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Some mathematics for nonlinear dynamics of complex systems

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Abstract

These lectures give an introduction to some mathematics that is useful for various problems of nonlinear dynamics for complex systems. In particular, applications are given to networks of bistable units, discrete breathers and synchronisation. The main mathematical tools used are the implicit function theorem, localisation bounds for Green functions, and normal hyperbolicity theory.

1 Implicit Function Theorem

The implicit function theorem gives sufficient conditions under which a solution of a system of equations persists as parameters are varied.

Examples of applications to nonlinear dynamics to be described include:

- an equilibrium of a vector field $v(x) = 0$
- a fixed point of a map $f(x) = x$
- a period-T orbit of a period-T vector field $\dot{x} = v(x,t) = v(x,t+T), x(t+T) = x(t)$
- a periodic orbit of an autonomous vector field $\dot{x} = v(x), x(t+T) = x(t)$, period T
- an equilibrium of a network of bistable units, e.g. $v_s(x_s) + y\sum_{r\in S} C_{sr}x_r = 0, s \in S$
- a discrete breather: spatially localised time-periodic solution of a Hamiltonian system of coupled oscillators
- response of an equilibrium to aperiodic forcing
- synchronisation of an oscillator to aperiodic forcing

Consider the general problem of finding solutions (x, y) of $F(x, y) = z_0$ for a function $F : X \times Y \to Z$, X, Y, Z manifolds, possibly infinite-dimensional, and some point $z_0 \in Z$, given a solution (x_0, y_0). We will look for the solutions near (x_0, y_0) so it is enough to consider the case of X, Y, Z Banach spaces.

A *Banach space* is a complete normed vector space. Recall that a *vector space* is a set V with an operation of addition making it into a commutative group, and an operation of multiplication by scalars (which we will take in the real numbers $\mathbb{R}$), such that for all $u, v \in V, \lambda, \mu \in \mathbb{R}$, then $\lambda(\mu v) = (\lambda\mu)v$, $\lambda(u + v) = \lambda u + \lambda v$, and $(\lambda + \mu)v = \lambda v + \mu v$. A *norm* on a vector space V is a function $\|.\| : V \to \mathbb{R}_+$ such that for all $u, v \in V, \lambda \in \mathbb{R}$ then $\|\lambda v\| = |\lambda|\|v\|$, $\|u + v\| \leq \|u\| + \|v\|$, and $\|v\| = 0$ implies $v = 0$. A normed vector space is *complete* if every Cauchy sequence converges. Recall that a sequence $(v_n)_{n\in\mathbb{N}}$, $v_n \in V$ is *Cauchy* if $\forall\varepsilon > 0\ \exists N \in \mathbb{N}$ such that $m, n \geq N$ imply $\|v_m - v_n\| \leq \varepsilon$; a sequence $(v_n)_{n\in\mathbb{N}}$ *converges* if $\exists v_\infty \in V$ such that $\forall\varepsilon > 0\ \exists N \in \mathbb{N}$ such that $n \geq N$ implies $\|v_n - v_\infty\| \leq \varepsilon$.

Examples of Banach space include $\mathbb{R}^n = \{x = (x_1, \ldots x_n) : x_m \in \mathbb{R}, m = 1, \ldots n\}$ with Euclidean norm $\|x\| = \sqrt{\sum_{m=1}^n x_m^2}$, $\mathbb{R}^n$ with maximum norm $\|x\| = \max_{1\leq m\leq n} |x_m|$, $\mathbb{R}^n$ with sum-norm $\|x\| = \sum_{m=1}^n |x_m|$. More generally, given a countable set S, which might label sites in a network, and for each $s \in S$ a Banach space V_s with norm $|.|_s$, we can define $\ell^\infty((V_s)_{s\in S}) = \{v = (v_s)_{s\in S} : v_s \in V_s, \sup_{s\in S} |v_s|_s < \infty\}$ and with norm $\|v\| = \sup_{s\in S} |v_s|_s$ it is a Banach space. Given two Banach spaces X, Y, the product $X \times Y$ is defined to be the set of pairs $(x, y), x \in X, y \in Y$, with the norm $\|(x, y)\| = \max(\|x\|_X, \|y\|_Y)$. Another example is $C^0(\mathbb{R}, V)$, the set of bounded continuous functions v from $\mathbb{R}$ to a Banach space $V, |.|$, when endowed with norm $\|v\|_{C^0} = \sup_{t\in\mathbb{R}} |v(t)|$. A last important example is $C^1(\mathbb{R}, V)$, the set of differentiable functions v from $\mathbb{R}$ to V with bounded continuous derivative v', endowed with norm $\|v\|_{C^1} = \max(\|v\|_{C^0}, \tau\|v'\|_{C^0})$, where $\tau > 0$ is a timescale which should be chosen relevantly.

Let us state what it means for a map $F : X \to Z$, from one Banach space to another, to be continuous or differentiable. F is *continuous* at $x \in X$ if $\forall\varepsilon > 0\ \exists\delta > 0$ such that $\|\xi\|_X \leq \delta$ implies $\|F(x+\xi)-F(x)\|_Z \leq \varepsilon$. F is *differentiable* at $x \in X$ if there is a bounded linear map $DF_x : X \to Z$ such that

$$\|F(x + \xi) - F(x) - DF_x\xi\|/\|\xi\| \to 0 \text{ as } \xi \to 0.$$

Recall that a map $A : X \to Z$ is *linear* if $\forall\xi, \eta \in X, \lambda, \mu \in \mathbb{R}$ then $A(\lambda\xi+\mu\eta) = \lambda A\xi+\mu A\eta$. It is *bounded* if $\exists K$ such that $\forall\xi \in X$ then $\|A\xi\|_Z \leq K\|\xi\|_X$. The infimum of such K is called the *operator norm* $\|A\|_{X\to Z}$ of A and makes the space $L(X, Z)$ of bounded linear maps from X to Z into a Banach space. Finally, F is C^1 (continuously differentiable) if it is differentiable and the derivative DF is continuous as a map from X to $L(X, Z)$.

Now we are almost ready to state the implicit function theorem. By shifting the origins, we take $x_0, y_0, z_0 = 0$. Note that for a differentiable map $F : X \times Y \to Z$, the derivative can be written as a pair of partial derivatives $DF = (D_X F, D_Y F)$. For $\varepsilon > 0$ we define the *ball* $B_\varepsilon(X)$ in a Banach space X to be $\{x \in X : \|x\| \leq \varepsilon\}$.

Theorem (IFT): If $F : X \times Y \to Z$ is C^1, $F(0, 0) = 0$ and $D_X F_{0,0}$ is invertible then $\exists\varepsilon, \delta > 0$ such that $\forall y \in B_\varepsilon(Y)$ there exists a unique $\bar{x}(y) \in B_\delta(X)$ such that $F(\bar{x}(y), y) = 0$.

The function $\bar{x} : B_\varepsilon(Y) \to B_\delta(X)$ is called the *implicit function*. Various remarks are useful before we go on to applications.

Firstly, given explicit information about the norm of $D_X F^{-1}$ (note that the inverse of an invertible bounded linear map between Banach spaces is always bounded) and how continuous is DF, one can obtain explicit numbers ε, δ and a function $\bar{\delta} : [0, \varepsilon] \to \mathbb{R}$ such that $\|\bar{x}(y)\| \leq \bar{\delta}(\|y\|)$. The case of Lipschitz-continuous DF is treated in Appendix A.

Secondly, the implicit function $\bar{x}$ is C^1, with

$$D_Y \bar{x}(y) = -(D_X F)^{-1} D_Y F, \tag{1}$$

evaluated at $(\bar{x}(y), y)$. Note that $D_X F$ remains invertible for $y \in B_\varepsilon(Y)$. One can use the following geometric series for the inverse; denoting $A = D_X F(x, y)$, $A_0 = D_X F(0, 0)$, and $\Delta = A_0 - A$, for $\|\Delta\| < \|A_0^{-1}\|^{-1}$ then

$$A^{-1} = A_0^{-1} + A_0^{-1} \Delta A_0^{-1} + A_0^{-1} \Delta A_0^{-1} \Delta A_0^{-1} + \dots$$

This also shows that $\|A^{-1}\| \leq 1/(\|A_0^{-1}\|^{-1} - \|\Delta\|)$.

Now we treat the examples listed in the introduction.

Let $v : X \times Y \to X$ be a vector field on a Banach space X depending on parameters in a Banach space Y. It is simplest to think of the finite-dimensional case. The equation for an equilibrium x (stationary point) of v at parameter point y is $v(x, y) = 0$. If x_0 is an equilibrium for y_0, v is C^1 and the derivative $D_X v$ is invertible at (x_0, y_0) then the IFT can be applied to deduce that for all y near enough to y_0 there is a locally unique equilibrium $\bar{x}(y)$ and it varies C^1 with y. For finite-dimensional X the invertibility condition is equivalent to 0 not being an eigenvalue of $D_X v$. An equilibrium at which $D_X v$ is invertible is called *non-degenerate*.

Let $f : X \times Y \to X$ be a C^1 map on X depending on parameters in Y. The equation for a fixed point x is $f(x, y) = x$. Define $F(x, y) = x - f(x, y)$. Then fixed points of f correspond to zeroes of F. If x_0 is a fixed point for y_0 and $I - D_X f$ is invertible there then the IFT can be applied to deduce that for all near enough y there is a locally unique fixed point $\bar{x}(y)$ and it varies C^1 with y. For finite-dimensional X the invertibility condition is equivalent to $+1$ not being an eigenvalue of $D_X f$. A fixed point at which $I - D_X f$ is invertible is called *non-degenerate*.

Let $v : X \times \mathbb{R}/T\mathbb{Z} \times Y \to X$ be a C^1 time-periodic vector field on X of period $T > 0$ depending on parameters in Y (there is a slight loss of generality here in taking X to be a Banach space, but for simplicity the case of a general manifold will not be explored). By scaling time, let us take $T = 1$. Suppose $x : \mathbb{R} \to X$ is a solution of period 1 for y_0, i.e. $\dot{x} = v(x(t), t, y_0), x(t + 1) = x(t)$. Define $F : X^1 \times Y \to X^0$ with $X^1 = C^1(\mathbb{R}/\mathbb{Z}, X)$, $X^0 = C^0(\mathbb{R}/\mathbb{Z}, X)$ by $F^y[x](t) = \dot{x}(t) - v(x(t), t, y)$ (square brackets are used to indicate that the argument is a function, not just a point of X, and the parameter dependence is indicated here by a superscript). Then period-1 orbits of v correspond to zeroes of F. So the IFT shows that a period-1 orbit at which $D_{X^1} F$ is invertible persists to a locally unique period-1 orbit for all nearby parameters. It can be checked that $D_{X^1} F$ is invertible iff $x(0)$ is a non-degenerate fixed point of the *Poincaré map* $f : X \times Y \to X$ defined by integrating the differential equation $\dot{x} = v(x,t,y)$ from $t = 0$ to $t = 1$. One may ask why we did not solve this problem by first defining the Poincaré map and then applying the result of the previous paragraph. That is a feasible approach, but requires proving that the Poincaré map is defined and C^1. Also if X is a product over sites in a network S and one wishes to deduce localisation results for the resulting period-1 orbits then one would need to first deduce localisation results for the Poincaré map, which again is feasible, but adds an additional step. Finally, this treatment extends easily to problems of aperiodic forcing where an approach via Poincaré maps would require constructing a sequence of maps from times t_n to t_{n+1} for some sequence $(t_n)_{n\in\mathbb{Z}}$ going from $-\infty$ to $+\infty$.

Let S be a countable set, representing sites in a network, and for each site suppose we have a vector field v_s on a finite-dimensional Banach space $X_s, |.|_s$ such that each v_s has two equilibria x_s^-, x_s^+. For every configuration of $\{-,+\}$ on S there is an attracting equilibrium. Then couple the units together, for example via coupling of the form

$$\dot{x}_s = v_s(x_s) + y \sum_{r\in S} C_{sr} x_r$$

for some matrix C and parameter y. Define $\mathcal{X} = \ell^\infty((X_s)_{s\in S})$ and $F : \mathcal{X}\times\mathbb{R} \to \mathcal{X}$ by setting $F((x_s)_{s\in S}, y)$ to the right hand side of the equations for $\dot{x}_s$. For this to really map into $\mathcal{X}$ we have to require $\sup_{s\in S} |v_s(x_s)|_s < \infty$ for $x = (x_s)_{s\in S} \in \mathcal{X}$, and $\sup_{s\in S} \sum_{r\in S} |C_{sr}| < \infty$ (the latter is the ℓ^∞-operator norm of C). Actually, we are interested inly in solutions in some bounded region of $\mathcal{X}$ so it would be enough to require the bound on v in such a region. To be sure that F is C^1 we furthermore require that $\sup_{s\in S} |Dv_s(x_s)| < \infty$ for $x \in \mathcal{X}$ and the derivatives to have a common module of continuity. $D_{\mathcal{X}}F$ is invertible at a configuration x if $\exists\delta > 0$ such that none of the Dv_s have eigenvalues in $|\lambda| < \delta$. Suppose there is a δ for which this holds for both equilibria of every uncoupled unit. Then there is $\varepsilon > 0$ such that the equilibria for all configurations of $\{-,+\}$ persist for $|y| \leq \varepsilon$. If the uncoupled equilibria are all attracting then one can prove that there is a possibly smaller $\varepsilon' > 0$ for which all the continuations of the equilibria remain attracting (a similar remark applies to all the other problems) [MS95].

Let $v : X\times Y \to X$ be an autonomous C^1 vector field (i.e. no explicit time-dependence) and $x : \mathbb{R} \to X$ be a periodic orbit of $\dot{x} = v(x,y)$ for some parameter value y_0, say 0, i.e. $x(t + T_0) = x(t)$ for some $T_0 > 0$ but x not constant (which would be an equilibrium). There are two obstacles to using the IFT to give conditions under which it persists to nearby y. The first is that if x is a period-T orbit of an autonomous system then so is x^τ defined by $x^\tau(t) = x(t+\tau)$ for any $\tau \in \mathbb{R}$, so the orbit is not locally unique (take τ small), contradicting the IFT which would give a locally unique solution (including for $y = 0$). The second obstacle is that we do not expect the period T to remain constant as parameters y are varied. A strategy to solve the first is to impose that some C^1 function $\phi : X \to \mathbb{R}$ is zero at $x(0)$; to work, this requires also that $D\phi\ v \neq 0$ at $x(0)$. A strategy to solve the second is to search for the period as well as the orbit. One way to do this is to formulate the equations in a time scaled to a putative period T, i.e. write $t = Ts$ and then the equations are $x'(s) = Tv(x(s), y)$, $x(1) = x(0)$, $\phi(x(0)) = 0$. So for $r = 0, 1$ let $X^r = C^r(\mathbb{R}/\mathbb{Z}, X)$ and define $F : X^1 \times \mathbb{R} \times Y \to X^0 \times \mathbb{R}$ by $F(x, T, y)(s) = (x'(s) - Tv(x(s), y), \phi(x(0)))$. Zeroes of F correspond to periodic orbits, given by scaling s by the obtained number T. The invertibility condition in the IFT is that the derivative of F with respect to the pair (x, T) is invertible. It turns out to be equivalent to $x(0)$ being a non-degenerate fixed point of the Poincaré map from $\phi = 0$ to $\phi = 0$ (taking time in a neighbourhood of T_0). Again, using the Poincaré map might be considered a simpler approach, but requires using the IFT to prove it is well defined and C^1, and the above approach has the same advantages as before.

Let $v : X \times Y \to X$ be an autonomous C^1 vector field with parameters in Y and conserving a C^1 function $H : X \times Y \to \mathbb{R}$ that we will call energy, for example an autonomous Hamiltonian system, and let $x : \mathbb{R} \to X$ be a periodic orbit for $y = 0$ with period T_0. The above strategy for continuation of the periodic orbit fails in general because $D_X H\xi$ is constant along every tangent orbit ξ (i.e. solution of the linearised equations about the orbit x) and thus $D_X H$ is an eigenform (left eigenvector) of the linearised Poincaré map with eigenvalue $+1$. The reason is that in general the periodic orbit belongs to a smooth one-parameter family of periodic orbits for the same value of y; one can often parametrise the family by the period T or energy or other quantities (in the Hamiltonian context it can be natural to use the action integral). This observation suggests its own solution. One can continue at fixed period T by considering $F : X^1 \times Y \to X^0 \times \mathbb{R}$ (with time scaled to the chosen period, or redefining $X^r = C^r(\mathbb{R}/T\mathbb{Z}, X)$) defined by $F(x, y)(t) = (\dot{x}(t) - v(x(t), t, y), \phi(x(0)))$, provided the eigenspace of the time-T map at the fixed point $x(0)$ with eigenvalue $+1$ is spanned by $v(x(0), 0)$ (the eigenvalue has multiplicity at least two, and this condition corresponds to non-isochronicity). One can continue at fixed energy E by considering $F : X^1 \times \mathbb{R} \times Y \to X^0 \times \mathbb{R} \times \mathbb{R}$ defined by $F(x, T, y)(s) = (x'(s) - Tv(x(s), y), \phi(x(0)), H(x(0)) - E)$, provided $x(0)$ is a non-degenerate fixed point of the Poincaré map on $\phi = 0$. These strategies can be used to prove persistence of time-periodic solutions for Hamiltonian systems of coupled oscillators under suitable conditions, including discrete breathers (spatially localised solutions) though the spatial localisation requires also the techniques of Section 2 [SM97].

Another class of autonomous vector field that can be relevant is the *reversible* ones. These are C^1 vector fields $v : X \to X$ such that there is a C^1 map $S : X \to X$ such that $S^2 = I$ and $\forall x \in X$, $v(Sx) = -DS_x v(x)$. Such an S is called a *reversing symmetry* for v. A periodic orbit $x : \mathbb{R} \to X$ is called *symmetric* if there is an origin of time $\tau \in \mathbb{R}$ such that $\forall t \in \mathbb{R}$, $x(\tau - t) = S(x(\tau + t))$. Symmetric periodic orbits are quite common for reversible systems: if $x(0)$ and $x(u) \in \mathrm{Fix}(S)$ (the set of fixed points for S) for some $u > 0$ then $x(-u) = x(u)$ so x is periodic with period $T = 2u$ and it is symmetric (with $\tau = 0$). It is common that $\mathrm{Fix}(S)$ has dimension half that of X and so there is in general a one-dimensional subset of $\mathrm{Fix}(S)$ whose orbits hit $\mathrm{Fix}(S)$ sometime in the future, and each point of this subset give a symmetric periodic orbit. If we allow v and S to depend on parameters $y \in Y$ then we can continue symmetric periodic orbits of given period T by applying the IFT to $F : X^1 \times Y \to X^0 \times X \times X$ defined by $F(x, y)(t) = (\dot{x}(t) - v(x(t), y), S(x(0), y) - x(0), S(x(T/2), y) - x(T/2))$. One can add T to the parameter space if desired to capture the continuation with respect to period simultaneously. This approach fails in the isochronous case, however: then it is better to use the vector field in a scaled time $s = t/T$, choose a function $\phi : X \to \mathbb{R}$ which has non-zero derivative along the anticipated curve on $\mathrm{Fix}(S)$ and is zero at the initial $x(0$, and solve for the pair (x, T) that satisfies the scaled equations of motion, the conditions at $s = 0, \frac{1}{2}$, and $\phi(x(0)) = \Phi$ (adding the number Φ to the parameter space).

Let us now consider the response of an equilibrium to aperiodic forcing, e.g. $\dot{x} = v(x) + yf(x, t)$ on X, with v, f C^1. For $r = 0, 1$ now let $X^r = C^r(\mathbb{R}, X)$ and define $F : X^1 \times \mathbb{R} \to X^0$ by $F(x, y)(t) = \dot{x}(t) - v(x(t)) - yf(x(t), t)$. For $y = 0$, if x_0 is an equilibrium of v then the constant function $x(t) = x_0$ is a zero of F. $D_{X^1}F$ is invertible there iff x_0 has no spectrum on the imaginary axis (i.e. $i\omega I - Dv$ invertible for all $\omega \in \mathbb{R}$). So there is a locally unique response $x : \mathbb{R} \to X$ for all small enough y [BM03].

Finally, consider the robustness of synchronisation of a limit cycle oscillator to forcing. It is standard that a limit cycle oscillator may synchronise to periodic forcing if the frequency difference is smaller than the forcing strength in a suitable dimensionless sense (though to treat this in general requires the normal hyperbolicity theory of Section 3). By this I mean that the oscillator performs a periodic motion of the same period as the forcing, with a definite phase relation. Let us write the system forced at period T as $\dot{x} = v(x, t)$ for a C^1 vector field on X with period T in t, and the synchronised solution as $x : \mathbb{R} \to X$. Then we ask what happens if v is modified to an aperiodic function $\tilde{v}(x, t, y)$ of t with parameters y. If $\tilde{v}$ is uniformly close to v in C^1 then the IFT gives a locally unique continuation under the condition that the initial periodic orbit has no spectrum on the unit circle. A much more general result is possible, however: we may consider forcing functions that are not uniformly close to v but such that there is a new time variable s with $dt/ds = \omega(s, y)$ near 1 so that $dx/ds = \omega(s, y)\tilde{v}(x, t(s), y)$ is uniformly close to $v(x, s)$. Think of s as a variable measuring the phase of the forcing. Then the IFT applied to this new equation gives persistence of the synchronised response for all small enough y under the same condition, and this can be converted back into the original time t at the end. This is the principle of phase-locked loops.

It is worth adding that the approach of applying the IFT to a map from X^1 to X^0 also extends to differential-delay equations, either with discrete delays or more general delay kernels.

2 Localisation Bounds for Green functions

In many circumstances, we wish to show that the continuation of some solution is exponentially localised in some sense. For example, we wish to show that discrete breathers are exponentially localised in space or that the response of an equilibrium to temporally localised forcing is localised in time.

The strategy is to prove localisation bounds for the linearised problem and then use (1) to extend them to the nonlinear problem.

For a linear problem, the basic player is the *Green function*. This is the response to a point source. We consider two contexts. The first is a network S of units with a metric d specifying a notion of distance between them. The second is differential equations in continuous time.

2.1 Networks

Let (S, d) we a countable metric space, and for each $s \in S$ let X_s, Z_s be Banach spaces, with norms denoted by $|.|$, for the state of unit s and the deviation at s from being a solution. Let $X = \ell^\infty((X_s)_{s\in S})$ and Z similarly. Let $L : X \to Z$ by a bounded linear operator of matrix type, meaning that for $x = (x_s)_{s\in S}$, $(Lx)_r = \sum_{s\in S} L_{rs}x_s$ for some bounded linear maps $L_{rs} : X_r \to Z_s$. L is bounded iff $\sup_{r\in S} \sum_{s\in S} |L_{rs}| < \infty$.

The Green function for L of the above form is the matrix elements of L^{-1}. Specifically, if $z \in Z$ has $z_u = 0$ for all $u \in S \setminus \{s\}$ then the bounded solution of $Lx = z$ can be written as $x_r = G_{rs}z_s$ for some bounded linear maps $G_{rs} : Z_s \to X_r$. By linearity, it gives the matrix expression for L^{-1}: $(L^{-1}z)_r = \sum_{s\in S} G_{rs}z_s$.

Suppose that L is a local operator in the sense that $|L_{rs}|$ decays as $d(r, s)$ grows. Specifically, say L is *exponentially local* if $\exists W > 1$ such that for $1 \le w < W$ then $\tilde{L}_{rs} = L_{rs}w^{d(r,s)}$ still defines a bounded operator. A special case is coupling of bounded range ($\exists D > 0$ such that $L_{rs} = 0$ for $d(r, s) > D$).

Say that $z \in Z$ is *exponentially localised* about some site $o \in S$ if there is $C > 0, \mu < 1$ such that $\forall s \in S$ then $|z_s| \le C\mu^{d(s,o)}$. Then one can prove that $x = L^{-1}z$ is also exponentially localised about $o \in S$, but in general with different C and μ, which depend on $\|L^{-1}\|, \|\tilde{L}\|, C$ and μ [BM97].

In particular, for L exponentially local, the Green function is exponentially local in the sense that there is $D > 0, \nu < 1$ such that $|G_{rs}| \le D\nu^{d(r,s)}$.

Let us treat some examples.

For the network of bistable units let us suppose the operator C is exponentially localised, and two equilibria which differ only by the sign $\{-, +\}$ at one unit $o \in S$. Then by integrating (1) from $y = 0$ and using the above exponential localisation result we obtain that the difference between the two equilibria decays exponentially with distance from o [MS95]. The same holds if $\{o\}$ is generalised to any bounded subset of S.

Similarly, for a Hamiltonian system of coupled oscillators, if the coupling is exponentially localised then the time-periodic solutions obtained in section 1 by starting with one excited site in the uncoupled case have amplitude decaying exponentially from that site [SM97].

As a final example, consider C^1 discrete-time dynamics $x_{t+1} = f(x_t)$, $t \in \mathbb{Z}$, on a Banach space V (we can also allow the map f to depend on t). The linearised dynamics about an orbit is $\xi_{t+1} = {f'}_t \xi_t$, where ${f'}_t = f'(x_t)$, which we suppose to be bounded. Let $X = \ell^\infty(V, \mathbb{Z})$ meaning the bounded doubly infinite sequences in V, with sup-norm. Then define $L : X \to X$ by $(L\xi)_t = \xi_t - {f'}_{t-1}\xi_{t-1}$. $L\xi = 0$ iff ξ is a bounded tangent orbit. This example is a special case of the network setting, in which $S = \mathbb{Z}$ and the coupling is only with the immediate left neighbour. If L is invertible then the unique bounded solution ξ of $L\xi = \phi$ for a bounded forcing sequence $\phi \in X$ can be written as $\phi_t = \sum_{s\in\mathbb{Z}} G_{ts}\phi_s$, and the above theory shows that $|G_{ts}| \le C\mu^|t-s|$ for some $C > 0$, $\mu < 1$. This exponential decay property is called *uniform hyperbolicity* of the trajectory x.

2.2 Differential equations in continuous time

For a system of linear ordinary differential equations in continuous time $\dot{x}(t) = A(t)x(t)$ on a Banach space X, define the linear map $L : X^1 \to X^0$ as before by $L[x](t) = \dot{x}(t) - A(t)x(t)$. If L is invertible then the bounded solution of $Lx = z$ for bounded forcing function $z \in X^0$ can be written as $x(t) = \int_{-\infty}^{+\infty} G(t,s)y(s)\ ds$, where the operator-valued function $G(t,s) : X \to X$ is called the *Green function*. It can be thought of as the response to the operator-valued delta-function $I\delta(t-s)$.

Again, it can be proved that the Green function decays exponentially $|G(t,s)| \le Ce^{-\mu|t-s|}$ for some $C > 0, \mu > 0$ (see for example, the lecture notes [M11]).

The basic case of such a linear system is the linearisation of a possibly time-dependent vector field $\dot{x} = v(x,t)$ about a trajectory x, in which case $A(t) = v_x(x(t),t)$ and the trajectory is called *non-autonomous uniformly hyperbolic* if L is invertible. Note that for an autonomous vector field, the only trajectories that can possibly satisfy this are the equilibria, because time-shift of a trajectory produces a vector in the kernel of L, so a modified notion of uniform hyperbolicity has to be formulated for autonomous vector fields but we shall not go into that here.

A consequence of the exponential decay of the Green function is a splitting of the tangent space at each time into the direct sum of backward and forward exponentially contracting subspaces.

For many applications, the backward contracting subspace is trivial because the trajectory is attracting. In this case the Green function is *causal*: $G(t,s) = 0$ for $t < s$.

3 Normal Hyperbolicity Theory

An introduction to this is given in [M11].

To be continued

Appendix A: Explicit IFT estimates for Lipschitz derivative

Let $F : X \times Y \to Z$ be a C^1 map between Banach spaces, with $D_X F$, $D_Y F$ Lipschitz-continuous. Explicitly, suppose $\|D_X F(x,y) - D_X F(x',y')\| \le a\|x-x'\| + b\|y-y'\|$, $\|D_Y F(x,y) - D_Y F(x',y')\| \le c\|x-x'\| + d\|y-y'\|$. Actually, I will use only the case (x',y') equal to the solution of the unperturbed problem, which we can take to be $(0,0)$. To simplify the analysis, take $b=c$ (one can just substitute $\max(b,c)$); this is quite natural because if F is in fact C^2 then the best choice for both b and c is $\sup \|D^2_{XY}\|$ over some neighbourhood of $(0,0)$.

To be completed.

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