho_0^2}\right)^2 \sum_{|q| \lesssim mc} e^{iqX} q\,\Pi(q, qV)$$

Quantum interference – possible cancellation

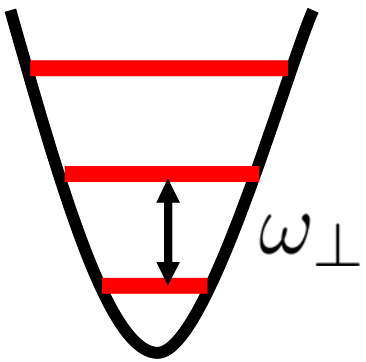
$$\Pi(q,\omega) = \frac{\rho_0^2}{4m^2c^3}\left(q^2 - \frac{\omega^2}{c^2}\right)\left(\coth\frac{cq-\omega}{4T} - \coth\frac{cq+\omega}{4T}\right)$$

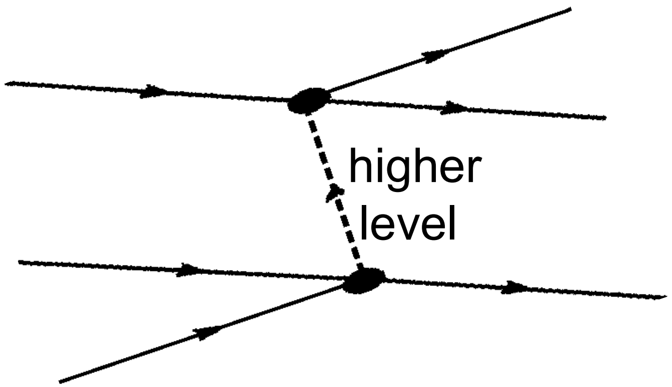
Integrable vs. Non-integrable model

$$\Gamma_\rho = \frac{\partial \mu}{\partial \rho_0}\frac{\partial N_s}{\partial \rho_0}\,; \qquad \Gamma_u = m N_s \left(1 + \frac{m N_s}{M^*}\right)$$

$$\left(\Gamma_\rho - \Gamma_u \frac{c^2}{\rho_0^2}\right) = 0$$

The amplitude of the Raman process is identically zero!





$$\frac{g}{2}|\phi|^4 + \frac{\alpha}{6}|\phi|^6$$

$$\alpha = -6 \ln\left(\frac{4}{3}\right)\frac{g^2}{\omega_\perp}$$

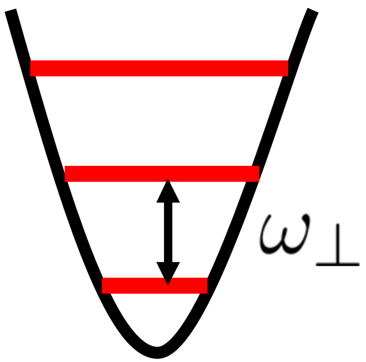
Mazets, et al 2008,
Muryshev, 2002.

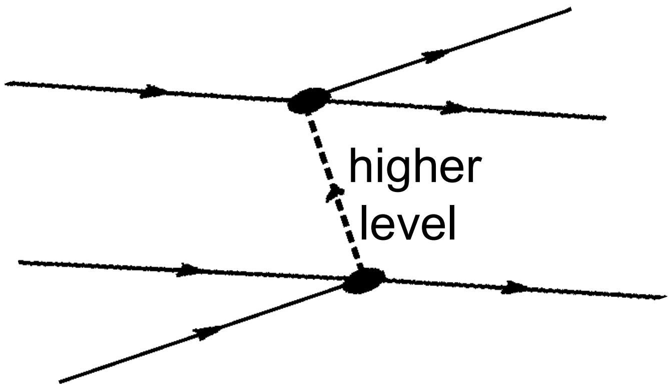
Integrable vs. Non-integrable model

$$\kappa(T) = \frac{1024\pi^3}{1215} \frac{\boxed{\alpha^2}\rho_0^4}{\hbar\, c^2} \left(\frac{T}{\mu}\right)^4$$

3-body scattering amp.
deviation from the exact integrability

The amplitude of the Raman process is identically zero!





$$\frac{g}{2}|\phi|^4 + \frac{\alpha}{6}|\phi|^6$$

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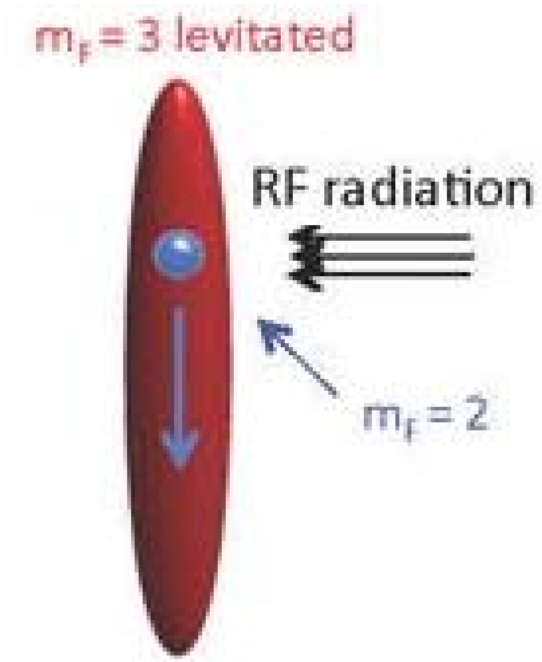
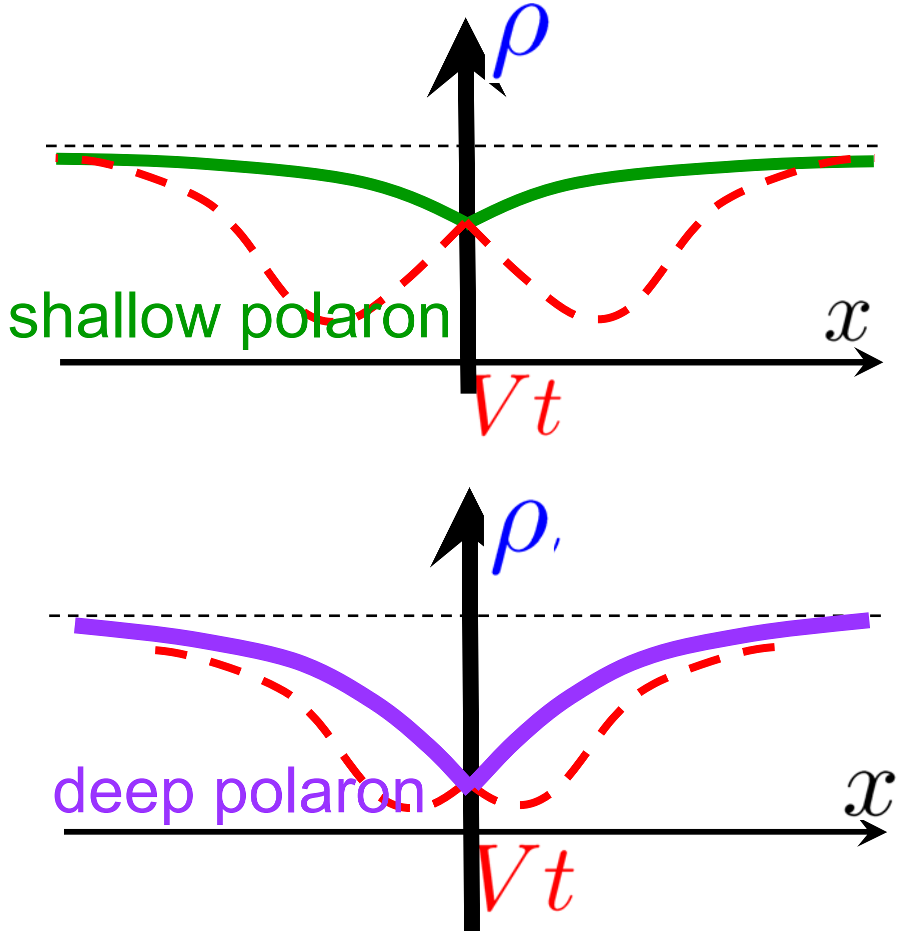
Mazets, et al 2008,
Muryshev, 2002.

Mobile Impurity

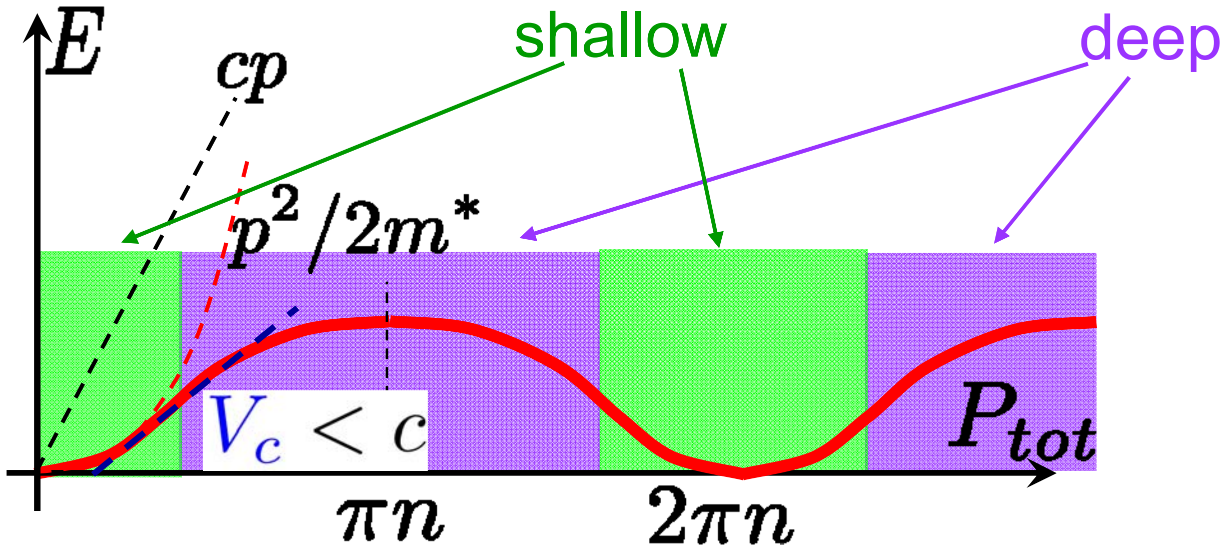
$$i\partial_t \phi = \left[-\frac{\nabla^2}{2m} + g|\phi|^2 - \mu + g_i \delta(x - X(t)) \right] \phi$$

$$\partial_x \sqrt{\rho}\,\Big|_{x=\pm 0} = \pm g_i \sqrt{\rho}\Big|_{x=0}$$





Impurity Dispersion



Ferro magnon

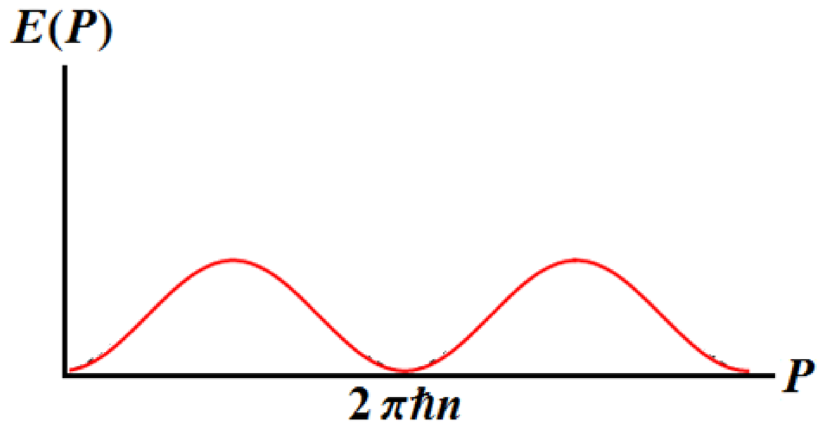
✓ T=0 superfluid: $V < V_c$: $V = const$

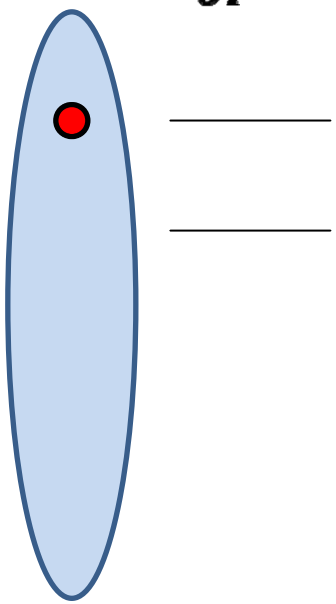
✓ T>0 friction: $m_i \partial_t V = F - \kappa V$ $\kappa \sim T^4$

Castro-Neto, M.P.A. Fisher, 1996; Gangardt, AK, 2009

Bloch Oscillations

- Apply a **Force** ✓ $\boldsymbol{P = Ft}$
- Small $\boldsymbol{F} \longrightarrow$ $\boldsymbol{E(Ft)}$
- $\boldsymbol{V = \dfrac{\partial E(P)}{\partial P}} \longrightarrow$ ✓ **Bloch Oscillations !**

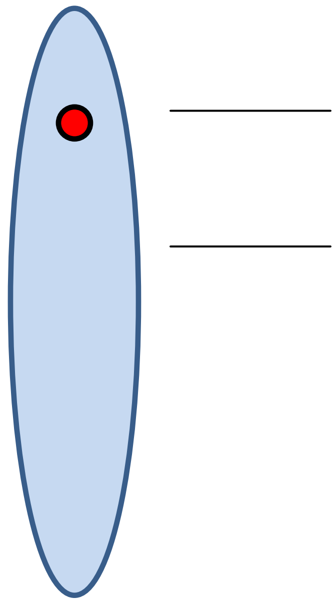




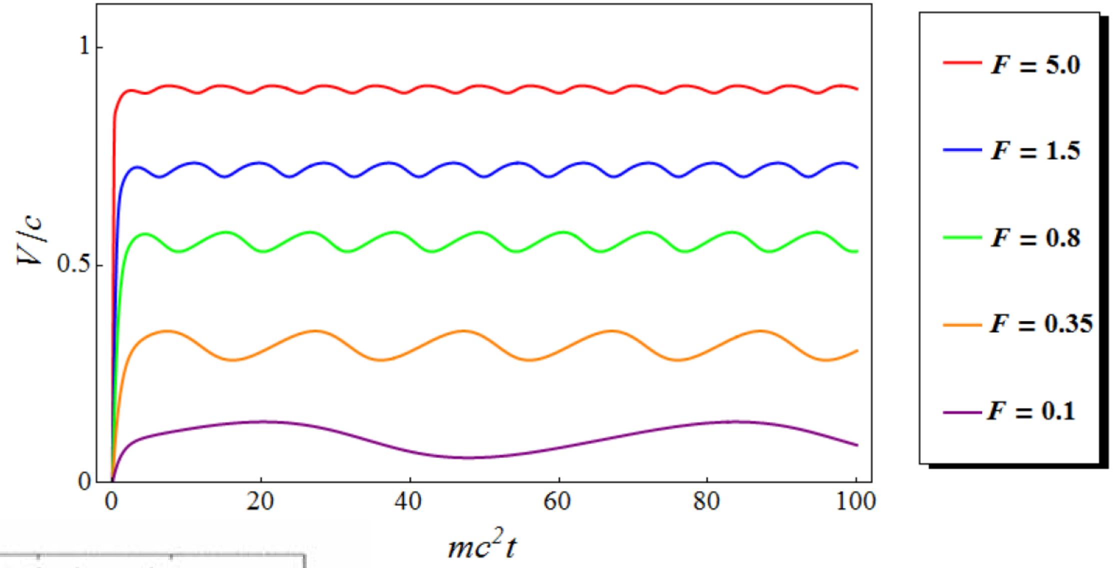
What about:

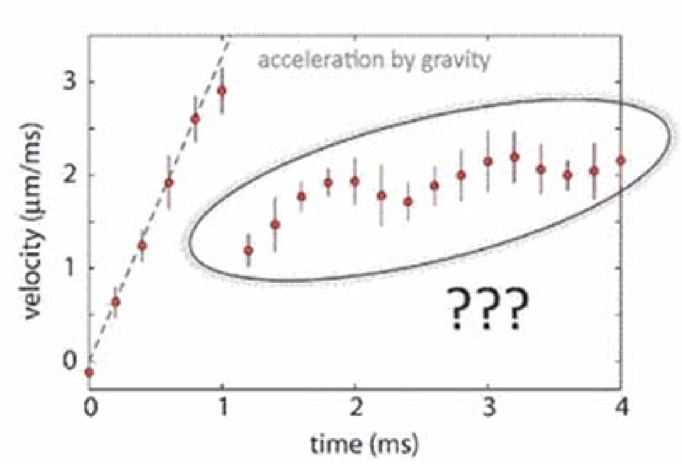
- **Acceleration**

✓ **Oscillations** + **Drift**



Drift and Mobility

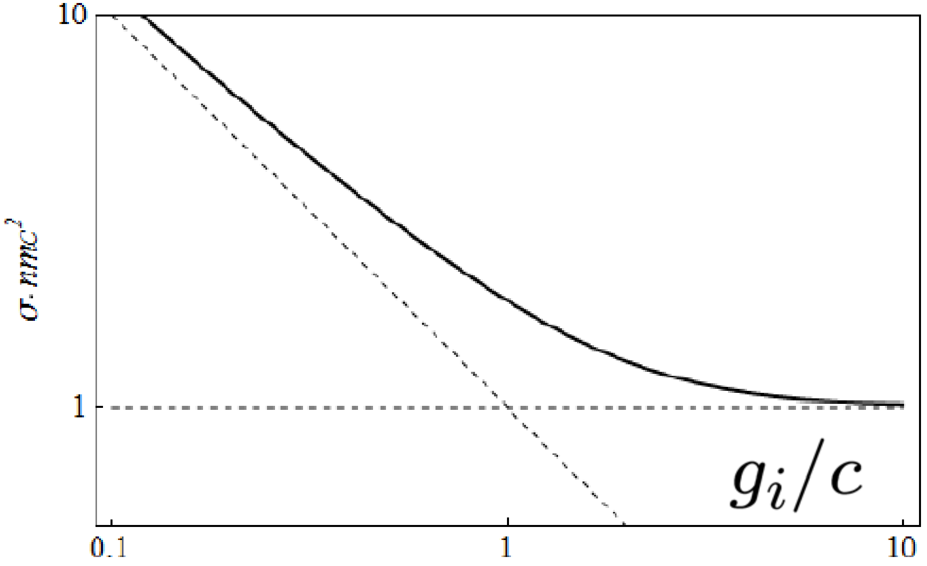


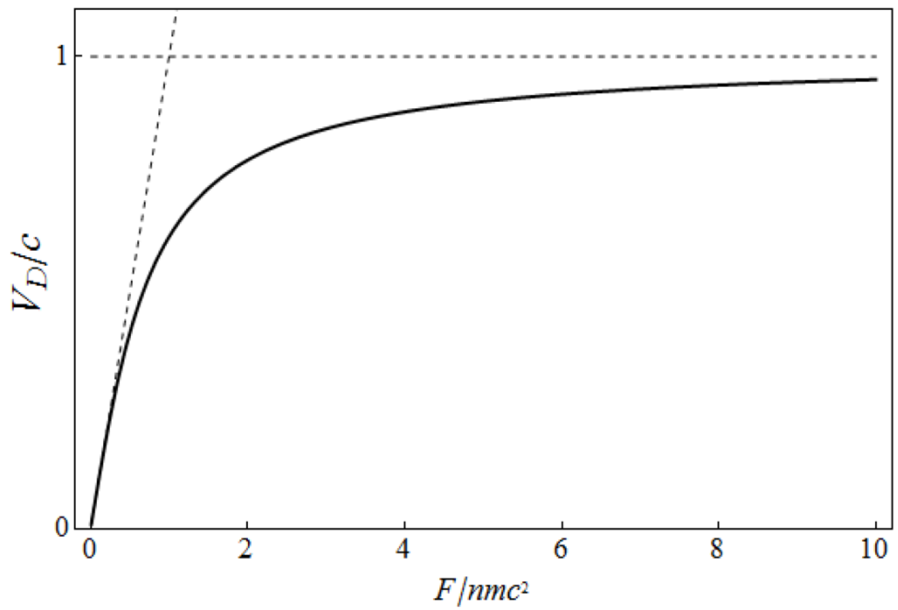


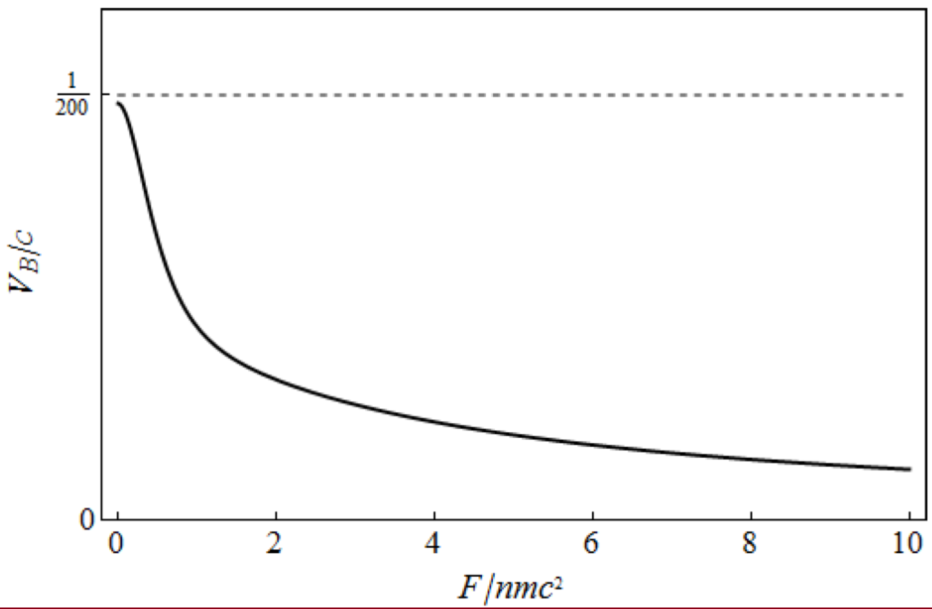
$$V_D = \sigma F$$

$$\sigma = \int_0^{2\pi n} \frac{dP}{2\pi n}\left[\frac{cK}{\pi}\left(\frac{\partial\theta_s}{\partial P}\right)^2 + 2V\left(\frac{\partial N_s}{\partial P}\right)\left(\frac{\partial\theta_s}{\partial P}\right) + \frac{c\pi}{K}\left(\frac{\partial N_s}{\partial P}\right)^2\right].$$

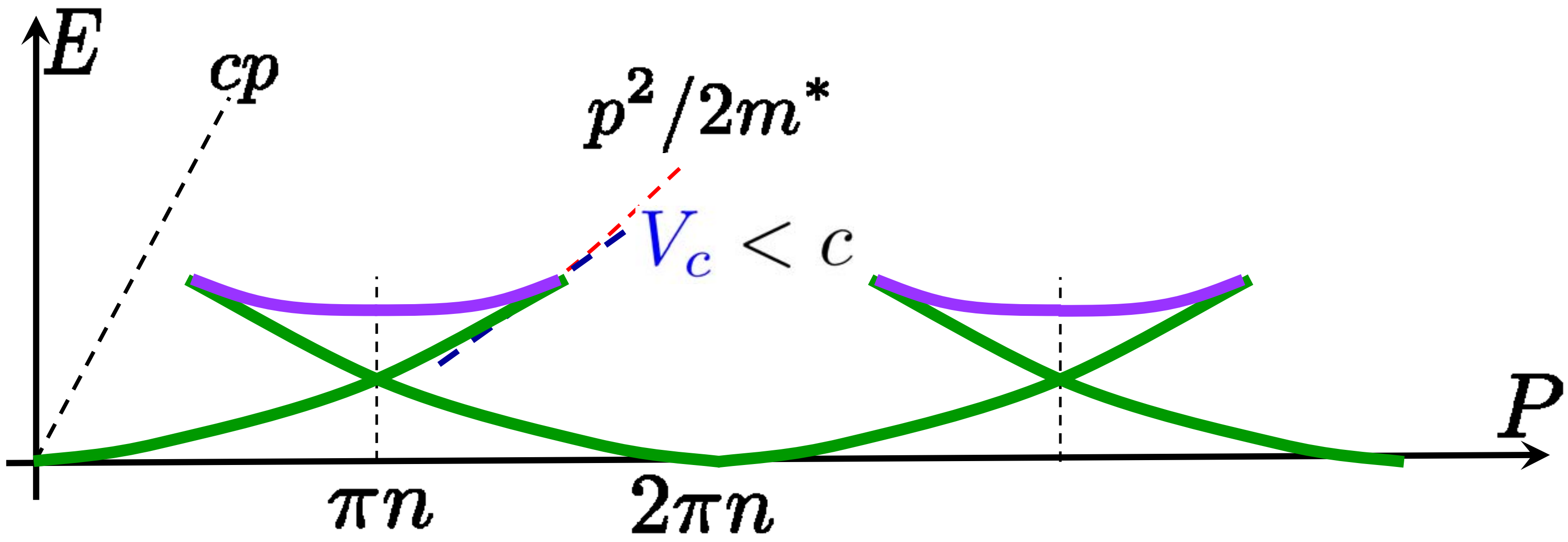
Drift and Mobility



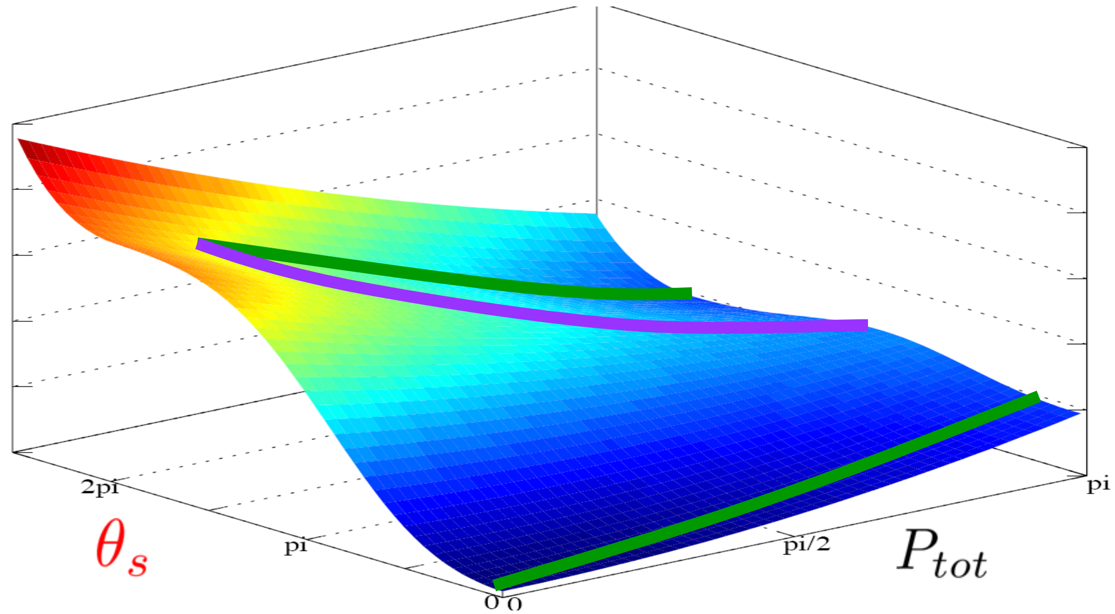




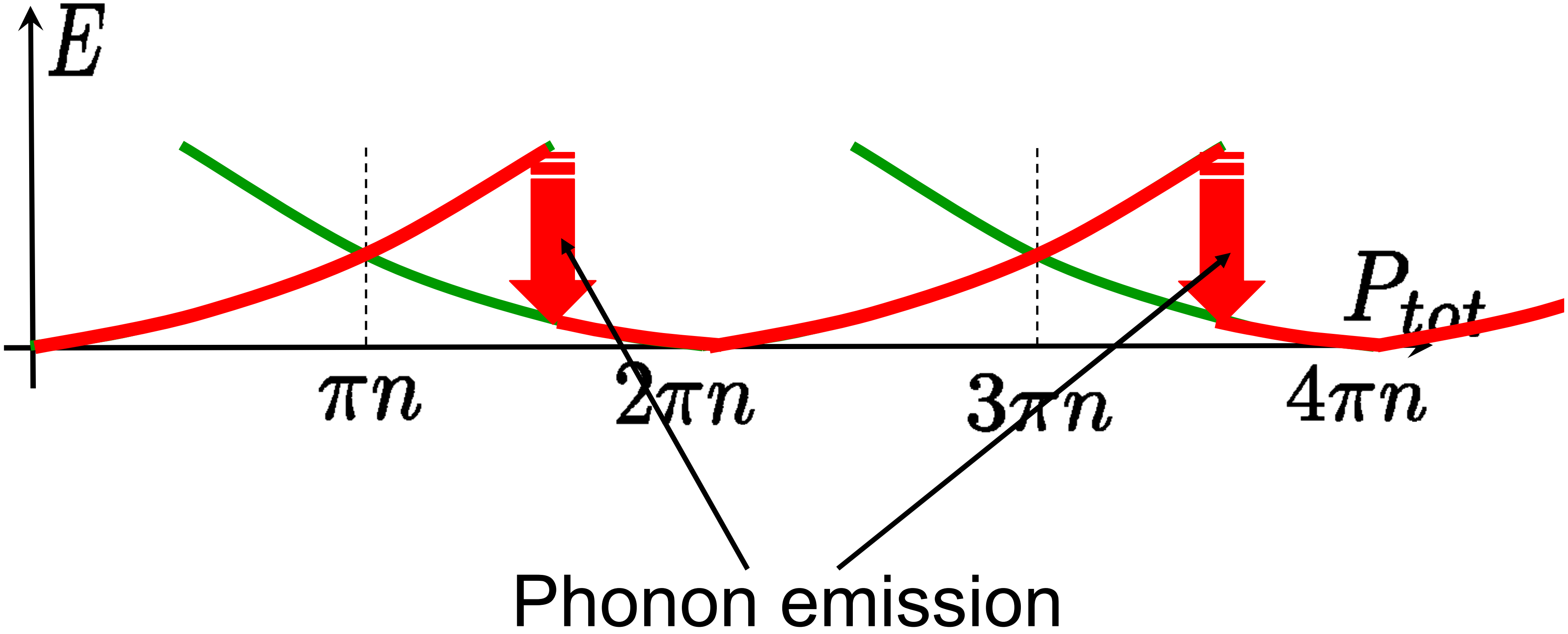
Heavy Impurity



Lamacraft 2009



Dissipative Bloch oscillations



$V_D \to \text{const}$ when $F \to 0$

Acknowledgments

Dima Gangardt, Birmingham

Michael Schecter, Minnesota

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Thank You