Cameroun

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Waves breaking, Burgers and Shocks

Far from the beach , i.e., when the sea is deep, water waves travel on the surface of the sea and their shape is not deformed too much. When the waves approach the beach their shape begins to change and eventually breaks upon touching the seashore.

Here we are interested in studying a simple model that captures the essential features of this physical system. More precisely we will examine the Burgers equation, a non-linear equation introduced as a toy model for turbulence. This equation can be explicitly integrated and, in the limit of zero viscosity, its solution generally displays discontinuities called shocks.

Model 0 : Deep sea

Consider the following equation for the shape of the wave:

$$\partial_t u(x,t) + v\,\partial_x u(x,t) = 0$$

where $u(x,t)$ is the height of the wave with respect to the unperturbed sea level, and v is constant. We will use the following initial condition $u(x,t=0) = u_{\text{init}}(x)$

$$u_{\text{init}}(x)\begin{cases}0 \quad \text{for } |x|>1\\ 1-x^2 \quad \text{for } |x|<1\end{cases} \qquad (0)$$

- Find the solution of the latter equation and compute the velocity of the wave
- Discuss why this model may mimic wave motion in deep sea.

Model 1 : Approaching the sea shore

It is possible to show that, when the height of the wave is negligible as compared to the sea depth, the velocity of the wave does not depend on u. On the contrary, when approaching the sea shore this is not anymore true and the velocity becomes proportional to u:

$$v(u) \simeq c_1 u + v_0$$

where c_1 is a constant and v_0 is the velocity of the bottom of the wave. Let us take $c_1 = 1$ and write the equation for the wave motion in the frame of the bottom of the wave, which yields:

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) = 0 \quad . \qquad (1)$$

This is the non linear equation originally introduced by Burgers. In Burgers' derivation a viscous term was also present

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) = \nu \partial_x^2 u(x,t), \qquad (2)$$

ν being the viscosity, and Eq. (1) is called the inviscid limit of Eq. 2.

Our strategy will be to investigate the properties of Eq1 by resorting to the method of characteristics. Moreover, we will integrate Eq 2 by using the Cole Hopf transformation, and then recover Eq.1 by taking the limit $\nu \to 0$.

The Method of characteristics

- Using the method of characteristics show that the solution of Eq.1 writes:

$$\begin{cases} u(x,t) = u_{\text{init}}(x_0, 0) \\ x(t) = x_0 + u_{\text{init}}(x_0, 0)\, t \end{cases}$$

- Give an interpretation for x_0
- Show that $u(x,t) = 0$ is alway a solution for $|x| > 1$. Denote by $u_0(x,t)$ this solution.
- Show that two other solutions exist, namely:

$$u_{\pm}(x,t) = 1 - \frac{1}{4t^2}\left(1 \pm \sqrt{\Delta(x,t)}\right)^2$$

where $\Delta(x,t) = 4t^2 - 4xt + 1$.

- Show that for $t < 0.5$ only $u_0(x,t)$ and $u_-(x,t)$ are acceptable.
- Draw the full solution for $t = 0.5$ and compute its derivative at $x = 1$
- Show that for $t > 0.5$ the solution is a multivalued function $u_{\text{mult}}(x,t)$

$$u_{\text{mult}}(x,t) = \begin{cases} u_0(x,t) & \text{for } |x| > 1 \\ u_-(x,t) & \text{for } -1 < x < \frac{1}{4t} + t \\ u_+(x,t) & \text{for } 1 < x < \frac{1}{4t} + t \end{cases}$$

and draw some examples of solutions for various values of t.

We look now for a solution which is a single-valued function. For $t > 0.5$ this function should display a discontinuity called shock. We will show in the second part of this exercise that the solution $u_+(x,t)$ is always unstable. The single valued solution for $t > 0.5$ takes the form

$$u_{\text{single}}(x,t) = \begin{cases} u_0(x,t) & \text{for } x < -1 \\ u_-(x,t) & \text{for } -1 < x < x_{\text{shock}}(t) \\ u_0(x,t) & \text{for } x > x_{\text{shock}}(t) \end{cases}$$

where $x_{\text{shock}}(t)$ indicates the shock location. The position $x_{\text{shock}}(t)$ is imposed by a conservation law that should be valid at all times.

- Find the conservation law for the wave and determine $x_{\text{shock}}(t)$.

The Cole Hopf Transformation

Consider now Eq. (2) with $\nu > 0$. This equation can be integrated via the Cole Hopf transformation which amounts to perform the following change of variable:

$$u = -2\nu \frac{\partial}{\partial x} \ln W$$

This can be done in three steps : (i) $u(x,t) = -2\nu\partial_x h(x,t)$, (ii) integrate the equation that you have obtained, (iii) substitute back $h(x,t) = \ln W(x,t)$.

The diffusion equation satisfied by $W(x,t)$ can be solved:

$$W(x,t) = \int dy e^{-\frac{(x-y)^2}{2\nu t}} W(y,0)$$

- Obtain an explicit expression for $u(x,t)$, solution of Eq. (2) with initial condition in Eq. (0).

- Show that, in the limit $\nu \to 0$, the profile $u(x,t)$ can be written as

$$u(x,t) = \frac{x - y_{\min}(x)}{t}$$

where $y_{\min}(x)$ is the location of the global minimum of the function

$$E_{x,t}(y) = \frac{1}{2t}(x-y)^2 + V(y)$$

with $\partial_y V(y) = u_{\text{init}}(y)$. The initial condition $u_{\text{init}}(y)$ is given in Eq.0.

- Show that the solutions u_0, u_+ and u_- can be interpreted as the points where $E_{x,t}(y)$ is stationary, i.e.

$$\partial_y E_{x,t}(y) = 0$$

- Draw $E_{x,t}(y)$ at $t = 1$ for different value of x:

$$x_1 < -1, \quad -1 < x_2 < 1, \quad 1 < x_3 < x_s(1), \quad x_4 = x_s(1), \quad x_s(1) < x_5 < 5/4, \quad x_6 > 5/4$$

- Comment on the obtained results.