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Workshop on New Materials for Renewable Energy

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Notes on localizations

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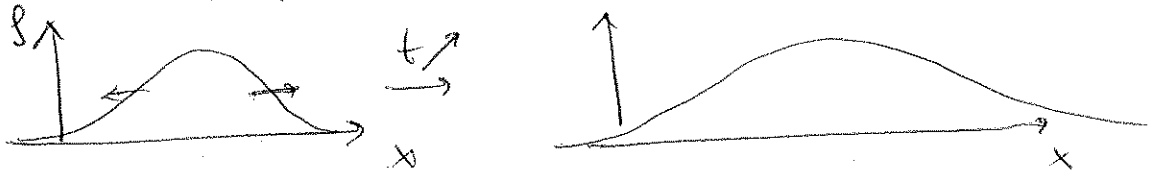
Lecture Notes on Localization

Diffusion

- Every day experience: density inhomogeneities even out over time



- Energy diffuses and distributes over all degrees of freedom:
  $\Rightarrow$ thermalization! / equilibration

$\Leftrightarrow$ 2nd law of thermodynamics: entropy increases and tends to its possible maximum! $\Rightarrow$ arrow of time! from diffusion and other entropy-increasing processes

$\Rightarrow$ In many systems, phase space is ergodically explored.

Basic phenomenon: Random walk:

particle takes random steps each unit of time; no memory, constraints
$\rightarrow$ diffusion (Einstein)

Einstein relation diffusion $\Leftrightarrow$ conductivity / transport: $\boxed{\sigma = e^2 D \frac{d\rho}{d\mu}}$ Density of states

BUT: Anderson: (1958) quantum random walks are different!

Energy conservation (no collisions) $\Rightarrow$ memory $\Rightarrow$ Can diffusion stop? $D = 0$?

$\Rightarrow$ Yes! Localization of particles due to quantum interference!

$\hookrightarrow$ • no diffusion
• no thermalization
• breakdown of ergodicity (equal energy surface is not explored uniformly, even for $t \to \infty$)
• no arrow of time at large times!

Main ingredient: (strong) disorder potential

Systems of relevance:

- spins (spin diffusion)
- light, microwaves
- sound waves
- cold atoms
- electrons in disordered metals / semiconductors

} subject to sufficient randomness

$\Rightarrow$ localization phenomena of waves

$\Longrightarrow$
- Concepts
- Phenomenology
- Approaches analytic / numerical

Electrons: (non-interacting)

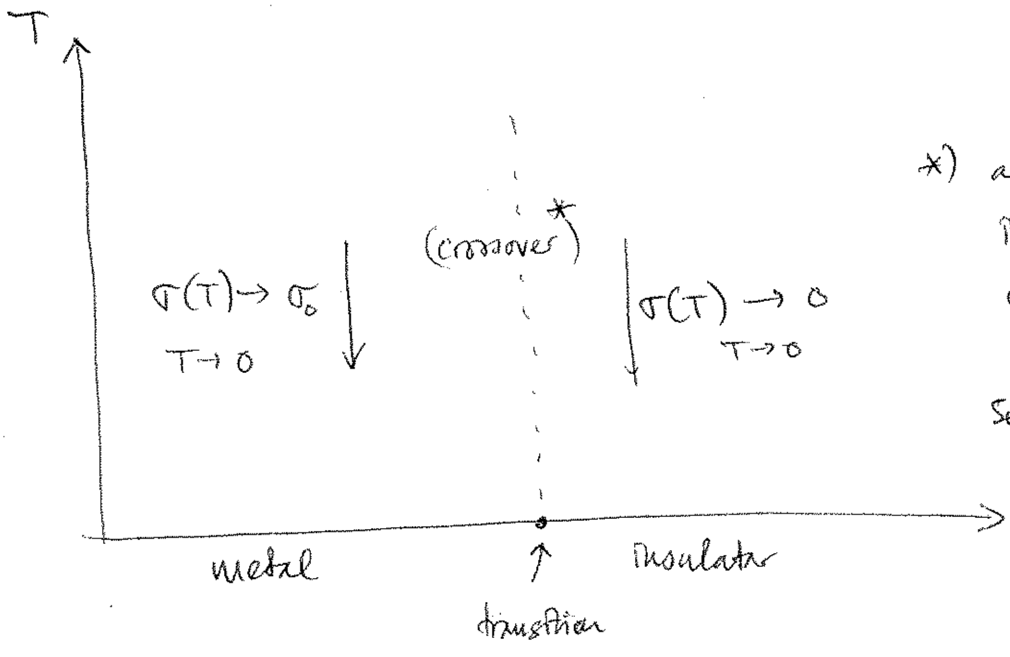
no diffusion $\qquad\qquad\qquad$ diffusion

$\sigma = D = 0$ $\qquad\qquad\qquad$ $\sigma > 0$, transport

$\Rightarrow$ insulator (Anderson) $\qquad \Uparrow \qquad$ $\Rightarrow$ metal

dynamical quantum phase transition!
(no thermodynamic signatures)

(second order continuous transition, critical phenomena etc.
but no simple order parameter)

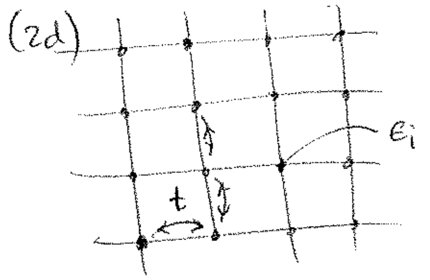


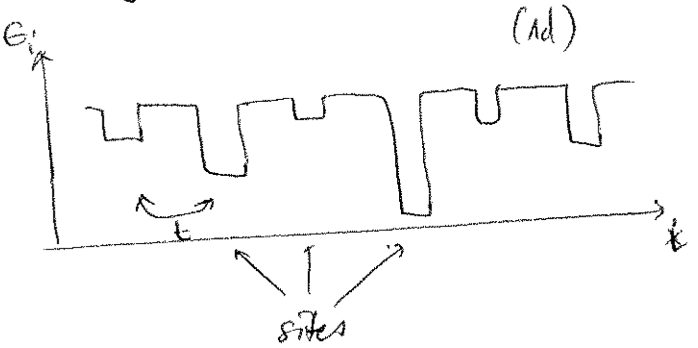
*) at finite T interactions with e.g. phonons destroy full localization ultimately.
See later for discussion of e-e interactions only.

Simple model (Anderson)

electrons hopping from site to site in a doped semiconductor:

$$H = \sum_i \epsilon_i c_i^\dagger c_i - t \sum_{\langle ij \rangle} (c_i^\dagger c_j + c_j^\dagger c_i)$$





ϵ_i: random energies (due to impurities, amorphicity etc)

ϵ_i independent, identically distributed

$P(\epsilon_i) = \frac{1}{W}$ for $\epsilon_i \in [-\frac{W}{2}, \frac{W}{2}]$ e.g.

t: nearest neighbor hopping (tight-binding) approximation

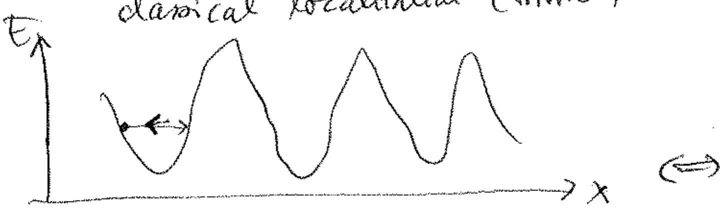
Only parameter in the problem: relative strength of disorder; $\boxed{\frac{W}{t}}$

General result: For $\frac{W}{t} > f(d) \leftarrow$ dimension-dependent critical threshold

$\Rightarrow$ particles are localized, no diffusion!

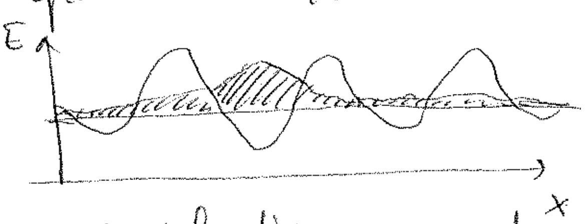
<u>Despite</u> quantum tunneling through classically forbidden regions!
Quantum localization is <u>more than</u> classical non-percolation!

classical localization (trivial)



$\Longleftrightarrow$

quantum localization



E = const surface is disconnected, non-percolating

wavefunction may spread across barriers, but ultimately decays exponentially!

Two approaches to understanding localization

→ strong disorder: locator expansion (expansion in hopping t)

→ understand stability of the (trivial) insulator ($t=0$)

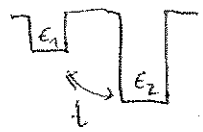
⇒ useful for generalizing to interacting systems. (similar to high dimensional single particle problem)

→ weak disorder: tuning up disorder, increase scattering, understand "proliferation of weak localization effects"

⇒ useful in understanding low dimensional localization, localization as critical phenomenon

Strong disorder approach: Basic understanding of localization

Instructive simple model: two wells



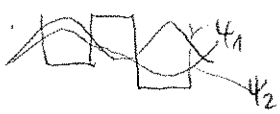
$$H = \epsilon_1 c_1^\dagger c_1 + c_2 c_2^\dagger c_2 - t\left(c_1^\dagger c_2 + c_2^\dagger c_1\right) \leftrightarrow H = \begin{pmatrix} \epsilon_1 & -t \\ -t & \epsilon_2 \end{pmatrix} \qquad \Delta\epsilon = \epsilon_2 - \epsilon_1$$

Two eigenfunctions: $\psi_1 = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ $\psi_2 = \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix}$

$$\frac{a_2}{a_1} = \frac{t}{\sqrt{t^2 + \Delta\epsilon^2}} \qquad E_{1,2} = \frac{\epsilon_1 + \epsilon_2}{2} \mp \sqrt{\Delta\epsilon^2 + t^2}$$

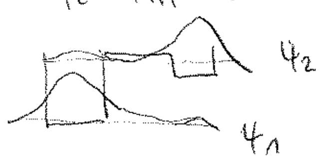
Two limits: "weak disorder": $|\Delta\epsilon| \ll t$ ⇒ delocalization over the wells (hybridization)

$|a_1| \simeq |a_2|$



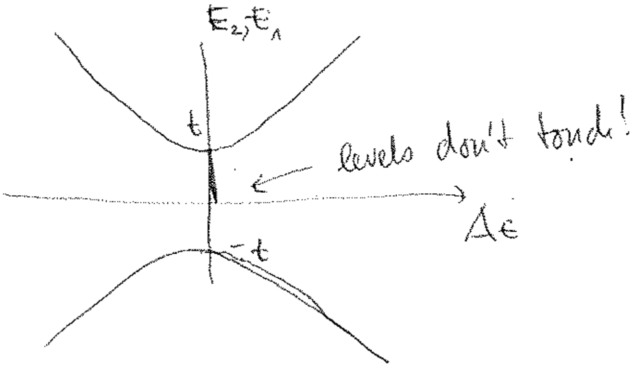
"strong disorder": $|\Delta\epsilon| \gg t$ ⇒ eigenfunctions localized on one of the wells, no hybridization

ψ_1: $|a_2| \ll 1$ ψ_2: $|a_1| \ll 1$



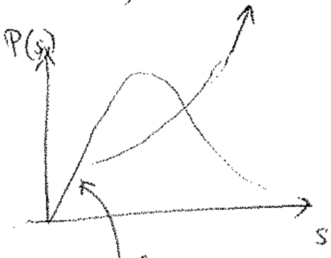
⇒ hybridization appreciable between wells if $\Delta\epsilon \lesssim t$

Level repulsion: $\epsilon_1 + \epsilon_2 = \text{const.}$



$s = E_2 - E_1$

$\text{Prob}(s) \sim s\, e^{-s^2/4}$ (for Gaussian distributed $t, \Delta\epsilon$)

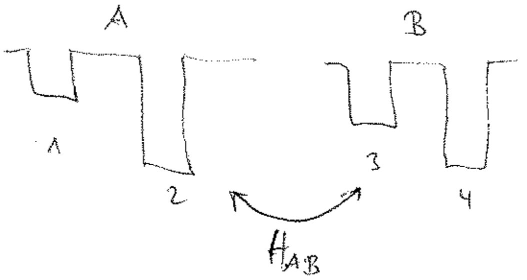


linear suppression of probability of small level spacing!

Need both $\Delta\epsilon$ and t small, since $s = 2\sqrt{(\Delta\epsilon)^2 + t^2}$

$$P(\Delta\epsilon \lesssim s) \cdot P(t \lesssim s) \sim s^2 \Rightarrow \underline{P(s)}\, ds \sim d(s^2) \sim \underline{s\,ds}$$

4 levels:



$$H = \left(\begin{array}{c|c} H_{AA} & H_{AB} \\ \hline H_{BA} & H_{BB} \end{array}\right)$$

If hopping H_{AB} comparable to t_{12}, t_{34} $\Rightarrow$ 4 levels with level repulsion
$\Rightarrow P(s) \sim s$ at small s for any adjacent levels
$\Rightarrow$ Wigner-Dyson statistics

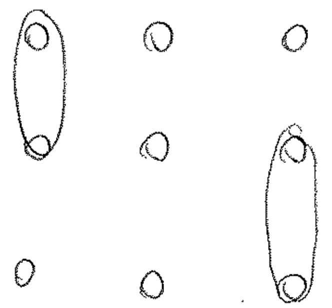
But: If $H_{AB} \approx 0 \Rightarrow$ independent two well systems.
Internally, energy levels repel, but the relative spacing between $E_{1,2}$ and $E_{3,4}$ is unconstrained:

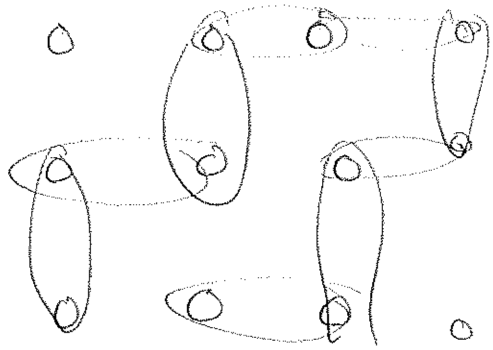
E_1 ——— E_3 ← unrestricted!
E_2 ——— E_4

$\Rightarrow P(s) \sim \text{const.}$ at small s

$\rightarrow$ Poisson statistics of levels

Many ~~levels~~ wells (Anderson model)





strong disorder | weak disorder

Consider neighbors hybridized when $|\Delta\epsilon| \lesssim t$

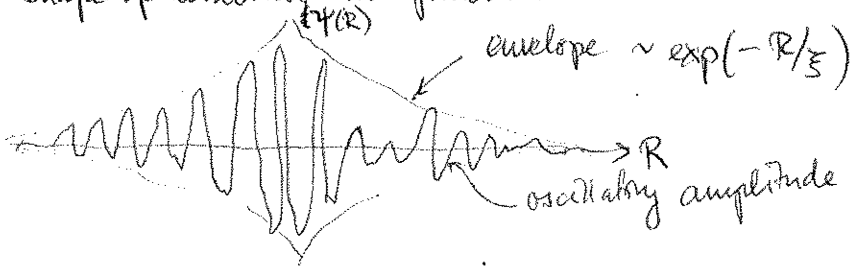
strong disorder	weak disorder
• Sparse hybridization bonds $\Rightarrow$ states / eigenfunctions are <u>localized</u>	hybridization bonds overlap & percolate $\Rightarrow$ <u>delocalization</u> of wavefunctions

More quantitative considerations give reasonable estimates for $\left(\frac{W}{t}\right)_c$ in 3d.
Numerically known value for 3d cubic lattice: $\left(\frac{W}{t}\right)_c = 16.8$.

strong disorder	weak disorder
• Level statistics: Eigenfunctions / eigenvalues far apart are independent $\Rightarrow$ Poissonian statistics $P(s) \underset{s\to 0}{\sim} \text{const}$	• Eigenfunctions occupy fraction of <u>all</u> volume $\Rightarrow$ overlap in space $\Rightarrow$ eigenvalues repel $\Rightarrow$ Wigner-Dyson statistics $P(s) \sim s$
Similar in <u>integrable systems</u>: many conserved quantities ensure that degrees of freedom do not mix / remain separated $\Rightarrow$ indep. degrees of freedom have uncorrelated energies.	(like in <u>random matrix ensembles</u>: disorder provides scattering between plane wave states $\Rightarrow$ large random matrices in k,k'-space)

<u>Phenomenology of localized phase</u>

Shape of localized wave function:



$\xi :=$ localization length

Local spectrum:

Green's function:

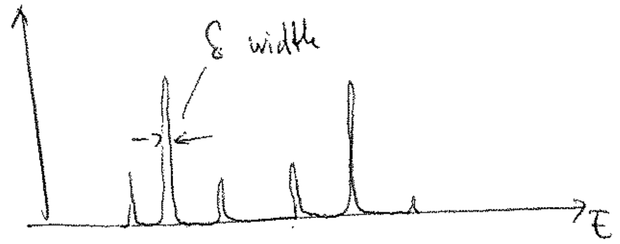
$$G_{R,R}(E+i\delta) := i\int_0^\infty \langle \psi(R,t)\, e^{i(E+i\delta)t}\, \psi^+(R,0)\rangle\, dt \equiv \langle R|\, \frac{1}{H-E-i\delta}\, |R\rangle$$

$$\underset{\uparrow}{=} \; i\sum_\alpha \frac{|\psi^\alpha(R)|^2}{E_\alpha - E - i\delta} \;;$$

decomposition in eigenbasis $\psi^\alpha(r)$

As $\delta\to 0$ $\Rightarrow$ see a bunch of δ-functions $\delta(E_\alpha - E)$ with weight $|\psi^\alpha(R)|^2$

Re $G_{RR}(E)$



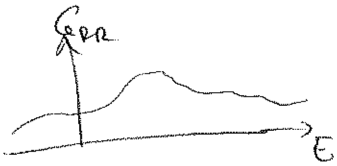
Note: $|\psi^\alpha(R)|^2$ is appreciable only for those eigenfunctions α which have a localization center within ξ from R!

$\Rightarrow$ # of visible peaks $\simeq \xi^d$!

$\Rightarrow$ discrete (pure point) spectrum

$\Leftrightarrow$ delocalized regime: infinitely many wavefunctions α contribute to $G_{RR}(E)$

$\rightarrow$ continuous spectrum; no delta-peaks
smooth limit $\delta \downarrow 0$.



Dynamics: Absence of diffusion and loss of arrow of time; non-ergodicity!

Initial condition: $\psi(t=0,R) = \delta_{R,0}$ $\Rightarrow$ $\psi(t,R) = ?$

Diffusion: at long times expect $|\psi(t,R)|^2 \approx \frac{1}{(Dt)^{d/2}}\, e^{-r^2/2Dt}$ $\Rightarrow \langle r^2\rangle \propto Dt$

$\Leftrightarrow$ Localized wavefunctions: expand in eigenfunctions:

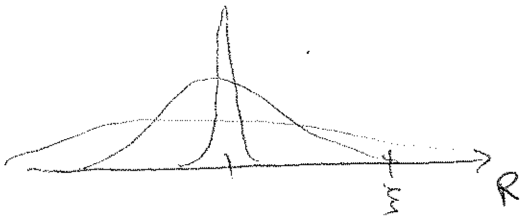
$$\psi(r,t) = \sum_\alpha \underbrace{\psi^\alpha(R=0)}_{c_\alpha} \cdot \psi^\alpha(r)\, e^{-iE_\alpha t}$$

$\sum_\alpha |c_\alpha|^2 = 1$ $\leftarrow$ only a finite number appreciable $|c_\alpha|^2 \sim \frac{1}{\xi^d}$

$\Rightarrow |\psi(t, r \gg \xi)| \lesssim e^{-2r/\xi}$ $\quad$ always, at all t, even when $t \to \infty$!

no tunneling escape! $\quad \Rightarrow$ $\underline{D=0}$.

• Short time dynamics:



Expansion of wavepacket up to $R \sim \xi$

in a time $\tau_{Th} \sim \frac{\hbar}{\delta_\xi}$ (Thouless time)

δ_ξ: mean spacing between eigenfunctions within the same localization volume, ξ^d

$$\delta_\xi = \frac{1}{\nu \cdot \xi^d}$$

ν: density of states

For $t \gg \tau_{Th}$: quasiperiodic motion, no further expansion

dynamics forward and backward in time indistinguishable

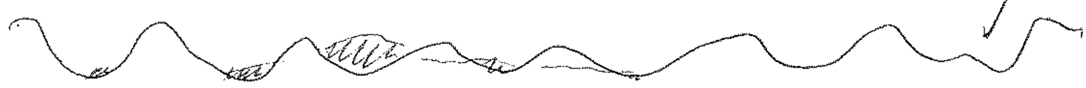
⟹ absence of diffusion and arrow of time!

2^{nd} law of thermodynamics not efficiently at work anymore

even though the system is not at equilibrium.

Entropy production stopped before S_{max} was reached!

• Ergodicity is broken:



phase space at large distance is unexplored even though it is not energetically forbidden.

Weak disorder approach

non-random system: $\rightarrow$ plane waves $\psi \sim e^{i\vec{k}\vec{r}}$

+ weak disorder $\Rightarrow$ scattering (elastic) between different states $\vec{k}, \vec{k}'$

small W $\longrightarrow$ mean free path between successive scatterings $l_{mf} \sim \left(\frac{t}{W}\right)^2 a$ (lattice spacing)

Fermi wavelength $\lambda_F \simeq \frac{1}{k_F} \sim \frac{1}{a}$ for a metal with 1 electron per site

$\Rightarrow \boxed{l_{mf} \gg \lambda_F}$ when $W \ll t$ $\Longrightarrow$ weakly scattered plane waves.

Ioffe-Regel limit: Disorder so strong that $k_F \cdot l_{mf} \sim \frac{l_{mf}}{\lambda_F} \lesssim O(1)$

$\rightarrow$ Breakdown of plane wave picture.

$\Rightarrow$ expect localization

measured by conductivity: $\sigma_{\text{Drude theory}} \simeq \underbrace{\frac{e^2}{\hbar} \cdot \lambda_F^{2-d}}_{\text{natural unit of conductivity}} \cdot \overbrace{(k_F l)}^{\text{measure of disorder}}$ and mean free path.

conductance: $G(L) = \sigma(L) \cdot \frac{L^{d-1}}{L}$ (Ohm's law)

$G(l) = \frac{e^2}{\hbar} (k_F l)^{d-1}$ $\quad g = G/e^2/\hbar$ dimensionless conductance

(counts number of channels contributing to transport)

Scale-dependence? $g(L)$? : A brief sketch of the scaling theory (1979)

Ohm's law: if $\sigma \simeq$ const. $\rightarrow g \sim L^{d-2} \rightarrow d>2$ grows more and more

$\rightarrow$ weak disorder is irrelevant

$d \leq 2$ $\quad g(2L) < g(L)$ due to quantum interference (enhanced quantum probability to go back to origin)

+ backscattering $\Longrightarrow$ always localized ($g \to 0$) at large scales!

One parameter scaling assumption: g is the only relevant parameter characterizing the conduction properties

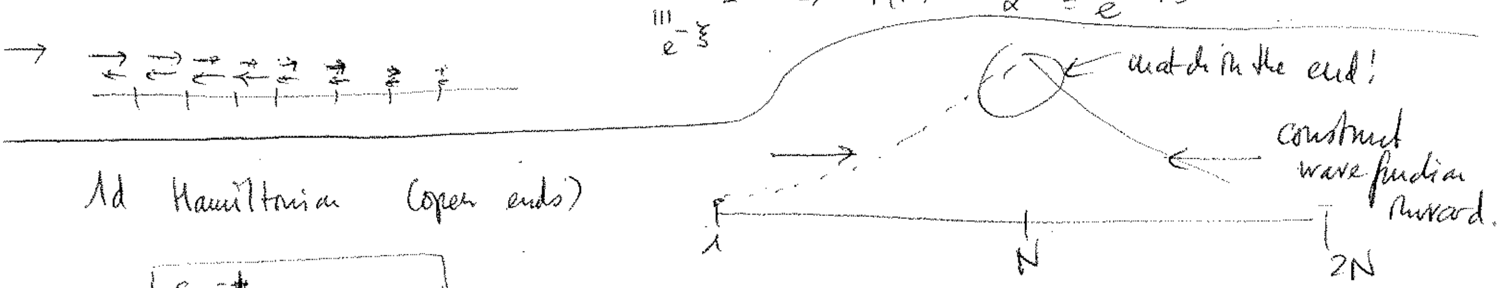
$\rightarrow$ expect that $\frac{\partial g}{\partial \ln L} = \beta(g)$ is function of g only $\quad \beta(g) < 0$ in $d=1,2$ $\forall g$!

Ohm: $d-2$ + q-correction

Localization in 1d (Mott + Twose)

1d systems always localize: transmission probability decreases due to piled up probability for backscattering:

$$T(N) = T(N-1)\cdot\alpha \qquad \underset{\underset{e^{-\xi}}{\parallel}}{\alpha} < 1 \;\to\; T(N) \sim \alpha^N = e^{-N/\xi}$$



1d Hamiltonian (open ends)

$$H = \begin{pmatrix} \varepsilon_1 & -t & & 0 \\ -t & \varepsilon_2 & \ddots & \\ & \ddots & \ddots & -t \\ 0 & & -t & \varepsilon_N \end{pmatrix}$$

Green's function: $G = (E-H)^{-1}$

$$G_{1N}(E) = ? = \left[\frac{1}{(E-H)}\right]_{1N} = \frac{(\text{minor})_{1N}}{\det(E-H)} \qquad \text{wavefunction } \psi_i(E) \; ?$$

$G_{1N}(E)$ and $\psi_1(E)$ both satisfy $(E-H)_{ij}\,\overset{G_{jN}}{\psi_j} = 0, \; j<N$

$$\Rightarrow \frac{G_{1N}(E)}{G_{NN}(E)} = \frac{\psi_1(E)}{\psi_N(E)}\,! = \frac{1}{\text{growth factor!}}$$

$$\frac{G_{1N}(E)}{G_{NN}(E)} = \frac{(-t)^{N-1}}{\det(E-H)}\,\frac{1}{G_{NN}(E)}$$

$$\Rightarrow \ln\left|\frac{\psi_N(E)}{\psi_1(E)}\right| = \underbrace{\ln|\det(E-H)|}_{\mathrm{Tr}\ln|E-H| = \sum_{\alpha=1}^{N}\ln|E-E_\alpha|} - (N-1)\ln|t| + \ln\Big(\underbrace{\sum_\alpha \frac{|\psi_N^\alpha|^2}{E-E_\alpha}}_{G_{NN}}\Big)$$

does not scale as $\sim N$

$$\Rightarrow \ln\left|\frac{\psi_N(E)}{\psi_1(E)}\right| = N\left[\int d\epsilon\, \rho(\epsilon)\ln|E-\epsilon| - \ln|t|\right] + o(N) = \frac{1}{\xi(E)}\cdot N + o(N)$$

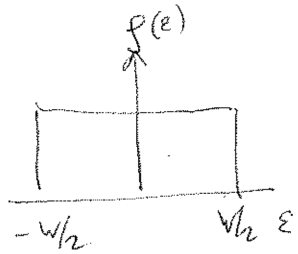
$\rho(\epsilon) = \frac{1}{N}\sum_{\alpha=1}^{N}\delta(\epsilon - E_\alpha)$ density of states

exponential growth!

$\frac{1}{\xi(E)}$ can be evaluated both in strong and weak disorder limit! Always: $\xi(E) < \infty$

$\xi(E)$ in 1d

A) strong disorder $t/W \ll 1$

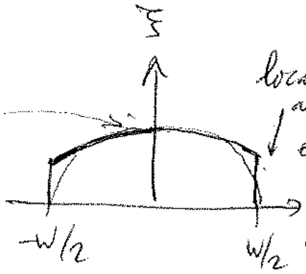


$\rightarrow \rho(\varepsilon) = \frac{1}{W}$ for $\varepsilon \in \left[-\frac{W}{2}, \frac{W}{2}\right]$

$$1/\xi(E) = \int_{-W/2}^{W/2} \frac{d\varepsilon}{W} \ln(|E-\varepsilon|) - \ln|t|$$

$$= \frac{1}{2} \ln\left[\frac{(W/2)^2 - E^2}{e^2 t^2}\right] + \frac{E}{W} \ln\left(\frac{W/2 + E}{W/2 - E}\right) > 0$$

$\Rightarrow$ for $E \ll W$ (band center) $\rightarrow \frac{1}{\xi(E)} \simeq \ln\left[\frac{W}{2et}\right]$



vanishing DOS. $\rightarrow$ suppresses hybridization

B) weak disorder

$$\frac{1}{\xi(E)} = \int d\varepsilon\, \rho(\varepsilon) \ln|(E-\varepsilon)| - \ln|t|$$

Trick: Calculate $\frac{d}{dE}\frac{1}{\xi(E)}$!

$$\frac{d}{dE}\left(\frac{1}{\xi(E)}\right) = \int d\varepsilon\, \rho(\varepsilon) \frac{1}{E-\varepsilon} \overset{!}{\simeq} \mathrm{Re}\,\overline{\left[\frac{1}{N}\sum_i G_{ii}(E+i\delta)\right]}^{\text{disorder}}$$

perturbation theory in ε_i:

$$\frac{1}{N}\sum_i G_{ii}(E) = \frac{1}{N}\left[\sum_i G^0_{ii} + \sum_{ij} G^0_{ij}\varepsilon_j G^0_{ji} + \sum_{ijk} G^0_{ij}\varepsilon_j G^0_{jk}\varepsilon_k G^0_{ki} + \dots\right]$$

average over ε_i: $\overline{\varepsilon_j \varepsilon_k} = \delta_{jk} \cdot \frac{W^2}{12}$

$$\Rightarrow \overline{\frac{1}{N}\sum_i G_{ii}(E)} \approx \frac{1}{N}\Big[\underbrace{\sum_i G^0_{ii}}_{\text{purely imaginary!}} + \frac{W^2}{12}\sum_{ij} G^0_{ij} G^0_{jj} G^0_{ji}\Big]$$

$$\boxed{\frac{1}{N}\sum_i G^0_{ii}(E) = \frac{1}{\sqrt{E^2-(2t)^2}}} \leftarrow \text{evaluate from direct knowledge of spectrum } E(k) = 2t\cos k$$

$$= \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{1}{E - 2t\cos k + i\delta}$$

$$\sum_i G^{0\,2}_{ij} = (G^0)^2_{jj} = \frac{1}{(E-H)^2}\Big|_{jj}$$

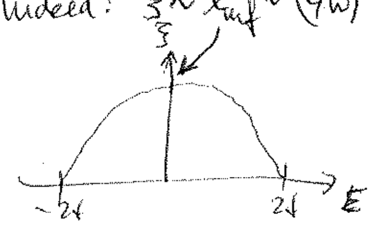
$$= -\frac{d}{dE} G^0_{jj}$$

$$\rightarrow \mathrm{Re}\,\frac{1}{N}\sum_i G_{ii}(E) = \mathrm{Re}\left[-\frac{W^2}{12}\sum_j G^0_{jj}\frac{d}{dE}G^0_{jj}\right]$$

$$= \frac{W^2}{12}\frac{E}{E^2-(2t)^2} \overset{!}{=} \frac{d}{dE}\left(\frac{1}{\xi(E)}\right)$$

$$\Rightarrow \frac{1}{\xi(E)} = \frac{W^2}{24}\frac{1}{(2t)^2 - E^2} \quad \text{(up to a constant } [=0])$$

Indeed: $\xi \sim l_{mf} \sim (t/W)^2$



Anderson's approach to strong localization

(Anderson Phys. Rev. 109, 1492 (1958); Thouless, J Phys. C. 3, 1559 (1970))

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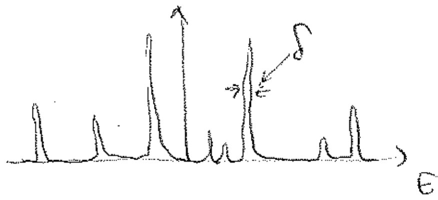
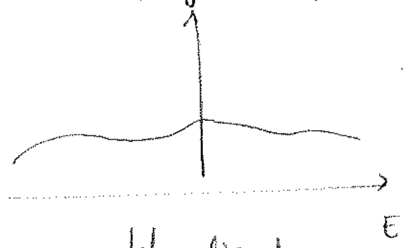
Locator expansion: Expand in small hopping, understand the physics

Local spectrum: $G_{RR}(E+i\delta)$ no hopping: $G_{RR}(E+i\delta)$





$$H = \sum_i \epsilon_i c_i^\dagger c_i - \sum_{\langle ij \rangle} t_{ij} \left(c_i^\dagger c_j + c_j^\dagger c_i \right)$$

→ consider time evolution:

$$i\partial_t \psi_i = \epsilon_i \psi_i - \sum_{j \text{ nghb. of } i} t_{ij} \psi_j \Rightarrow \text{eigenfunctions: } \epsilon_i \psi_i^\alpha - \sum_j t_{ij} \psi_j^\alpha = E^\alpha \psi_i^\alpha$$

Now add an infinitesimal decay rate (coupling to a reservoir) to each site

$$\Rightarrow i\partial_t \psi_i = (\epsilon_i - i\delta)\psi_i - \sum_j t_{ij}\psi_j = H\psi - i\delta \mathbb{1}\psi$$

⇒ Take the infinite volume limit $V \to \infty$ fixed, then $\delta \to 0$:

Does some decay rate survive in this limit? (⇔ Does a particle escape to infinity, or is it localized despite $V \to \infty$?)

Look at Green's function: $= \int_0^\infty dt\, e^{i(E+i\delta)t} \psi_k(t)$ with initial condition $\psi_j(0) = \delta_{ji}$

$$\langle i | \frac{1}{E-(H-i\delta)} | k \rangle \equiv G_{ik}(E+i\delta) = \underset{\substack{\uparrow \\ \text{eigenbasis}}}{} \sum_\alpha \frac{(\psi_i^\alpha)^* \psi_k^\alpha}{E - E^\alpha + i\delta}$$

← bunch of near δ-functions in real part. if α sufficiently sparse

Anderson's recipe: Study self-energy (renormalized energies) $S_i(E)$:

local Green's function
$$\left[G_{ii}(E)\right]^{-1} =: E^{+i\delta} - \epsilon_i - S_i(E+i\delta)$$

close around some E^α where $E - \epsilon_i - S_i(E) = 0$ $\to [(E-E_\alpha)+i\delta]\frac{1}{z_\alpha}$ ↖ $|\psi_i^\alpha|^2$

↖ correction due to hopping t_{ij}

But: if $z_\alpha \to 0$? → many α's combine → finite decay rate!

$$\exists \text{ finite decay rate} \Leftrightarrow \mathrm{Im}(S_i(E+i\delta)) \xrightarrow[\delta \to 0]{} \neq 0 \quad (<0)$$

(from any finite volume)

physical interpretation: particle can escape to ∞ and decay there.

Note: $\delta \gtrless 0$ distinguishes $\left\{ \begin{matrix} \text{retarded} \\ \text{advanced} \end{matrix} \right\}$ propagation. If infinitesimal δ induces finite $\mathrm{Im}(S_i)$ → spontaneous symmetry breaking with respect to the direction of time.
⇒ ∃ an arrow of time!

Note: $G_{ii}(E) = \sum_\alpha \frac{|\psi_i^\alpha|^2}{E - E^\alpha}$: poles at $E = E^\alpha$ (localized phase) / branchcuts at $E = E^\alpha$ (delocalized phase)

[$\Rightarrow$ E^α are solutions of $\underline{E^\alpha - \epsilon_i - S_i(E^\alpha) = 0}$ (loc)]

Advantage of $S_i(E)$ over $G_{ii}(E)$: $S_i(E \to E^\alpha)$ has no singularity!

Behaviour of $S_i(E)$ in the localized / delocalized regimes

(i) loc: $G_{ii}(E = E^\alpha + i\delta) = \frac{|\psi_i^\alpha|^2}{i\delta}$ + small terms from $\beta \neq \alpha$ (either $|E^\beta - E^\alpha| \gg \delta$ or if $|E^\beta - E^\alpha| \lesssim \delta$ $\Rightarrow$ $|\psi_i^\beta|$ will be exp. small $\sim e^{-(1/\delta)^{\frac{1}{d}}}$)

$\hookrightarrow (G_{ii})^{-1} \approx \frac{i\delta}{|\psi_i^\alpha|^2} \overset{!}{=} E^\alpha - \epsilon_i - S_i(E^\alpha + i\delta)$

$\rightarrow \mathrm{Im}\, S_i(E^\alpha + i\delta) \propto \delta \xrightarrow[\delta \to 0]{} 0$

(ii) deloc: $G_{ii}(E + i\delta) \simeq \int d\epsilon \frac{\rho_i(\epsilon)}{E - \epsilon + i\delta} = -i\pi \rho_i(\epsilon) + \int d\epsilon\, \rho_i(\epsilon)\, P \frac{1}{E - \epsilon}$

local D.O.S., smooth

$\Rightarrow \mathrm{Im}\left[(G_{ii})^{-1}\right] \neq 0 \Rightarrow \mathrm{Im}\, S_i(E) \neq 0$ for any δ, also when $\delta \to 0$.

Hopping expansion for self-energy $S_i(E)$

$H = \underset{\text{local}}{H_0} - \underset{\text{hopping terms}}{V}$

$G = \frac{1}{E - H_0 + V} = \frac{1}{G_0^{-1} + V}$

$\Rightarrow G = G_0 - G_0 V G_0 + G_0 V G_0 V G_0 - \dots$ $\qquad (G_0)_{ij} = \frac{\delta_{ij}}{E - \epsilon_i + i\delta}$

$G_{ii} = ?$ Look for first index i within series!

$\Rightarrow G_{ii} \overset{!}{=} G_{0,ii} + \left[-G_{0,ii} V_{ii} + G_{0,ij} V_{ij} G_{0,jj} V_{ji} - \dots\right] G_{ii}$ $\qquad \boxed{\text{"Dyson Equation"}}$

$= G_{0,ii} + \underbrace{G_{0,ii}[\underset{0 = V_{ii}}{-V} + VG_0V - VG_0VG_0V +]_{ii}}_{=: \tilde{S}_i(E)} G_{ii}$ no internal i indices!

$\Rightarrow G_{ii}(1 - G_{0,ii} S_i) = G_{0,ii}$

$\rightarrow \frac{1}{G_{ii}} = \frac{1}{G_{0,ii}} - S_i = E - \epsilon_i - \tilde{S}_i(E) \Longrightarrow \tilde{S}_i \equiv S_i = [VG_0V - VG_0VG_0V + \dots - \dots]_{ii}$

with no return to i!

Physical content of $S_i(E)$

$$S_i(E+i\delta) = \sum_{j\neq i} V_{ij} \frac{1}{E-\epsilon_j+i\delta} V_{ji} + O(V^3)$$

2nd order in t_{ij} ($\equiv V_{ij}$)

$$\xrightarrow[\delta\to 0]{} \sum_{\substack{j\neq i \\ \epsilon_j \neq E}} \frac{V_{ij}^2}{E-\epsilon_j} - i\pi \sum_j \delta(E-\epsilon_j) V_{ij}^2 - i\delta \sum_{j\neq i} \frac{V_{ij}^2}{(E-\epsilon_j)^2}$$

2nd order perturbation of energy ϵ_i

essentially always zero (with probability 1)

Looks like Fermi Golden rule decay; but decay is not possible here because exact energy conservation cannot be satisfied.

$\boxed{\text{No average should be taken over } \epsilon_j!}$ $\Rightarrow$ Study $P(\mathrm{Im}\, S)$ not $\langle \mathrm{Im}\, S \rangle$!

(Exception: if V_{ij} is long ranged, $V_{ij} \sim \frac{1}{r_{ij}^\alpha}$ with $\alpha \leq d$

$\Rightarrow$ always resonances with $E-\epsilon_j < \delta$ can be found.)

$\Rightarrow$ decay at arbitrarily small δ.

Note: at all finite orders of perturbation theory $\mathrm{Im}[S_i(E+i\delta)]$ is $\propto \delta$!

(V_{ij} and $G_{0,ii} = \frac{1}{E-\epsilon_i+i\delta}$ are real up to the part from δ)

$\Rightarrow$ $\boxed{\mathrm{Im}\, S_i \neq 0 \text{ can only occur if the perturbation theory diverges!}}$

*) $\Rightarrow$ Proof of localization by showing that the expansion converges!

Delocalization: $\Leftrightarrow$ spontaneous emergence of $\mathrm{Im}\, S_i < 0$ (decay rate) due to infinitesimal coupling to a bath/reservoir. ($\delta > 0$)

*) Finite dimension: physically motivated resummation of diagrams (locator expansion) + assumptions of non-importance of correlations between different terms

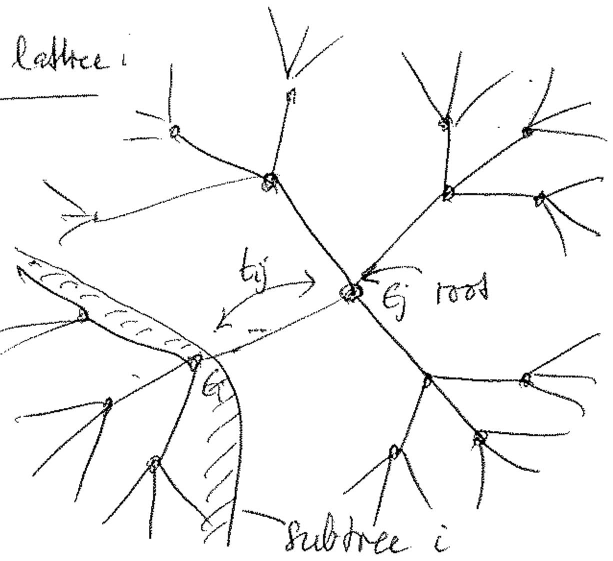
$\Rightarrow$ plausible argument for localization at small hopping.

Later: Mathematical proofs by Fröhlich + Spencer, Molchanov + Aizenmann.

(rigor gained, physics much less clear)

Exactly solvable case: Localization on the Bethe lattice

Bethe lattice:



hierarchical lattice without loops, constant connectivity ($c=4$ here)

Solution: single out a site as "root".

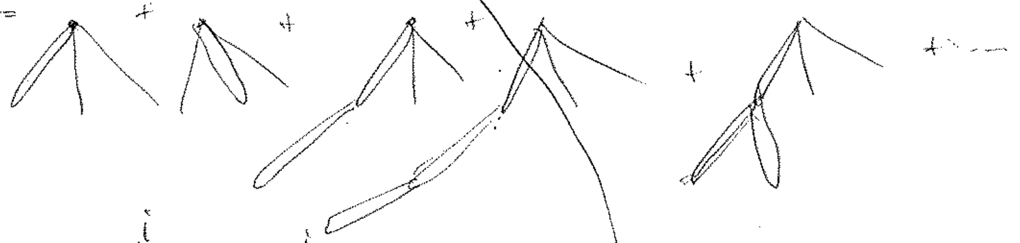
Define subtrees rooted at site i as the sublattice containing i and all sites further away from the root than i, on the same subbranch as i.

Recall: $S_i = +(V G_0 V - V G_0 V G_0 V + \dots)_{ii}$
$(-V)+$

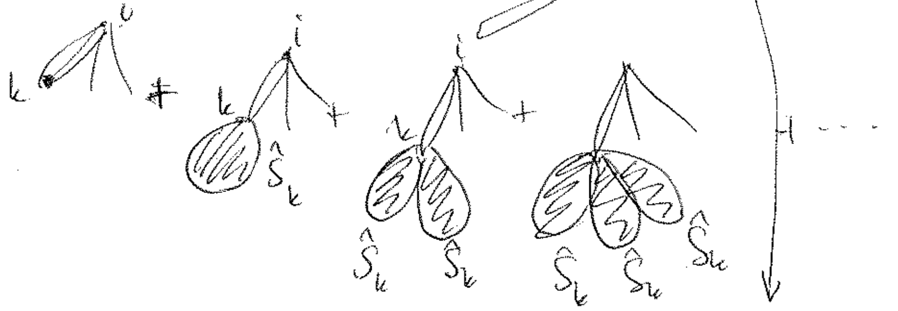
no internal i-index $\Rightarrow$ never come back to site i on the tree

Define $\hat{S}_i$ = subset of all terms belonging to the subtree i.

$\hat{S}_i =$



$=$



$$\rightarrow \hat{S}_i = \sum_{k} V_{ik}\left(G_{0,kk} + G_{0,kk}\hat{S}_k G_{0,kk} + G_{0,kk}\hat{S}_k G_{0,kk}\hat{S}_k G_{0,kk} + \dots\right)V_{ki}$$

k = neighbours of i further from root

$$= \sum_k V_{ik}\frac{1}{E-\epsilon_k+i\delta-\hat{S}_k}V_{ik}$$

$\Rightarrow$ closed recursion relation!

Write $\hat{S}_i = E_i - i\Delta_i$ (real + imaginary part)

Recursion relation:

$$\hat{S}_i = \sum_k \frac{V_{ik}^2}{E - \epsilon_k + i\delta - \hat{S}_k}$$

$$\Rightarrow E_i = \sum_k \frac{V_{ik}^2 (E - \epsilon_k - E_k)}{(E - \epsilon_k - E_k)^2 + (\delta + \Delta_k)^2}$$ real part.

$$\Delta_i = \sum_k \frac{V_{ik}^2 (\Delta_k + \delta)}{(E - \epsilon_k - E_k)^2 + (\delta + \Delta_k)^2}$$

Anderson: Check for stability of localized regime!

$\Rightarrow$ Is $\Delta_i \sim \delta$ as $\delta \to 0$?

or $\Delta_i (\delta \to 0)$ finite

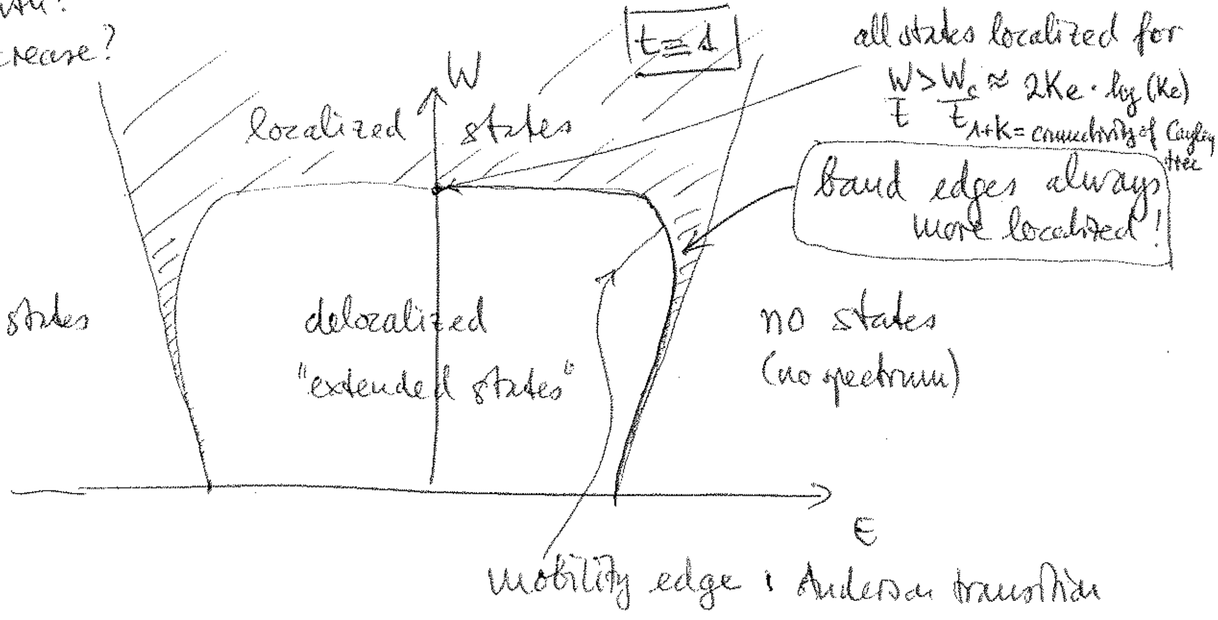
$\Rightarrow$ Linear stability analysis of the recursion relation!

$\Rightarrow$ Instability: $\Leftrightarrow$ For $\delta = 0$, small Δ_k's, Δ's grow indefinitely under iteration! $\Rightarrow$ time reversal symmetry is spontaneously broken!

$$\Delta_i = \sum_k \frac{V_{ik}^2 \Delta_k}{(E - \epsilon_k - E_k)^2} \qquad E_i = \sum_k \frac{V_{ik}^2}{E - \epsilon_k - E_k}$$

growth? or decrease?

Result:



Many body systems

• Assume strong disorder ⇒ full localization of single particle states.

• Can interactions, whatever weak, restore diffusion, thermalization etc?

Answer: No, not in general:

Idea: • Start in a many body eigenstate,

• Make a local excitation

• question: does the local excitation diffuse?

(⇒ analyze problem perturbatively in the interaction between localized single particle states

e.g. electrons: $H_{int} = \int d^3r\, d^3r'\, \psi^\dagger(r)\psi^\dagger(r')\, V(r-r')\, \psi(r')\psi(r)$

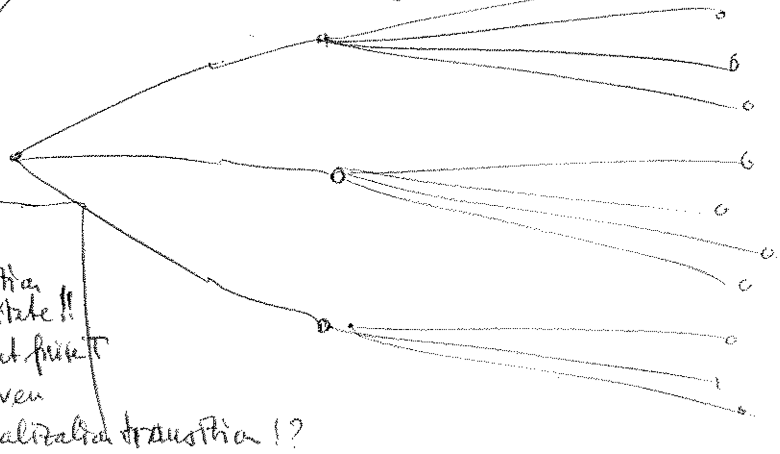
⇒ in single particle (localized basis) : $\sum_{\alpha\beta\gamma\delta} U_{\alpha\beta\gamma\delta}\, \underbrace{c^\dagger_\alpha c^\dagger_\beta c_\gamma c_\delta}$

creation/annihilation of single particle states

Assume U small.

(Construct a lattice of states (many body)

$c^\dagger_\alpha |GS\rangle \xrightarrow{c^\dagger_\gamma c^\dagger_\delta c_\beta c_\alpha} c^\dagger_\gamma c^\dagger_\delta c_\beta |GS\rangle \longrightarrow c^\dagger_\gamma c^\dagger_\varepsilon c^\dagger_\varphi c_\psi c_\beta |GS\rangle$ etc



→ map problem onto (almost) equivalent Cayley tree problem, with $\underline{T}$-dependent K!
(# of thermally facilitated scatterings grows with T)

Conclusion:
- For weak U: transition to fully localized state!! $\sigma = 0$ even at finite T
- Possibility of T-driven localization-delocalization transition !?