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Optical Analogy of Quantum Particles Interaction in a Lattice

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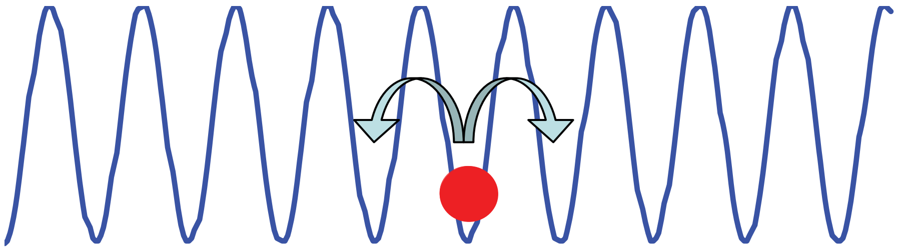
Plan of the Presentation

1. Bose-Hubbard model for one and two particles
2. Analogy with one and two dimensional waveguide lattices
3. Conventional quantum billiards
4. Discrete quantum billiards with waveguide lattices
5. Assymmetric and disordered billiards
6. Conclusions

Bose-Hubbard Model – one particle

$$\hat{H} = \sum_{j=1}^{N-1} g\left(\hat{a}_{j+1}^{+}\hat{a}_{j} + \hat{a}_{j}^{+}\hat{a}_{j+1} + \hat{b}_{j+1}^{+}\hat{b}_{j} + \hat{b}_{j}^{+}\hat{b}_{j+1}\right) + U\sum_{j=1}^{N}\hat{a}_{j}^{+}\hat{b}_{j}^{+}\hat{a}_{j}\hat{b}_{j}$$

$$\left[\hat{a}_m, \hat{a}_n^{+}\right] = \left[\hat{b}_m, \hat{b}_n^{+}\right] = \delta_{mn} \qquad \left[\hat{a}_m, \hat{a}_n\right] = \left[\hat{b}_m, \hat{b}_n\right] = \left[\hat{a}_m^{\pm}, \hat{b}_n^{\pm}\right] = 0$$



$$|n\rangle = \hat{a}_n^{+}|0\rangle \qquad |\Psi(t)\rangle = \sum_{n=1}^{N} c_n(t)|n\rangle$$

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$$

$$\left(\sum_{j=1}^{N-1}\hat{a}_{j+1}^{+}\hat{a}_{j}\right)\hat{a}_n^{+}|0\rangle = \hat{a}_{n+1}^{+}|0\rangle = |n+1\rangle \qquad \left(\sum_{j=1}^{N}\hat{a}_{j}^{+}\hat{b}_{j}^{+}\hat{a}_{j}\hat{b}_{j}\right)\hat{a}_n^{+}|0\rangle = 0$$

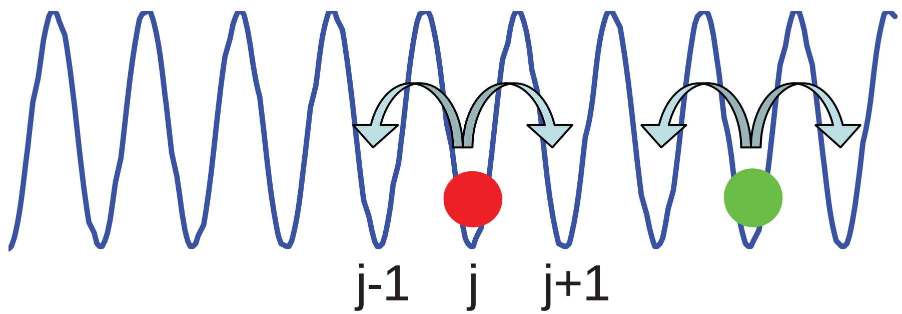
$$\hat{H}|n\rangle = g\left(|n+1\rangle + |n-1\rangle\right)$$

$$i\frac{\partial}{\partial t}c_n(t) = g\left[c_{n+1}(t) + c_{n-1}(t)\right]$$

Bose-Hubbard Model – two particles

$$\hat{H} = \sum_{j=1}^{N-1} g\left(\hat{a}_{j+1}^{+}\hat{a}_{j} + \hat{a}_{j}^{+}\hat{a}_{j+1} + \hat{b}_{j+1}^{+}\hat{b}_{j} + \hat{b}_{j}^{+}\hat{b}_{j+1}\right) + U\sum_{j=1}^{N}\hat{a}_{j}^{+}\hat{b}_{j}^{+}\hat{a}_{j}\hat{b}_{j}$$

$$\left|m,n\right\rangle = \hat{a}_{m}^{+}\hat{b}_{n}^{+}\left|0\right\rangle$$



$$\left|\Psi(t)\right\rangle = \sum_{m,n=1}^{N} c_{mn}(t)\left|m,n\right\rangle$$

$$\left(\sum_{j=1}^{N-1}\hat{a}_{j+1}^{+}\hat{a}_{j}\right)\hat{a}_{m}^{+}\hat{b}_{m}^{+}\left|0\right\rangle = \hat{a}_{m+1}^{+}\hat{b}_{n}^{+}\left|0\right\rangle = \left|m+1,n\right\rangle$$

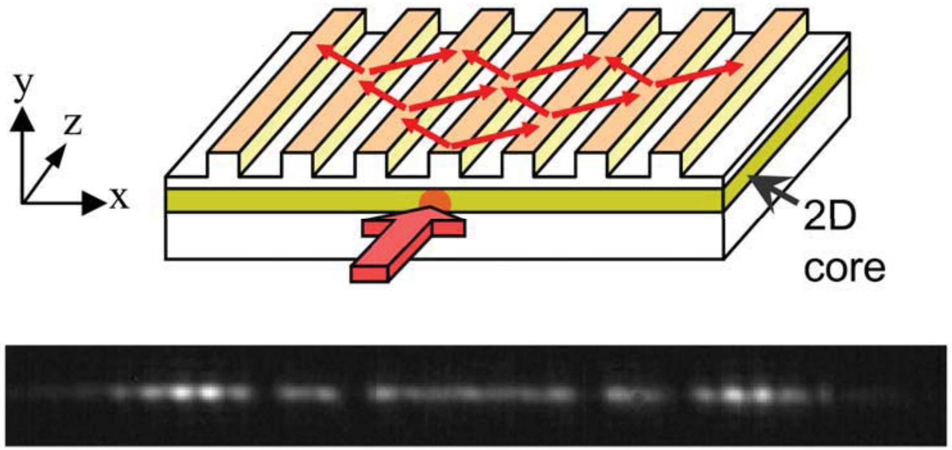
$$\left(\sum_{j=1}^{N}\hat{a}_{j}^{+}\hat{b}_{j}^{+}\hat{a}_{j}\hat{b}_{j}\right)\hat{a}_{m}^{+}b_{n}^{+}\left|0\right\rangle = \delta_{mn}\hat{a}_{m}^{+}b_{n}^{+}\left|0\right\rangle = \delta_{mn}\left|m,n\right.$$

$$\hat{H}\left|m,n\right\rangle = g\left(\left|m,n+1\right\rangle + \left|m,n-1\right\rangle + \left|m+1,n\right\rangle + \left|m-1,n\right\rangle\right) + \delta_{mn}U\left|m,n\right\rangle$$

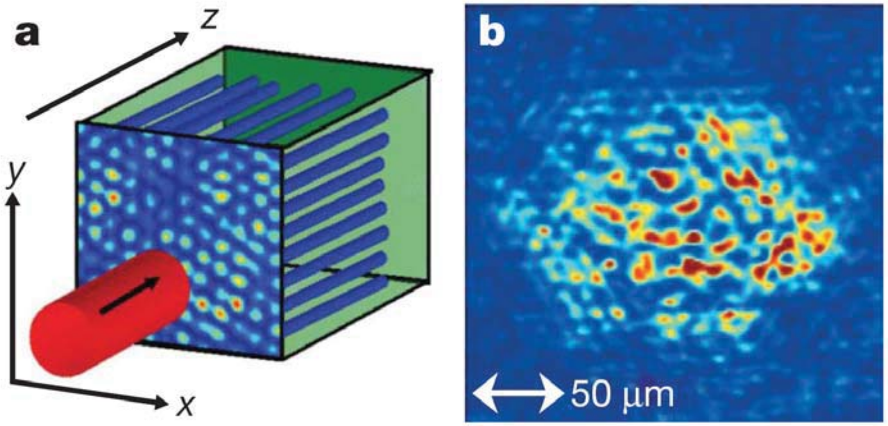
$$i\frac{\partial}{\partial t}c_{mn} = g\left[c_{m,n+1} + c_{m,n-1} + c_{m+1,n} + c_{m-1,n}\right] + \delta_{mn}Uc_{mn}$$

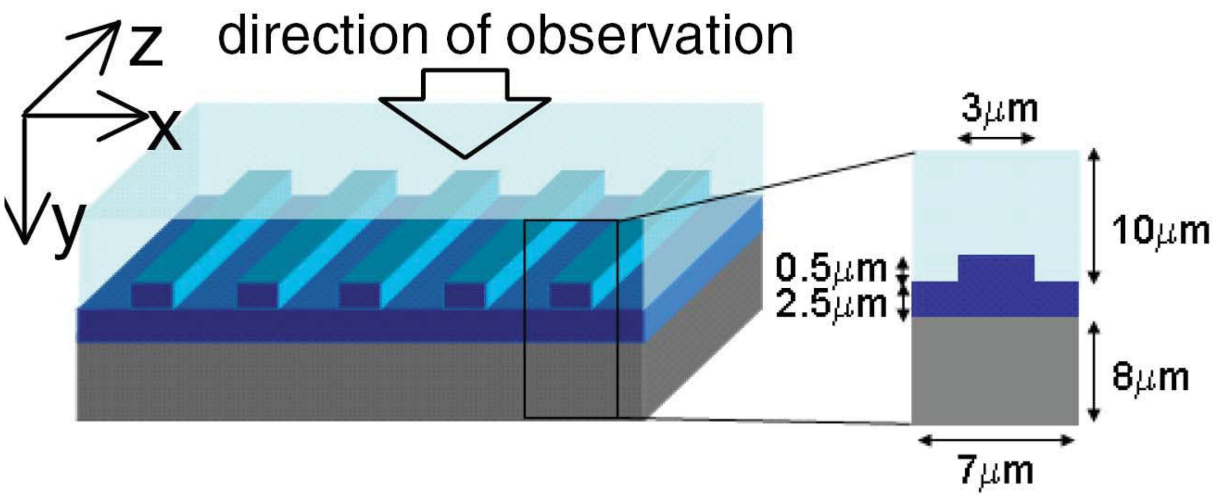
Optical Waveguide Arrays and Lattices

Perets, et.al., PRL **100,** 170506 (2008)



Schwartz, et.al., Nature, **446**, 52 (2007)





Trompeter et.al., PRL **96,** 023901 (2006)

Maxwell Equations

$$\vec{\nabla}\times\vec{E}(\vec{r},t)=-\frac{1}{c}\frac{\partial\vec{B}(\vec{r},t)}{\partial t};\quad \vec{\nabla}\times\vec{H}(\vec{r},t)=\frac{1}{c}\frac{\partial\vec{D}(\vec{r},t)}{\partial t};\quad \vec{\nabla}\cdot\vec{D}(\vec{r},t)=0;\quad \vec{\nabla}\cdot\vec{B}(\vec{r},t)=0$$

$$\vec{B}(\vec{r},t)=\vec{H}(\vec{r},t)\quad\Rightarrow\quad \vec{\nabla}\times\vec{\nabla}\times\vec{E}(\vec{r},t)=-\frac{1}{c^2}\frac{\partial^2\vec{D}(\vec{r},t)}{\partial t^2}$$

$$\vec{E}(\vec{r},t)\equiv E_y(y,z,t)$$

$$\Downarrow$$

$$\frac{\partial^2 E_y(x,z,t)}{\partial x^2}+\frac{\partial^2 E_y(x,z,t)}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 D_y(x,z,t)}{\partial t^2}=0$$

$$n(x)=n_0+n_0\delta n(x)\qquad D_y=n(x)^2E_y=\left[n_0^2+2n_0\delta n(x)\right]E_y$$

Paraxial Approximation

$$\frac{\partial^2 E_y(x,z,t)}{\partial x^2} + \frac{\partial^2 E_y(x,z,t)}{\partial z^2} - \frac{n_0^2}{c^2}\frac{\partial^2 E_y(x,z,t)}{\partial t^2} - \frac{2n_0\delta n(x)}{c^2}\frac{\partial^2 E_y(x,z,t)}{\partial t^2} = 0$$

$$E_y = \Psi e^{-i(\omega t - kz)} + \Psi^* e^{i(\omega t - kz)} \qquad \omega = kc/n_0$$

$$E_y = \Psi\left[\varepsilon x, \varepsilon^2 z\right] e^{i(\omega t - kz)} + \left(\Psi\left[\varepsilon x, \varepsilon^2 z\right]\right)^* e^{-i(\omega t - kz)};$$

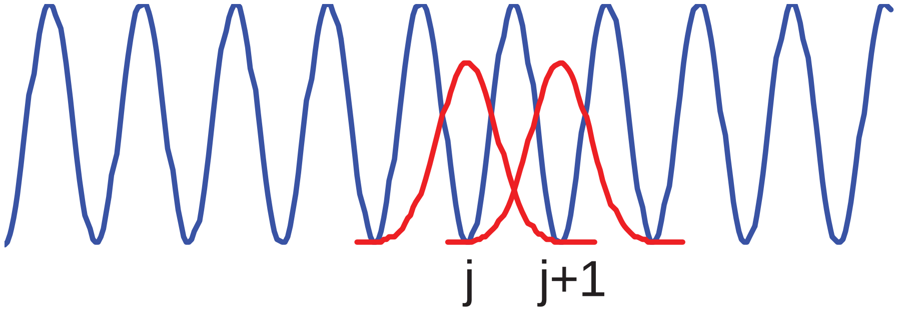
$$\frac{\partial}{\partial x}\Psi \sim \varepsilon; \qquad \frac{\partial}{\partial z}\Psi \sim \varepsilon^2$$

$$2ik\frac{\partial \Psi}{\partial z} + \frac{\partial^2 \Psi}{\partial x^2} + \omega^2 \frac{2n_0\delta n(x)}{c^2}\Psi = 0$$

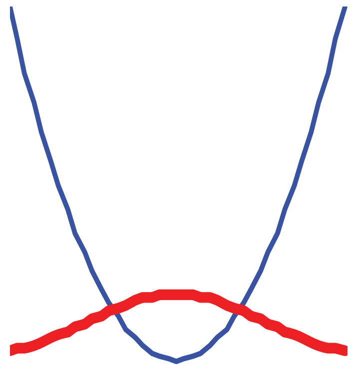
Tight-Binding Approximation

$$i\frac{\partial \Psi}{\partial z}+\frac{\partial^2 \Psi}{\partial x^2}-V(x)\Psi=0$$

$$V(x)=-\frac{\omega}{c}\delta n(x)$$



$$a\sin^2 x \approx ax^2$$



$$V(x)=ax^2$$

$$\Psi(t,x)=\frac{a^{1/8}}{\pi^{1/4}}e^{-it\sqrt{a}}e^{-x^2\sqrt{a}/2}$$

$$\Phi_n(x)=\frac{a^{1/8}}{\pi^{1/4}}e^{-(x-n\pi)^2\sqrt{a}/2}$$

$$\Psi(t,x)=e^{-it\sqrt{a}}\sum_{n=-\infty}^{\infty}\psi_n(t)\Phi_n(x)$$

$$i\frac{\partial \psi_j}{\partial t}+g\left(\psi_{j+1}+\psi_{j-1}\right)+\varepsilon\psi_j=0$$

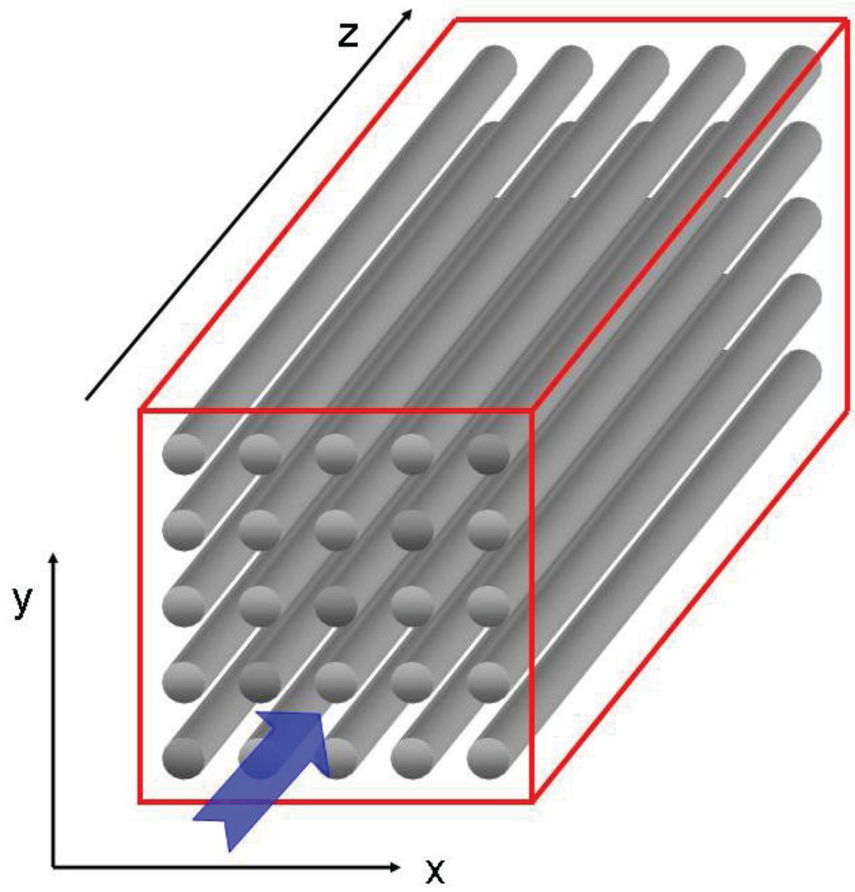
$$\int_{-\infty}^{\infty}\Phi_j\Phi_{j+1}dx\approx 0 \qquad \varepsilon=\int_{-\infty}^{\infty}\left[-a\Phi_j\sin^2 x\Phi_j+\Phi_j\frac{\partial^2\Phi_j}{\partial x^2}\right]dx$$

$$\mathrm{g}=\int_{-\infty}^{\infty}\left[-a\Phi_j\sin^2 x\Phi_{j+1}+\Phi_j\frac{\partial^2\Phi_{j+1}}{\partial x^2}\right]dx=\int_{-\infty}^{\infty}\left[-a\Phi_j\sin^2 x\Phi_{j-1}+\Phi_j\frac{\partial^2\Phi_{j-1}}{\partial x^2}\right]dx$$

$$\psi_j\to\psi_j\exp\left(i\varepsilon z\right)$$

$$i\frac{\partial \psi_j}{\partial t}+g\left(\psi_{j+1}+\psi_{j-1}\right)=0$$

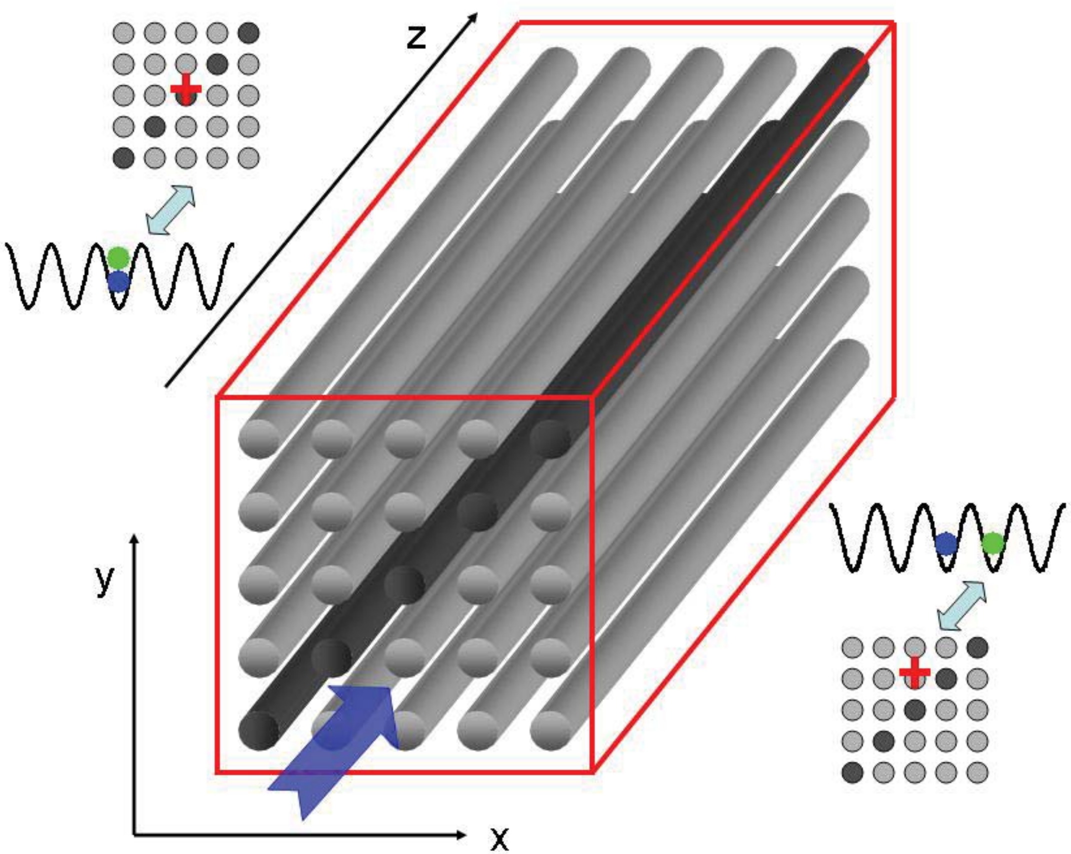
Square Lattice of
Optical Waveguides



$$i\frac{\partial}{\partial t}\psi_{mn} = g\left[\psi_{m,n+1} + \psi_{m,n-1} + \psi_{m+1,n} + \psi_{m-1,n}\right]$$

Analogy between Bose Hubbard and Waveguide Lattices

$$\hat{H} = \sum_{j=1}^{N-1}\left(\hat{a}_{j+1}^{+}\hat{a}_{j} + \hat{a}_{j}^{+}\hat{a}_{j+1} + \hat{b}_{j+1}^{+}\hat{b}_{j} + \hat{b}_{j}^{+}\hat{b}_{j+1}\right) + U\sum_{j=1}^{N}\hat{a}_{j}^{+}\hat{b}_{j}^{+}\hat{a}_{j}\hat{b}_{j}$$



$$i\frac{\partial}{\partial t}\psi_{mn} = Q\left[\psi_{m,n+1} + \psi_{m,n-1} + \psi_{m+1,n} + \psi_{m-1,n}\right] + \delta_{mn}U\psi_{mn} + \varepsilon_{mn}\psi_{mn}$$

Discrete Quantum Billiard with Optical Waveguides

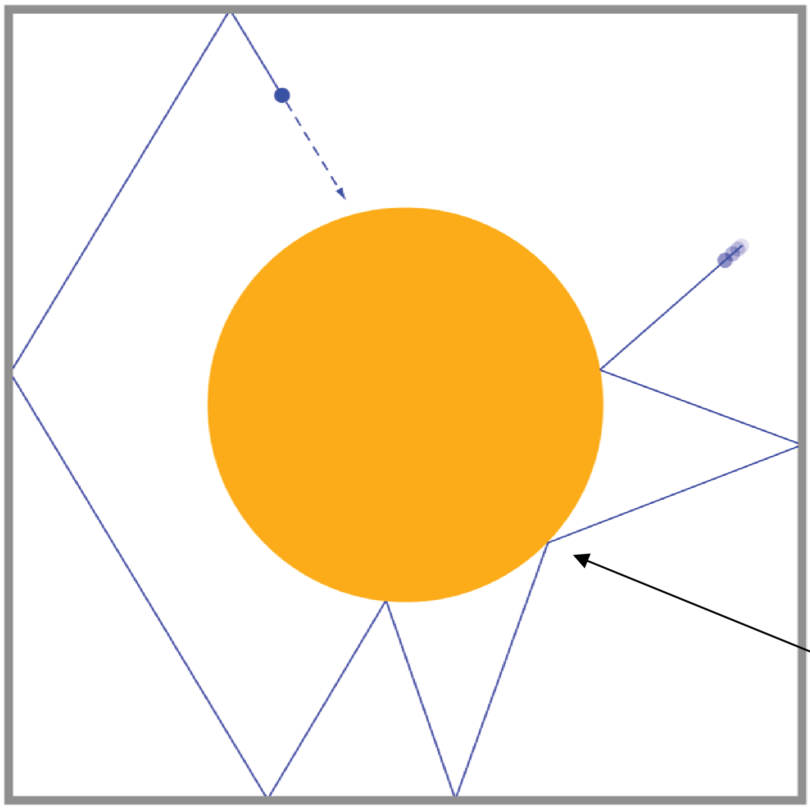
Phys. Rev. A, 001800(R) (2011)

Chaotic Classical Billiards

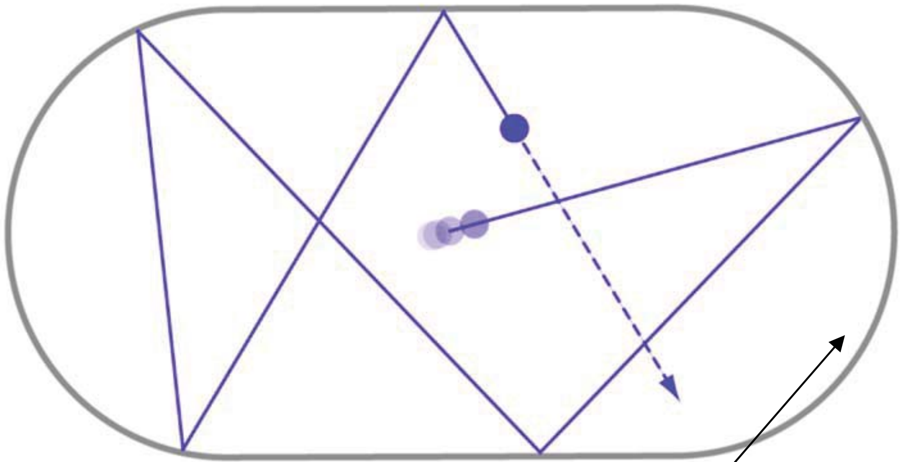
$$H(\vec{p},\vec{r}) = \frac{p^2}{2m} + V(\vec{r})$$

$$V(\vec{r}) = \begin{cases} 0, & \vec{r} \in \mathbf{S} \\ \infty, & \vec{r} \notin \mathbf{S} \end{cases}$$

Sinai Billiard



Bunimovich Stadium



Convex and Concave scatters

Quantum Billiards

$$i\frac{\partial\Psi}{\partial t}=\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}, \qquad \Psi(\vec{r},\mathrm{t})\big|_{\mathrm{s}}=0$$

$$\Psi(\vec{r},t)=\sum_n e^{i\lambda_n t}A_n(\vec{r}); \qquad -\lambda_n A_n(x,y)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)A_n(x,y)$$

$$\text{sorting } \lambda_n \text{ we define } s_n=\left|\lambda_{n+1}-\lambda_n\right|$$

Regular Billiards – Poisson Distribution

$$R(s)=\exp[-s]$$

Chaotic Billiards – Wigner Distribution

$$R(s)=\frac{\pi}{2}s\cdot\exp\left[-\pi s^2/4\right]$$

Discrete Quantum Billiards

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \to \Psi_{m,n+1} + \Psi_{m,n-1} + \Psi_{m+1,n} + \Psi_{m\text{-}1,n} - 4\Psi_{m,n}$$

$$i \frac{\partial \Psi_{m,n}}{\partial t} = \Psi_{m,n+1} + \Psi_{m,n-1} + \Psi_{m+1,n} + \Psi_{m\text{-}1,n} - 4\Psi_{m,n}$$

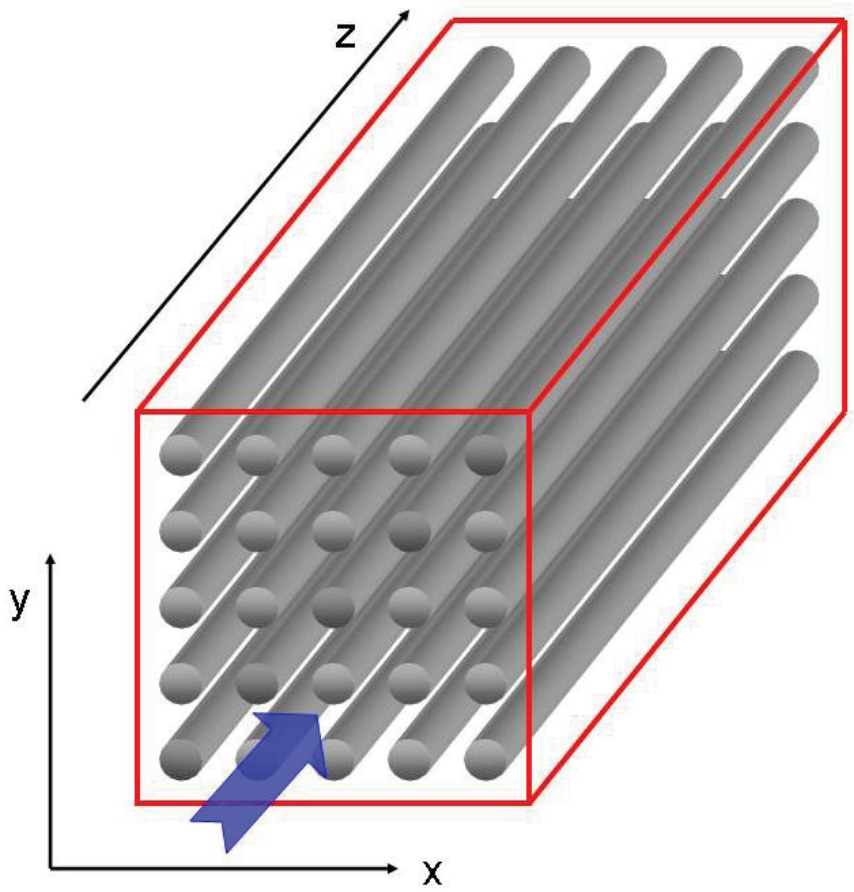
$$\Psi_{m,n} = \Phi_{mn}^{(f)} \exp\left[-i\left(\lambda_f - 4\right)t\right]$$

$$\lambda_f \Phi_{m,n}^{(f)} = \Phi_{m,n+1}^{(f)} + \Phi_{m,n-1}^{(f)} + \Phi_{m+1,n}^{(f)} + \Phi_{m-1,n}^{(f)}$$

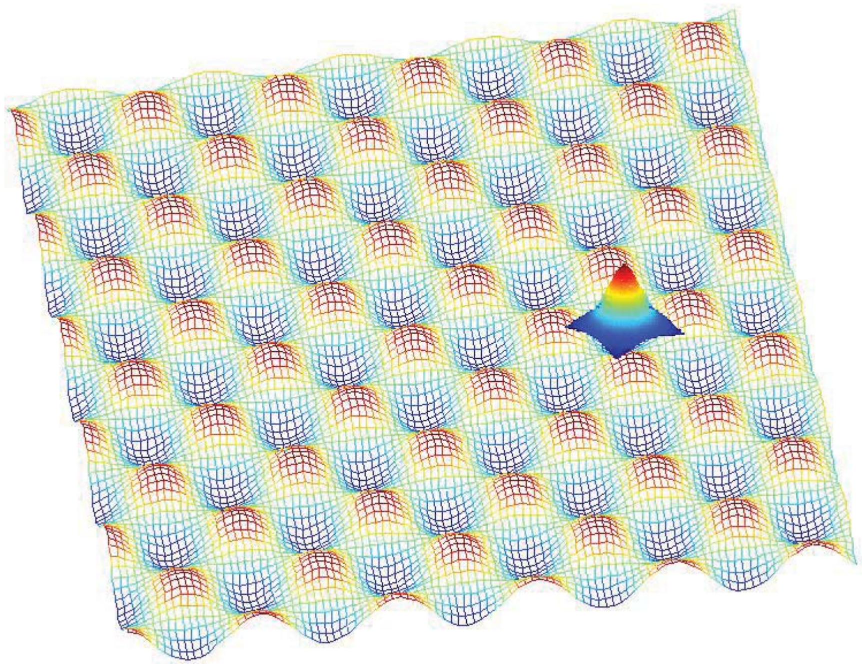
Square Boundaries: $\Psi_{m,n} = 0$ if $m > N$ or $n > N$

Realizations for Discrete Quantum Billiards

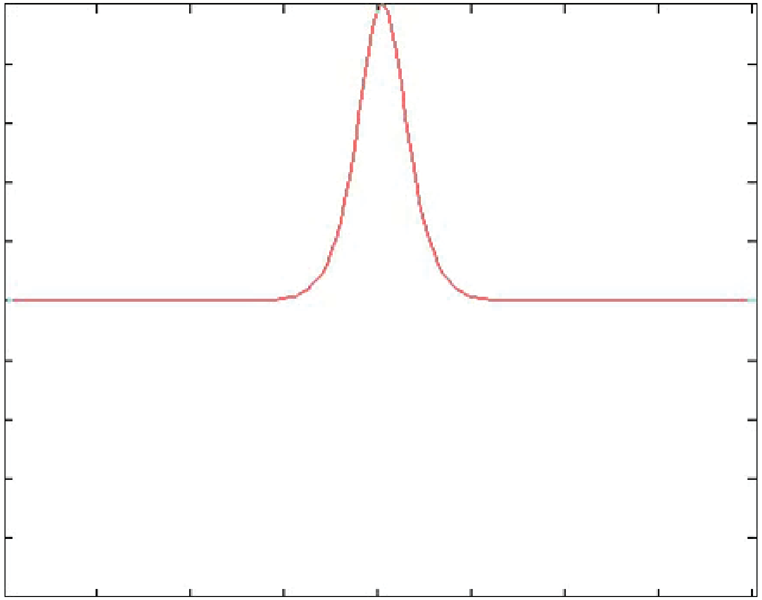
2D waveguide arrays

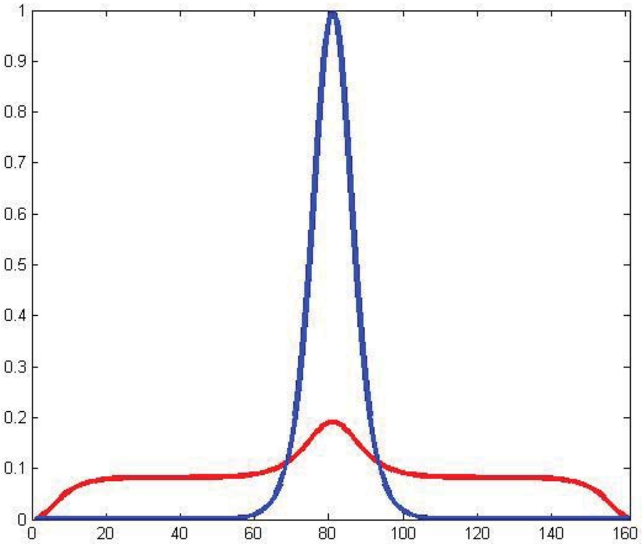


BEC in 2D optical lattices

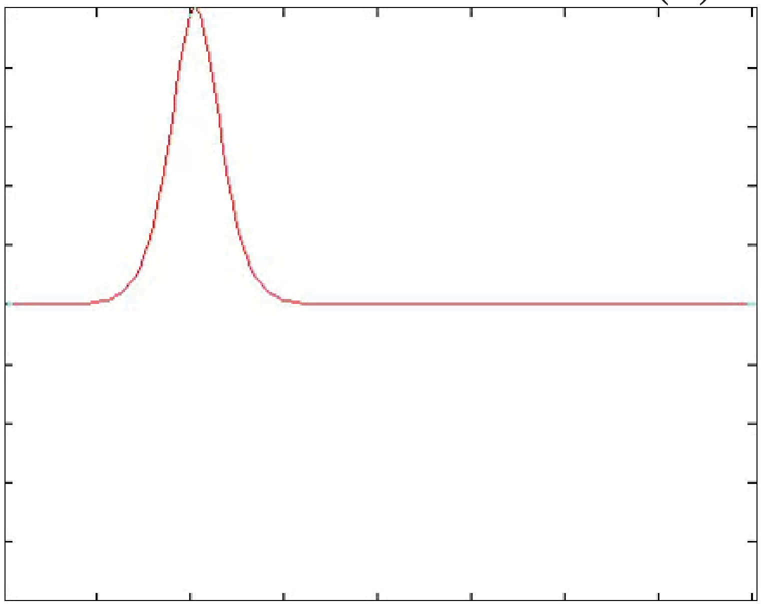


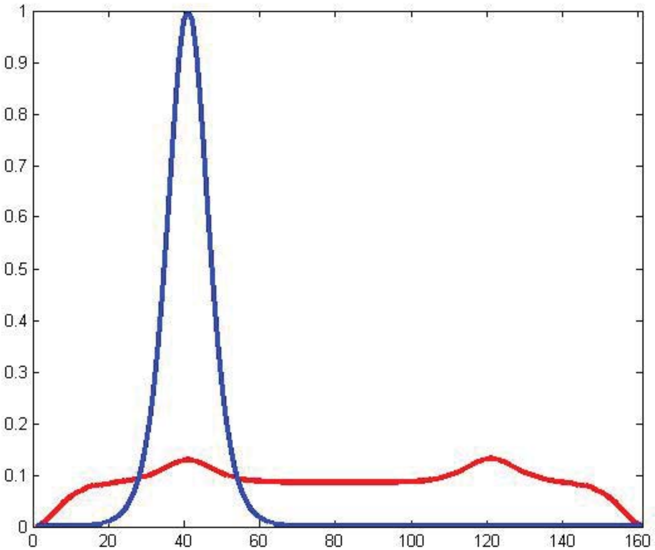
Continuous 1D Quantum Billiard



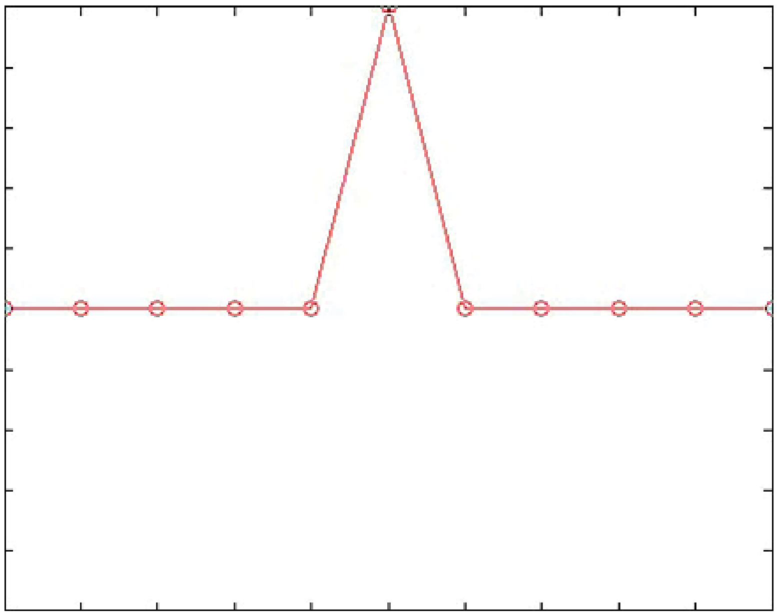


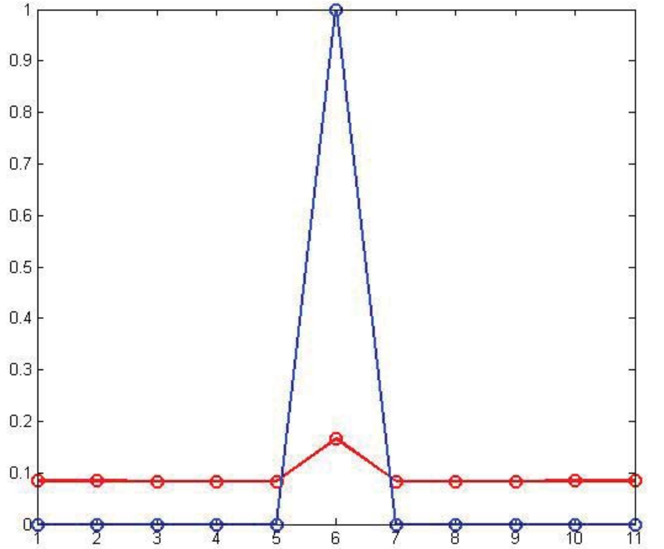
$$P(r) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left| \Psi(r,t) \right|^2 dt$$



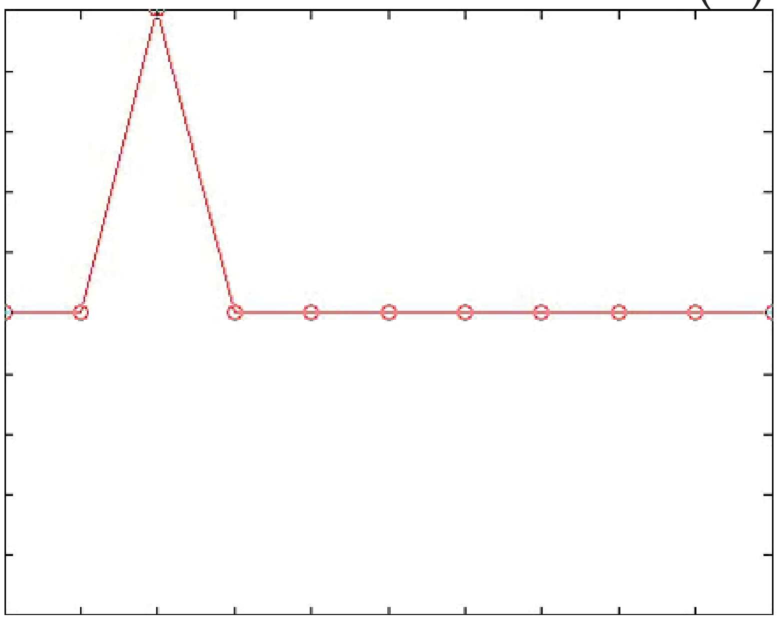


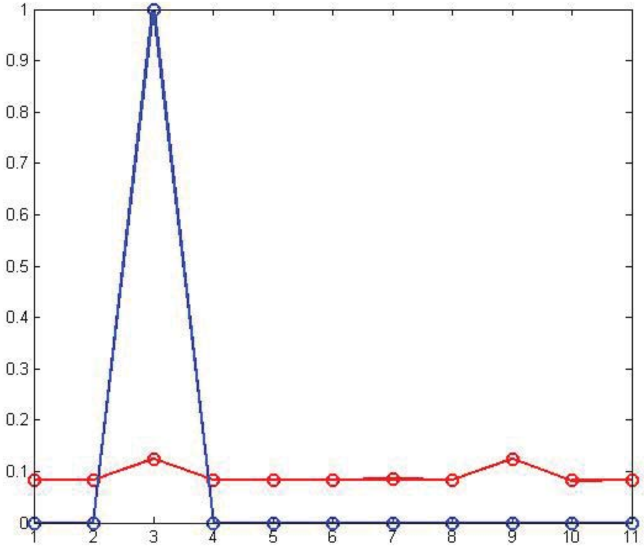
Discrete 1D Quantum Billiard





$$P(n) = \lim_{T \to \infty} \frac{1}{T} \int_0^T |\Psi(n,t)|^2 \, dt$$





Calculation of Probability Distribution Function (PDF)

$$P_{m,n} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \left|\Psi_{m,n}\right|^2 dt$$

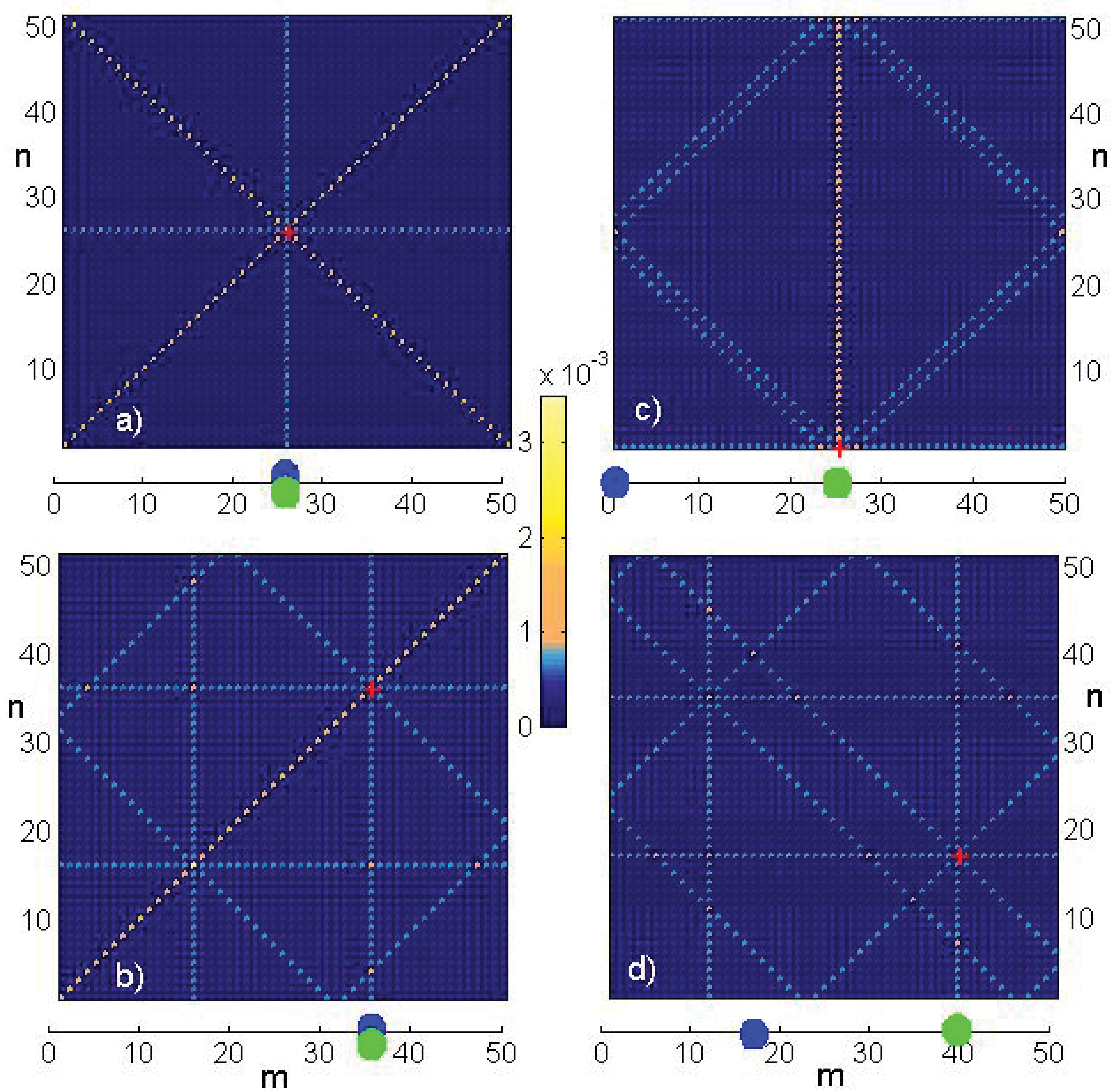
$$\left.\begin{aligned} \hat{H}\left|q\right\rangle = \lambda_q\left|q\right\rangle \quad \Rightarrow \quad & \left|\Psi(t)\right\rangle = \sum_q \varphi_q e^{i\lambda_q t}\left|q\right\rangle \\ & \left|\Psi(t)\right\rangle = \sum_{m,n=1}^{N} \Psi_{mn}(t)\left|m,n\right\rangle \end{aligned}\right\}$$

$$\Psi_{mn}(t) = \sum_q \varphi_q L_{mn}^q e^{-i\lambda_q t}\left|m,n\right\rangle \quad \text{where} \quad L_{mn}^q = \left\langle m,n \mid q\right\rangle$$

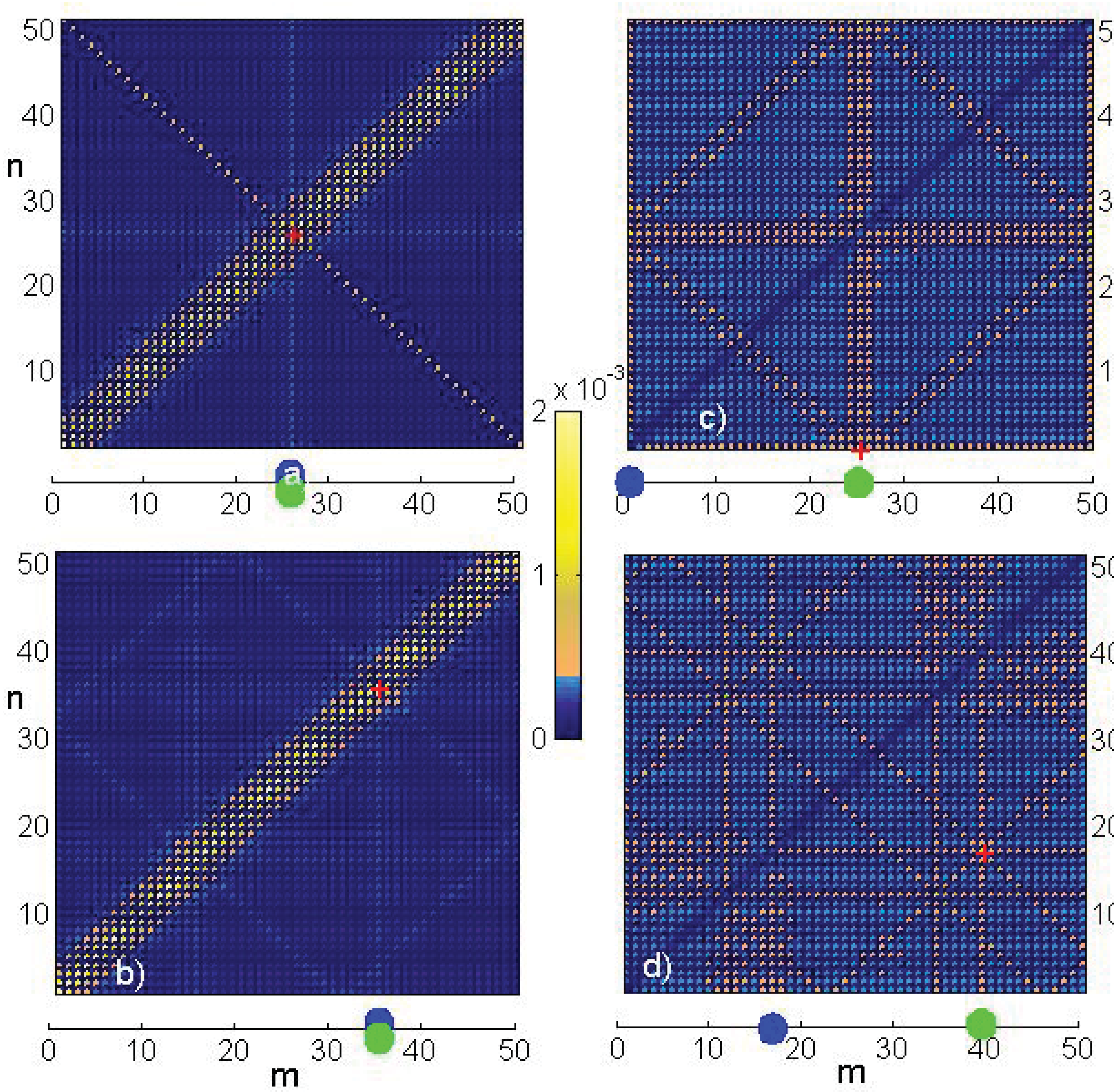
$$\varphi_q = \sum_{m,n=1}^{N} L_{mn}^q \Psi_{mn}(0)$$

$$P_{mn} = \sum_{q'} \left|\varphi_{q'}\right|^2 \left(L_{mn}^{q'}\right)^2 + \sum_i \left|\sum_{q_i=1}^{r_i} \varphi_{q_i} L_{mn}^{q_i}\right|^2$$

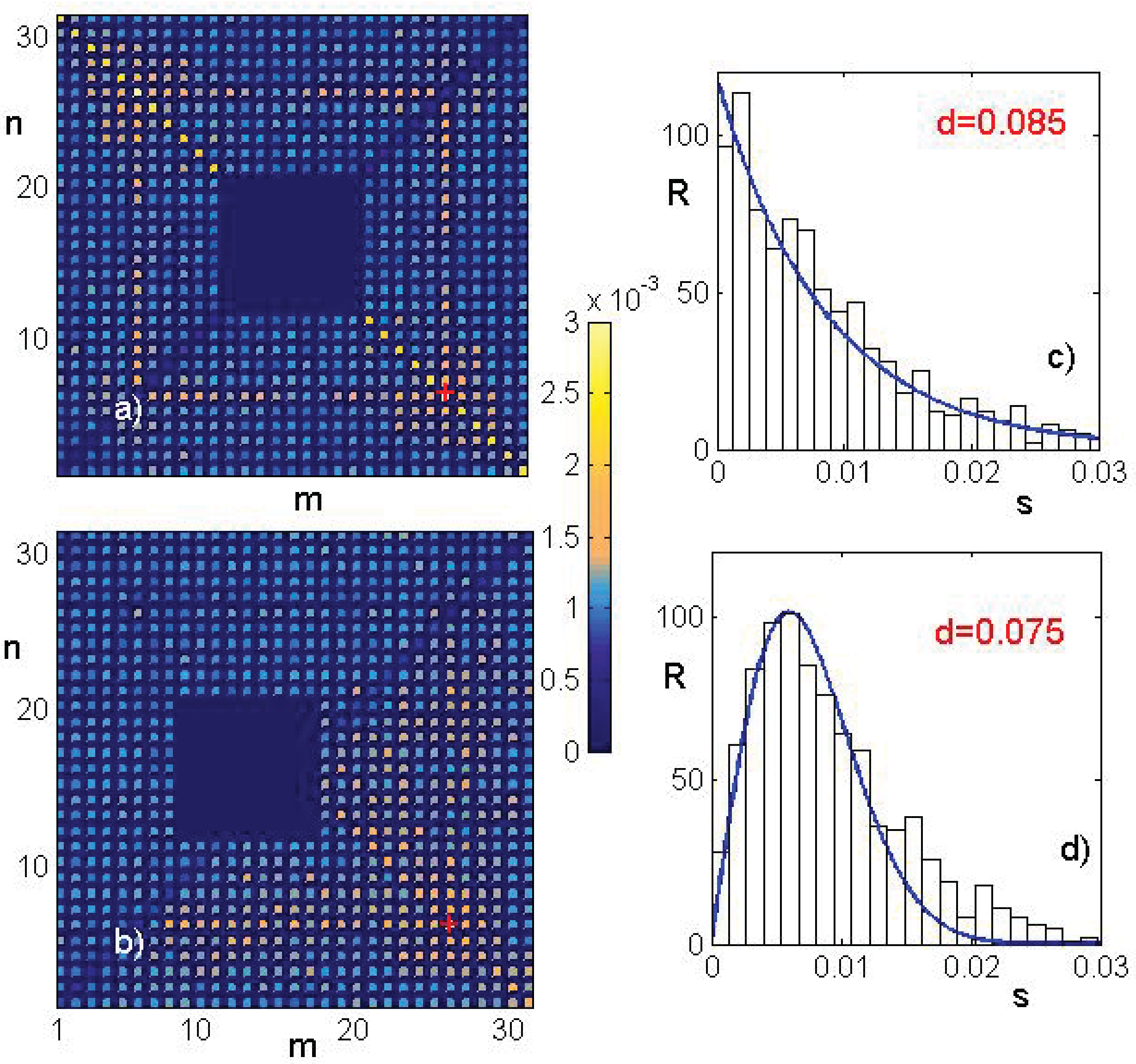
PDF for the case U=0



PDF for the case U=1



PDF for the case U=2



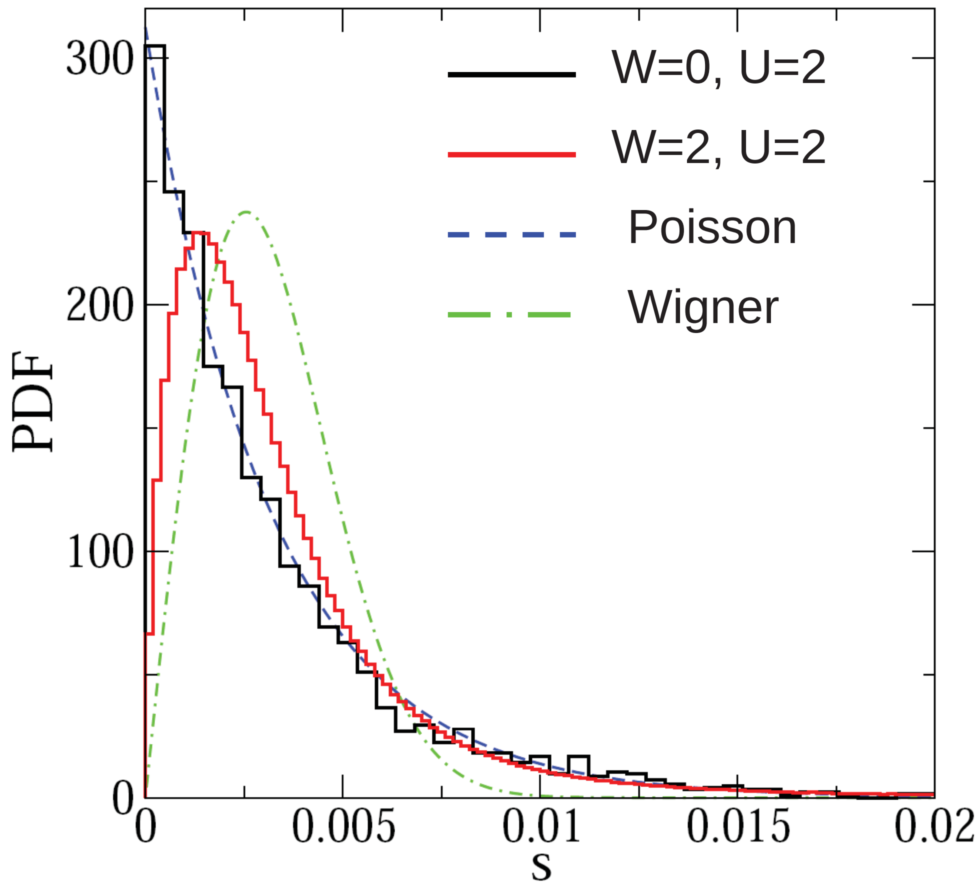
$$R(s) = \frac{1}{d}\exp\left[-\frac{s}{d}\right]$$

$$R(s) = \frac{\pi s}{2d^2}\exp\left[-\frac{\pi s^2}{4d^2}\right]$$

Disordered 2D Billiard

$$\hat{H} = \sum_{j=1}^{N-1}\left(\hat{a}_{j+1}^{+}\hat{a}_{j} + \hat{a}_{j}^{+}\hat{a}_{j+1} + \hat{b}_{j+1}^{+}\hat{b}_{j} + \hat{b}_{j}^{+}\hat{b}_{j+1}\right) + U\sum_{j=1}^{N}\hat{a}_{j}^{+}\hat{b}_{j}^{+}\hat{a}_{j}\hat{b}_{j} + \sum_{j=1}^{N}\varepsilon_{j}\left(\hat{a}_{j}^{+}\hat{a}_{j} + \hat{b}_{j}^{+}\hat{b}_{j}\right)$$

$$i\partial_t \Psi_{mn} = \left(W_{mn} + U\delta_{mn}\right) + \sum_{m',n'=1}^{N} R_{mn}^{m'n'}\Psi_{m'n'}$$



Correlated Disorder

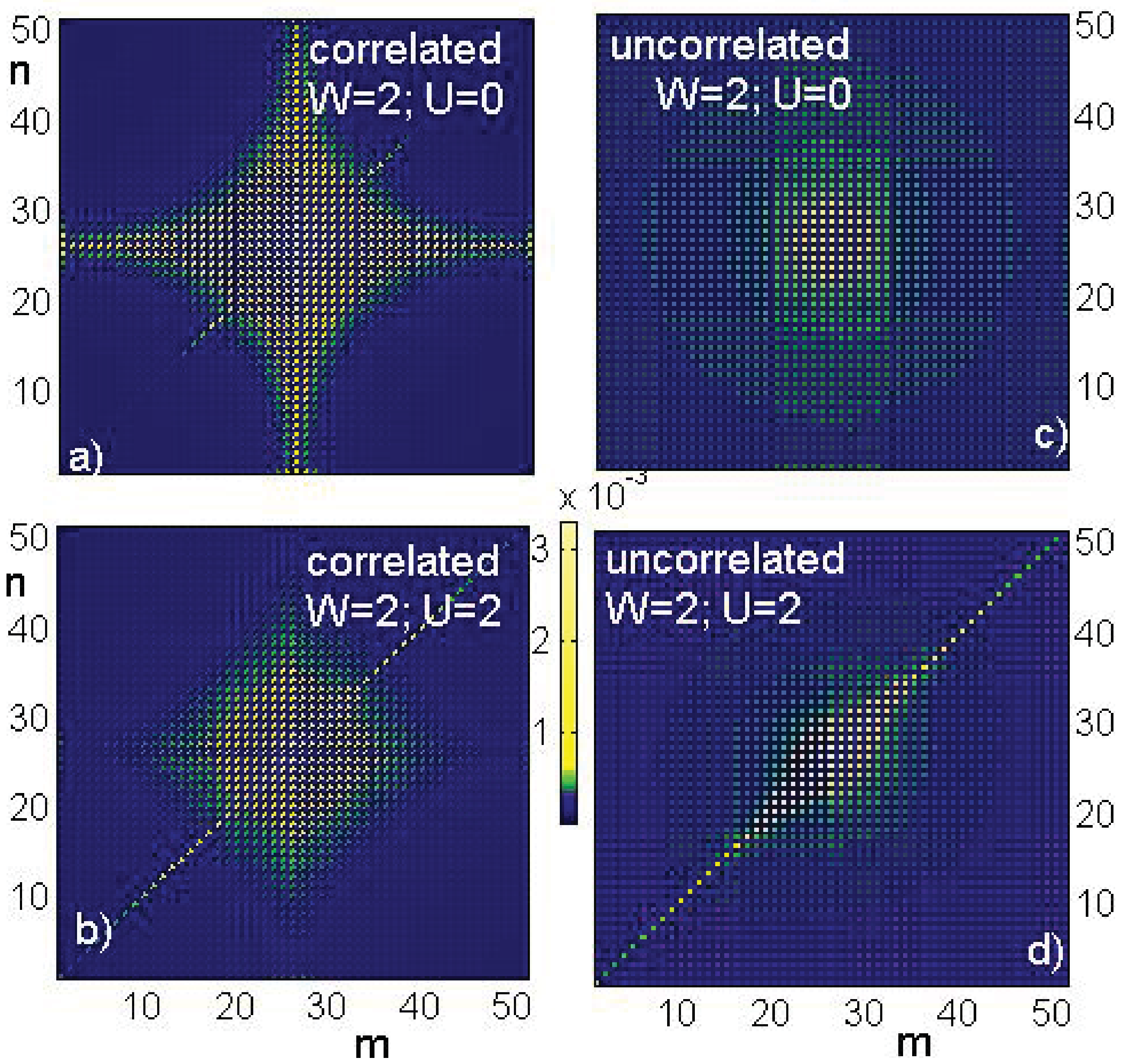
$$W_{mn} = \varepsilon_m + \varepsilon_n$$

$$\varepsilon_j \in \left[-W/2, W/2\right]$$

Uncorrelated Disorder

Each W_{mn} is a sum of two random numbers

PDF for the case W=2, U=2



Conclusions

THANK YOU