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Synchronisation in oscillator networks: an approach via normal hyperbolicity theory for non-autonomous oscillators

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Abstract

Much has been written about synchronisation in oscillator networks. Yet it tends to deal with rather idealised settings, like identical oscillators. I propose here a general approach. Although at present somewhat hard to implement, it is nevertheless conceptually straightforward.

The idea is that synchronisation is a collapse in dimension. For example, the state of two uncoupled oscillators generally explores a two-torus, but on adding coupling they may synchronise onto an attracting periodic orbit, which has one dimension less.

To treat the phenomenon of synchronisation of a set of oscillators forming part of a network, one needs to deal with the fact that the rest of the network in general applies a non-trivial forcing function of time to the chosen subset. Temporarily one can regard the forcing as given and then the problem reduces to defining and studying synchronisation for non-autonomous oscillator systems.

This is a worthwhile problem in its own right, even for a single oscillator. Although often idealised as autonomous, many oscillators are subject to time-dependent forcing, in general aperiodic. Examples include an AC electricity grid, an analogue phase-locked loop and a bipedal walker. Normal hyperbolicity theory shows that a non-degenerate limit cycle of an autonomous system persists to an invariant cylinder in extended state space for weak time-dependent perturbations. Our first goal here is to derive realistic estimates for this cylinder. We achieve this by a refined approach to proving persistence of normally hyperbolic submanifolds. This allows us to construct safety criteria for the effects of time-dependent forcing on oscillators.

Our second goal is to make precise the notion of synchronisation of an oscillator to its forcing. If the forcing is time-periodic the concept is clear – a periodic orbit with the period of the forcing – but synchronisation of an oscillator can also occur to an aperiodic forcing. Indeed this is crucial to the operation of various signal decoding mechanisms such as quadrature phase shift keying. We propose to say that an oscillator synchronises to its forcing if the flow on its invariant cylinder contains a uniformly hyperbolic trajectory (in the non-autonomous sense). We derive realistic sufficient conditions for synchronisation.

Finally, we will present an aggregation approach to synchronisation in networks of oscillators, based on the same non-autonomous normal hyperbolicity ideas.

1 Introduction

The concept of a limit cycle of an autonomous system of ordinary differential equations is crucial to the understanding of a wide range of physical, biological and technological phenomena (e.g. [HS]). Yet in reality most systems are not autonomous. Instead they are subject to external time-dependent influences. The case of time-periodic forcing has been extensively studied (e.g. [TS]), giving rise to phenomena like Arnol'd tongues. Periodic forcing is a poor representation of many real situations, however, and the case of general bounded time-dependent forcing of a general limit cycle has received little attention.

In principle, the qualitative effect of weak time-dependent forcing on a hyperbolic limit cycle oscillator is clear from the theory of normal hyperbolicity [HPS, Fe] (and its heuristic versions such as slaving theory [Ha]): it persists to a normally hyperbolic invariant cylinder in the extended state space.

To spell this out, let us parametrise the unperturbed limit cycle by an angle $\theta \in \mathbb{R}/\mathbb{Z}$ equal to the fraction of the period T from a reference point on the cycle and extend θ to a coordinate on a tubular neighbourhood. There is a choice of parametrisation for which $\dot{\theta} = \omega = 1/T$, the cyclic frequency[1] of the limit cycle (its level sets are called "isochrons") but we do not need this choice and its construction would add unnecessary estimates. In an orientable manifold the normal bundle to a circle is trivialisable, so the coordinate system can be completed in the neighbourhood by a coordinate $r \in \mathbb{R}^{n-1}$, n being the dimension of the manifold, with the unperturbed limit cycle being $r = 0$ (for the non-orientable case it suffices to take a double cover of the limit cycle and to remember the resulting $\mathbb{Z}_2$ symmetry). Then we can write the perturbed system in the neighbourhood of the unperturbed limit cycle as

$$\begin{aligned} \dot{\theta} &= \Theta(\theta, r, t), \\ \dot{r} &= R(\theta, r, t). \end{aligned} \tag{1}$$

In the unperturbed case, Θ and R are independent of t, $\Theta(\theta, 0, t) = \omega$ and $R(\theta, 0, t) = 0$. We suppose Θ and R to be C^1 in (θ, r) and continuous in (θ, r, t) (the latter could be relaxed to allow some cases of discontinuous forcing). Note that the forcing is not restricted to be additive: it can be state dependent. The unperturbed limit cycle is *hyperbolic* if the time-T map ϕ of the linearised unperturbed dynamics

$$\dot{\xi} = R_r(\omega t, 0, t)\xi, \quad \xi \in \mathbb{R}^{n-1}, \tag{2}$$

where R_r denotes the matrix of partial derivatives of R with respect to r, has no eigenvalue on the unit circle. The case of most practical interest is when the spectrum of ϕ is inside the unit circle (exponentially attracting limit cycle), but the theory applies equally well if there is some spectrum outside too. To study non-autonomous systems it is convenient to extend the state space to include time t as an additional dimension.

[1] Many engineers would use the symbol f for cyclic frequency and reserve ω for the angular frequency $f/2\pi$, but for us f will often denote a forcing function.

Then by normal hyperbolicity theory [HPS, Fe] the straight cylinder $r = 0$ (θ, t arbitrary) in extended state space, representing the product of the unperturbed limit cycle with time, persists to a C^1-nearby normally hyperbolic invariant submanifold $r = \rho(\theta, t)$. The vector field on the cylinder is C^1-close to $\dot\theta = \omega$, $\dot t = 1$.

This is fine as theory, but in practice one would like to know how close is the cylinder to the unperturbed one and to what extent the dynamics on the cylinder changes.

To achieve realistic estimates, we present a proof of persistence of normally hyperbolic invariant submanifolds (for the case of trivialisable normal bundles which suffice here) which has the advantage that the graph transform it uses can be made an arbitrarily strong contraction if coordinates are chosen appropriately. In contrast, the usual graph transform requires choice of a time step and contracts by only the ratio of normal to tangential contraction factors. It also has the advantage that it deals with normally contracting and expanding directions simultaneously, whereas in usual treatments one has to construct stable and unstable manifolds first and then intersect them. The method is a hybrid between those of Hadamard and Perron, in the spirit of Irwin [Ir]. In the light of Anosov's 1967 remark that "Every five years or so, if not more often, someone 'discovers' the theorem of Hadamard and Perron, proving it by Hadamard's method of proof or by Perron's" [An], we hesitate to claim our proof to be new (though we think Anosov was referring to the uniformly hyperbolic case rather than the normally hyperbolic one), yet ours seems more appropriate for our purposes than others of which we are aware (references in [HPS] plus subsequent works [Ir, LW, BOV, Ch]), partly because many references treat only discrete-time systems, but mainly because the arbitrarily strong contraction of our graph transform gives us more accuracy with fewer iterations.

Our graph transform maps one graph $r = \rho(\theta, t)$ to another $r = \tilde\rho(\theta, t)$. We allow θ to be in an m-torus, rather than just the circle, in order to deal with the case of groups of oscillators. For any (θ_0, t_0), $\tilde\rho(\theta_0, t_0)$ is defined by taking the trajectory of $\dot\theta = \Theta(\theta, \rho(\theta, t), t)$ from (θ_0, t_0), then solving for the locally unique bounded trajectory of $\dot r = R(\theta(t), r, t)$ and setting $\tilde\rho(\theta_0, t_0) = r(t_0)$. The solution of the r-equation exists and is locally unique because the dynamics in the r direction is assumed to be uniformly hyperbolic in a neighbourhood of $r = 0$. Under a normal hyperbolicity assumption, the graph transform is a contraction on the space of Lipschitz graphs with a suitable Lipschitz constant, so has a unique fixed point and this is an invariant submanifold. With additional work, the submanifold can be shown to be C^1 (and indeed for a C^r vector field, $r > 1$, it can be shown to be C^r under an r-normal hyperbolicity assumption). The autonomous case is included as the special case without the explicit time-dependence.

The strategy for obtaining realistic estimates is analogous to that presented in [BM] for the case of aperiodically forced hyperbolic equilibria, but needs normal hyperbolicity theory rather than just uniform hyperbolicity theory.

The dynamics on the invariant cylinder may collapse onto special trajectories. This is familiar in the case of periodic forcing where collapse onto an attracting periodic orbit is called "synchronisation", "frequency-locking", "mode-locking" or "phase-locking". We propose to broaden the use of the term "synchronisation" to existence of a uniformly hyperbolic trajectory (in the non-autonomous sense, definition recalled in next section) on the invariant cylinder. In [BM] the theory was extended from persistence of hyperbolic equilibria to non-autonomous uniformly hyperbolic trajectories. It is used here to derive realistic sufficient conditions for synchronisation of an oscillator to its forcing.

Similarly, for a weakly coupled group of m oscillators subject to weak forcing functions there is an invariant m-torus$\times$time in the extended state space. The dynamics on this m-torus$\times$time may possess an attracting k-torus$\times$time for some $k < m$, in which case we say the group of oscillators partially synchronises. If $k = 1$ then the group has totally synchronised together. If $k = 0$ it has totally synchronised to its forcing. In all cases there is a reduction in dimension, which allows us to replace the group of oscillators by an effective group of fewer oscillators. In the case $k = 0$, the group can be eliminated entirely. This provides at least conceptually, a hierarchical aggregation scheme for oscillator networks. Given some assumptions about the likely shape of forcing of the rest of the network on a given group, one can investigate whether the group would synchronise to some extent and if so aggregate the group together into a lower dimensional one. At a later stage one would have to check consistency of the assumptions about the shape of the forcing functions.

Parts of this work were done in collaboration with Stephen Gin.

2 Basics

2.1 Linear systems

Uniform and normal hyperbolicity theory centre on linearisation of various aspects of the dynamics around trajectories or candidates for trajectories. Thus we consider linear dynamics of the form

$$\dot{x}(t) = A(t)x(t) \tag{3}$$

for $x(t)$ in a normed linear space V and $A(t)$ a bounded linear map from V to V. For each $s \in \mathbb{R}$ we write the matrix solution of (3), i.e.

$$\partial_1 X(t,s) = A(t)X(t,s)$$

from initial condition $X(s,s) = I$ as $X(t,s)$, where ∂_j denotes derivative with respect to the j^{th} argument. Thus $x(t) = X(t,s)x(s)$. We will take A to be bounded and measurable, which is a sufficient condition for X to be globally and uniquely defined.

Note that by differentiating the identity $X(t,s)X(s,t) = I$ with respect to s,

$$\partial_2 X(t,s)X(s,t) = -X(t,s)\partial_1 X(s,t),$$

and thus $X(t,s)$ also satisfies $\partial_2 X(t,s) = -X(t,s)A(s)$.

More generally, one could consider V to be a normed linear bundle over $\mathbb{R}$ and $A(t)$ an appropriate linear map involving a connection on the bundle to allow to compare vectors at nearby t, but our applications will not require this generality.

The simplest example is for a trajectory y of a vector field $\dot{y} = v(y,t)$ on a manifold M to take $A(t) = v_y(y(t),t)$, the derivative of v along the trajectory, so (3) is the linearised dynamics around the trajectory.

In this paper we will consider three other instances of (3) related to dynamical systems of the form (1), but allowing θ to be in a general submanifold M with trivialised normal bundle NM:

- linearised normal dynamics: given a path (not necessarily trajectory) $(\theta(), r())$, let

  $$A(t) = R_r(\theta(t), r(t), t)$$

  on NM (the linearisation of $\dot{r} = R(\theta, r, t)$ in the r direction).

- slope dynamics linearised about 0: a *slope* is a linear map from TM to NM. Slope S evolves by the Ricatti equation

  $$\dot{S} = R_\theta + R_r S - S\Theta_\theta - S\Theta_r S. \tag{4}$$

  Linearising about $S = 0$ gives

  $$\dot{\sigma} = A(t)\sigma = R_r\sigma - \sigma\Theta_\theta,$$

  for an infinitesimal slope σ.

- modified linearised slope dynamics: given a C^1 graph $r = \rho(\theta, t)$ it will be convenient to consider the modified slope dynamics with

  $$A(t)\sigma = R_r\sigma - \sigma(\Theta_\theta + \Theta_r\rho_\theta).$$

We shall often consider families of linear systems of the form (3), for example linearised normal dynamics for trajectories $\theta()$ with different initial conditions.

2.2 Uniform hyperbolicity

We say a family of linear systems (3) is *uniformly hyperbolic* if there exists $K > 0$ such that for each choice of A from the family and bounded continuous function $f : \mathbb{R} \to V$ the forced linear system

$$\dot{x}(t) = A(t)x(t) + f(t) \tag{5}$$

has a unique bounded solution $x()$ and $|x|_1 \leq K^{-1}|f|$, where we endow $C^0(\mathbb{R}, V)$ with the norm

$$|f| = \sup_{t\in\mathbb{R}} |f(t)| \tag{6}$$

and $C^1(\mathbb{R}, V)$ with the norm

$$|x|_1 = \max(|x|, \tau|\dot{x}|). \tag{7}$$

Here $\tau > 0$ is a timescale chosen to make the C^1 norm scale sensibly with change of unit of time. We will restrict attention to families for which A is bounded and then it will be convenient to choose τ so that $\tau|A| \leq 1$, to simplify some of the estimates.

Equivalently, there exists $K > 0$ such that for each member of the family the linear operator $L : C^1(\mathbb{R}, V) \to C^0(\mathbb{R}, V)$ defined by

$$L[x](t) = \dot{x}(t) - A(t)x(t) \tag{8}$$

is invertible with $\|L^{-1}\|^{-1} \geq K$, using the operator norm of L^{-1} from $C^0 \to C^1$.

Sometimes it is convenient to consider L as mapping $W^{1,\infty}$ (bounded Lipschitz functions) to $W^{0,\infty}$ (L^∞ functions) instead. The norms of L and its inverse remain unchanged.

Uniform hyperbolicity is robust to small perturbation, by the following result which is valid for any linear operator L from one Banach space X to another Y with bounded inverse, that we recall without proof.

Lemma 2.1. *If ΔL is a perturbation to an invertible operator $L : X \to Y$ with $\|\Delta L\| < \|L^{-1}\|^{-1}$ then $L - \Delta L$ is invertible and $\|(L - \Delta L)^{-1}\|^{-1} \geq \|L^{-1}\|^{-1} - \|\Delta L\|$.*

Note as one consequence that for (8) $\|L^{-1}\|^{-1} \leq |A|$, else taking $\Delta L = -A$ the lemma would give that $\partial_t : C^1 \to C^0$ is invertible, which is not the case. Thus $K \leq |A|$.

The above definition of uniform hyperbolicity is not the usual one, which is in terms of an invariant splitting of the tangent bundle into backwards and forwards contracting subspaces with uniform exponential estimates and angle between them bounded away from zero. Nevertheless, it implies the usual one. This result goes under the name "exponential dichotomy", e.g. [Co]. Since we need the estimates, we give a statement and proof.

Theorem 2.2. *If a linear system (3) is uniformly hyperbolic with $\|L^{-1}\|^{-1} \geq K$ and A bounded, there are complementary projections $P_\pm$ of V at each point $s \in \mathbb{R}$, bounded uniformly in s, invariant under the linear flow, and for any $\mu < K$ there is $C \in \mathbb{R}$, such that for $x(s) \in E_\pm(s) = ran\ P_\pm(s)$ the trajectory of $x(s)$ satisfies*

$$|x(t)| \leq Ce^{-\mu|t-s|}|x(s)| \tag{9}$$

for $t \geq s$, respectively $t \leq s$.

Proof. As mentioned above, choose the timescale $\tau > 0$ such that $|A(t)| \leq 1/\tau$ for all t. Given $s \in \mathbb{R}$ and any vector x_0 at s, we wish to split x_0 into components with bounded forwards, respectively backwards, orbit. Without loss of generality, we take $s = 0$. Let x be the solution of the free linearised system $\dot{x} = Ax$ from the initial condition $x(0) = x_0$. Let

$$\phi(t) = 1 - \frac{|t|}{\tau} \text{ for } |t| < \tau$$

and 0 otherwise. Let

$$\Phi(t) = \int_0^t \phi(u)\ du = t(1 - \frac{|t|}{2\tau}) \text{ for } |t| < \tau,$$

$\mathrm{sgn}(t)\tau/2$ otherwise (see Figure 1).

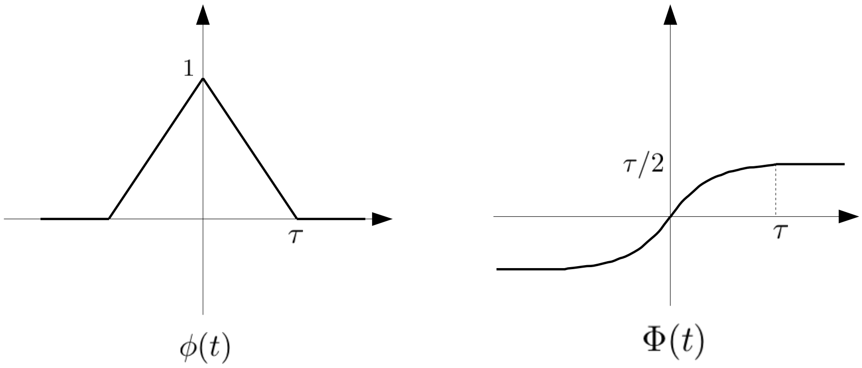

Figure 1: Graphs of the functions ϕ and Φ.

Let $\zeta = \Phi x$ and $\eta = \phi x$. Then $\dot{\zeta} = A\zeta + \eta$. By hypothesis there is a unique bounded solution $\beta = L^{-1}\eta$ of this equation. Let $x_+ = (\beta - \zeta)/\tau + \frac{1}{2}x$. It satisfies $\dot{x}_+ = Ax_+$ and equals β/τ for $t > \tau$, so is bounded for $t \geq 0$. Similarly, $x_- = (\beta - \zeta)/\tau - \frac{1}{2}x$ satisfies $\dot{x}_- = Ax_-$ and equals β/τ for $t < -\tau$ so is bounded for $t \leq 0$. Then $x = x_+ - x_-$. Define $P_+x_0 = x_+(0)$ and $P_-x_0 = -x_-(0)$.

By construction, $P_\pm$ are linear and sum to the identity. To see that they are projections ($P^2 = P$), take $Px_0 = x_+(0)$ as new initial condition and define ζ_+, η_+, β_+ to be the corresponding functions above. Then $(\beta_+ - \zeta_+)/\tau - \frac{1}{2}x_+$ is bounded not only for $t < 0$ but also for $t > 0$ because each of its terms is bounded for $t > 0$. By the invertibility hypothesis, however, the only solution of the free linearised equation is 0. So $P_-P_+x_0 = 0$ for all x_0. But $P_+ + P_- = I$ hence $P_\pm^2 = P_\pm$.

To show that the projections are uniformly bounded, note that

$$x_+(0) = \frac{\beta(0) - \zeta(0)}{\tau} + \frac{1}{2}x_0, \ \zeta(0) = 0$$

and

$$|\beta(0)| \leq |\beta|_0 \leq |\beta|_1 \leq K^{-1}|\eta|_0 = K^{-1} \sup_{|t|<\tau} \left((1 - \frac{|t|}{\tau})|x(t)| \right). \tag{10}$$

Now from the choice of τ and Gronwall's estimate on (5), $|x(t)| \leq e^{|t|/\tau}|x_0|$, so the supremum in (10) is at most the value at $t = 0$, viz. $|x_0|$. Thus $|\beta(0)| \leq K^{-1}|x_0|$ and so $|x_+(0)| \leq (\frac{1}{K\tau} + \frac{1}{2})|x_0|$. So $|P_+| \leq \frac{1}{K\tau} + \frac{1}{2}$. Similarly for $|P_-|$.

To prove invariance of $P_\pm$ under the linear flow, let $\tilde{P}_\pm(t) = X(t,0)P_\pm(0)X(0,t)$. They are complementary projections at t and the forward, respectively backward orbits from ran $\tilde{P}_\pm(t)$ are bounded. The latter condition determines $P_\pm(t)$ uniquely, however, so $P_\pm(t) = \tilde{P}_\pm(t)$. Hence the invariance condition $X(t,0)P_\pm(0) = P_\pm(t)X(t,0)$.

To derive exponentially decaying bounds for $x_\pm$, consider the modified operator

$$L_\mu\zeta = \dot{\zeta} - A\zeta - \mu\psi\zeta, \tag{11}$$

where

$$\psi(t) = t/\tau \text{ for } |t| < \tau$$

and sgn(t) otherwise (see Figure 2). For $|\mu| < K$, L_μ is invertible by Lemma 2.1, with $\|L_\mu^{-1}\|^{-1} \geq K - |\mu|$. We will take $\mu \geq 0$ here. Let

$$\Psi(t) = \int_0^t \psi(u)\ du = \frac{t^2}{2\tau} \text{ for } |t| < \tau, \tag{12}$$

and $|t| - \frac{\tau}{2}$ otherwise.

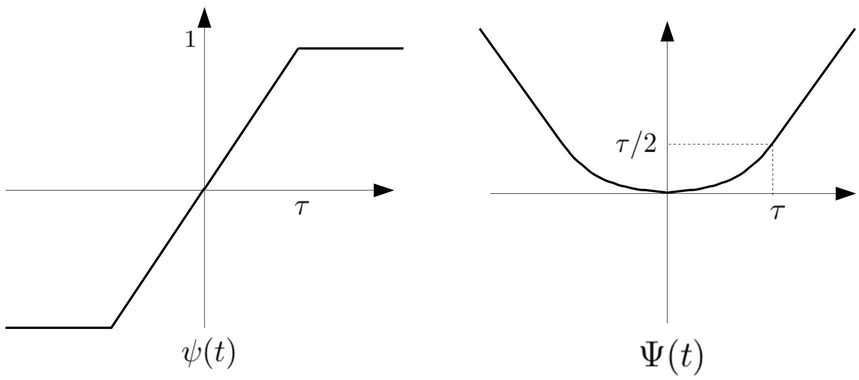

Figure 2: Graphs of the functions ψ and Ψ.

Let $\tilde{x} = xe^{\mu\Psi}$, $\tilde{\eta} = \phi\tilde{x}$. Then being of compact support, $\tilde{\eta}$ is bounded. Let $b\tilde{e}ta = L_\mu^{-1}$ $tilde\eta$, so $|\tilde{\beta}|_1 \leq |\tilde{\eta}|_0/(K-\mu)$. Since $\tilde{\beta}$ is bounded then so is $\beta = \tilde{\beta}e^{-\mu\Psi}$, and $L_\mu\tilde{\beta} = \tilde{\eta}$ implies that $L\beta = \eta$. Now $\tilde{\eta} = \phi xe^{\mu\Psi}$. The restriction $\mu < K$ implies also $\mu < 1/\tau$ because $K \leq |A| \leq 1/\tau$. Thus $\tilde{\eta}$ is maximum at 0 and $|\tilde{\eta}| = |x_0|$. So $|\tilde{\beta}|_1 \leq |x_0|/(K-\mu)$. In particular

$$|\beta(t)| \leq \frac{e^{-\mu\Psi(t)}|x_0|}{K-\mu}. \tag{13}$$

Now if $x_0 \in$ ran P_+ then $x_+ = \frac{\beta}{\tau} + (\frac{1}{2} - \frac{\Phi}{\tau})x_+$, so

$$x_+ = \frac{\beta}{\Phi + \tau/2}$$

and thus

$$|x_+(t)| \leq \frac{e^{-\mu\Psi}|x_0|}{(K-\mu)(\Phi+\tau/2)}.$$

So

$$|x_+(t)| \leq C(t)e^{-\mu t}|x_0| \text{ with } C(t) = \frac{e^{-\mu(\Psi - t)}}{(K-\mu)(\Phi+\tau/2)}. \tag{14}$$

For $t \geq 0$, $\Psi(t) - t = \Phi(t)$, so $C(t)$ is maximised over $t \geq 0$ at $t = 0$, giving the bound

$$C(t) \leq C = \frac{2}{(K-\mu)\tau} \text{ for } t \geq 0. \tag{15}$$

Note that this bound can be improved to

$$C(t) \leq \frac{e^{\mu\tau/2}}{(K-\mu)\tau} \tag{16}$$

for $t \geq \tau$.

Proceed similarly for $x_0 \in \text{ran } P_-$ and negative time. $\square$

Remark 2.3.

(i) One can optimise the decay estimate (14) over μ. For example, using the bound (15) the optimum over μ is at $\mu = K - \frac{1}{t}$ for $t \geq 1/K$ so

$$|x_+(t)| \leq \frac{2t}{\tau} e^{1-Kt} |x_0| \text{ for } t \geq \frac{1}{K}$$

(and $\leq \frac{2}{K\tau}|x_0|$ for $t \in [0, K^{-1}]$ by taking $\mu = 0$). Or one can use the improved bound (16), valid for $t \geq \tau$, to obtain

$$|x_+(t)| \leq (\frac{t}{\tau} - \frac{1}{2}) e^{1-K(t-\tau/2)} |x_0| \text{ for } t \geq \frac{3}{2K}.$$

(ii) The functions ϕ and ψ could be chosen asymmetrically, and one could use different values of μ for positive and negative time; if the resulting operator (call it L_{μ_+,μ_-}) happens to remain invertible for larger values of one or both of $\mu_\pm$ then stronger decay estimates follow.

(iii) Of particular importance is the *attracting case* defined by $P_- = 0$; then much of the analysis simplifies; we shall deal with the simplifications in a later section.

Definition 2.4. The *Green function* for a uniformly hyperbolic linear system is the matrix function on $\mathbb{R}^2$ defined by

$$\begin{aligned} G(t,s) &= X(t,s)P_+(s) \text{ for } t > s, \\ &\quad -X(t,s)P_-(s) \text{ for } t < s. \end{aligned} \tag{17}$$

Given $s \in \mathbb{R}$, $G(t,s)$ is the unique bounded solution of $\dot{x}(t) = A(t)x(t)$ for $t \neq s$ with $G(s+,s) - G(s-,s) = I$. Note that by invariance of the projections, $G(t,s)$ can also be written as $P_+(t)X(t,s)$ for $t > s$, $-P_-(t)X(t,s)$ for $t < s$, and that $\partial_2 G(t,s) = -G(t,s)A(s)$ for $s \neq t$, $G(t,t-) - G(t,t+) = I$.

Theorem 2.5. *The following results hold:*

(i) *The unique bounded response $x = L^{-1}[f]$ of (5) to bounded forcing f can be written*

$$x(t) = \int G(t,s)f(s)\ ds. \tag{18}$$

(ii) *For any $\mu \in [0,K)$ there exists $D(\mu)$ such that $|G(t,s)| \le De^{-\mu|t-s|}$.*

(iii) *If $|f(s)| \le e^{\mu|s|}$ for some $\mu \in [0,K)$ then $|x(0)| \le (K-\mu)^{-1}$.*

(iv) *If $\mu \in [0,K), T > 0$ and $f(s) = 0$ for all $s \in (-T,T)$ then $|x(0)| \le \frac{e^{-\mu T}}{K-\mu}|f|$; optimising over $\mu \in [0,K)$ yields*

$$|x(0)| \le Te^{1-KT}|f| \text{ for } T \ge 1/K.$$

Proof.

(i) The Green function G behaves sufficiently nicely that the formula defines a bounded function x. Splitting the integral at $s = t$ we can differentiate the integrand and with respect to the limits to obtain

$$\dot{x}(t) = \int A(t)G(t,s)f(s)\ ds + P_+(t)f(t) + P_-(t)f(t) = A(t)x(t) + f(t),$$

but $L^{-1}[f]$ is the unique bounded solution of $\dot{x} = Ax + f$.

(ii) A bound on $|G(t,s)|$ of the given form can already be obtained by composition of those of the previous theorem for the projections and the evolution of vectors in their ranges, but it will be useful to sharpen the estimate as follows. Repeat the estimates using L_μ as in the proof of the previous theorem to obtain (13). Then $x_+ = \frac{\beta}{\tau} + (\frac{1}{2} - \frac{\Phi}{\tau})x$ implies

$$e^{\mu t}|x_+(t)| \le \left(\frac{e^{\mu(t-\Psi(t))}}{\tau(K-\mu)} + (\frac{1}{2} - \frac{\Phi}{\tau})e^{\mu t}e^{t/\tau} \right) |x_0| \le D|x_0|, \tag{19}$$

for $t \ge 0$, with $D = \frac{e^{\mu\tau/2}}{\tau(K-\mu)} + \frac{1}{2}$ (because the expression in parentheses is bounded by its value at $t = 0$). So $|x_+(t)| \le De^{-\mu t}|x_0|$ for $t \ge 0$. Similarly $|x_-(t)| \le De^{-\mu|t|}|x_0|$ for $t \le 0$. This result could be optimised over μ if desired.

(iii) If $|f(s)| \le e^{\mu|s|}$ then $Lx = f$ is equivalent to $L_{-\mu}\tilde{x} = \tilde{f}$ with $\tilde{x} = e^{-\mu\Psi}x$ and $\tilde{f} = e^{-\mu\Psi}f$, where $L_{-\mu}$ is as defined in (11) but using the opposite sign of μ and Ψ is as defined in (12) except now we shall allow its value of τ to differ from that in the definition of the norm $|.|_1$. Then $\|L_{-\mu}^{-1}\|^{-1} \ge K - \mu$ for $\mu \in [0,K)$, so

$$|\tilde{x}| \le |\tilde{f}|/(K-\mu). \tag{20}$$

This holds for all $\tau > 0$ so we can take τ to 0 to obtain $|x(0)| \le 1/(K-\mu)$.

(iv) If f is a bounded function with $f(s) = 0$ for all $s \in (-T, T)$ then again (20) with $\tau \to 0$ gives $|x(0)| \le \frac{e^{-\mu T}}{K-\mu}|f|$. The minimum over $\mu \in [0, K)$ is achieved at $\mu = K - 1/T$ for $T \ge 1/K$, giving the optimised result.

□

It will be useful to combine scenarios (iii) and (iv) of Theorem 2.5.

Theorem 2.6. *If $\alpha < K \le \|L^{-1}\|^{-1}$, $|f(s)| \le F$, $|f(s)| \le \varepsilon e^{\alpha|s|}$ for $|s| \le T$, $x = L^{-1}[f]$, $|t| \le T - 1/K$, then*

$$|x(t)| \le \frac{\varepsilon}{K-\alpha} e^{\alpha|t|} + (T - |t|)e^{1-K(T-|t|)}F. \tag{21}$$

Proof. Firstly, consider $f_{<T} = f\chi_{[-T,T]}$, where χ_A denotes the indicator function of a subset A. This may be discontinuous at $\pm T$, but we can consider L as mapping $W^{1,\infty}$ to $W^{0,\infty}$ and L remains invertible with the same bound. Then applying (iii) above, $|L^{-1}[f_{<T}](t)| \le \frac{\varepsilon}{K-\alpha}e^{\alpha|t|}$. Next consider $f_{>T} = f\chi_{\{s:|s|\ge T\}}$. Applying (iv) above, $|L^{-1}[f_{>T}](t)| \le (T - |t|)e^{1-K(T-|t|)}F$ for $T - |t| \ge 1/K$. Adding the two provides the result for $|t| \le T - 1/K$. □

Note that one can obtain a similar bound on $\dot{x}(t)$ if desired, but using $e^{\alpha\Psi(t)}$ instead of $e^{\alpha|t|}$.

The use of this result is to suppose that ε is small and that we can take $T = \frac{1}{\gamma}\log\frac{F}{\varepsilon}$ for some $\gamma \in (\alpha, K)$. Then roughly speaking the first term of (21) dominates for $|t|/T < \frac{K-\gamma}{K-\alpha}$, which is a positive fraction. T goes to infinity as $\varepsilon \to 0$. Hence $|x(t)| \le \frac{\varepsilon+o(\varepsilon)}{K-\alpha}e^{\alpha|t|}$, uniformly on any bounded interval of t.

Let us do this properly. The ratio ρ of the second term of (21) to the first can be written as $ye^{1-y}x$, where $y = (K - \alpha)(T - |t|)$ and $x = \frac{F}{\varepsilon e^{\alpha T}}$. So $\rho \le 1$ when $y \ge g(x)$ where g is the inverse function to e^{y-1}/y on $y \ge 1$. An upper bound for $g(x)$ is $\bar{g}(x) = \log(2ex\log(ex))$ (exercise). So $y \ge \bar{g}(x)$ implies the second term is at most the first. Using the above choice of T we have $\log x = (\gamma - \alpha)T$. Thus the second term is at most the first when $(K - \alpha)(T - |t|) \ge (\gamma - \alpha)T + \log(2e(1 + (\gamma - \alpha)T))$, i.e.

$$\begin{aligned}|t| &\le T - \frac{1}{K-\alpha}\left((\gamma-\alpha)T + \log(2e(1+(\gamma-\alpha)T))\right)\\ &= \frac{K-\gamma}{K-\alpha}T - \frac{1}{K-\alpha}\log(2e(1+(\gamma-\alpha)T)).\end{aligned}$$

Similarly, for any $p > 0$, we obtain $\rho \le p$ if $y \ge g(x/p)$, which is true if

$$|t| \le \frac{K-\gamma}{K-\alpha}T - \frac{1}{K-\alpha}\log(\frac{2e}{p}(1 + (\gamma-\alpha)T + \log\frac{1}{p})).$$

2.3 Continuity of the splitting

If a uniformly hyperbolic family of linear systems (3) is generated by $A(t) = \mathcal{A}(y(t), t)$ for some Lipschitz matrix function $\mathcal{A}$ evaluated along the orbits $y()$ of a Lipschitz vector field $\dot{y}(t) = u(y, t)$ then it can be necessary to know how the projections $P_\pm(t)$, without loss of generality $t = 0$, depend on the initial condition $y(0)$. The difficulty is that trajectories from two nearby initial conditions y_0, y_1 may separate arbitrarily far and the bound on how much A changes:

$$|A_1 - A_0| \leq Var(\mathcal{A}) = \sup_{y_1, y_0, t} |\mathcal{A}(y_1, t) - \mathcal{A}(y_0, t)|$$

is in general insufficient to apply Lemma 2.1 and in any case is insensitive to $|y_1 - y_0|$. Thus we have to work harder. The best that one can obtain in general is Hölder continuity. We will present one such estimate. Continuity of the splitting can be proved in more general contexts (e.g. for a general closed invariant subset for $\dot{y} = u(y, t)$), but the above suffices for present purposes.

Before proving Hölder continuity of the splitting, we note one simple consequence of the continuity of $P_\pm$ in the finite-dimensional case: their ranks are constant on connected components, because if P' has greater rank than P, then by counting dimensions $\ker P \cap \operatorname{ran} P'$ contains a non-zero v and we have $Pv = 0$, $P'v = v$, so $|P - P'| \geq 1$, which is not arbitrarily small.

The simplest continuity estimate for the splitting is obtained by using the formula

$$L_1^{-1} - L_0^{-1} = L_1^{-1}(L_0 - L_1)L_0^{-1} \tag{22}$$

for the difference between the inverses of two invertible linear operators. In our case $L_0 - L_1 = A_1 - A_0$, thus

$$\Delta P_+(0) = \Delta G(0+, 0) = \int G_1(0+, s)\Delta A(s) G_0(s, 0)\; ds, \tag{23}$$

where ΔQ denotes the change $Q_1 - Q_0$ in any quantity Q. Now $|\Delta A| \leq V = Var(\mathcal{A}) \leq 2|\mathcal{A}|$ but for s near 0 we can do better. Specifically, if λ and α are Lipschitz constants with respect to y for u and $\mathcal{A}$ respectively then $|\Delta y(t)| \leq e^{\lambda|t|}|\Delta y(0)|$, so $|\Delta A(t)| \leq \alpha e^{\lambda|t|}|\Delta y(0)|$. We can now employ the estimate $|G(t, s)| \leq De^{-\mu|t-s|}$ of Theorem 2.5(ii) to obtain

$$|\Delta P_+(0)| \leq \int D^2 e^{-2\mu|s|} \min(\alpha e^{\lambda|s|}|\Delta y(0)|, V)\; ds$$

(one could optimise over μ). Supposing $|\Delta y_0| \leq V/\alpha$, let $s^* \geq 0$ be the value such that $\alpha e^{\lambda s^*}|\Delta y(0)| = V$, so $e^{\lambda s^*} = \frac{V}{\alpha|\Delta y(0)|}$. Then

$$\begin{aligned}
|\Delta P_+(0)| &\leq D^2 \left(\int_{-s^*}^{s^*} \alpha|\Delta y_0| e^{(\lambda - 2\mu)|s|}\; ds + \int_{|s| > s^*} V e^{-2\mu|s|}\; ds \right) \\
&= 2D^2 \left(\alpha|\Delta y_0| \frac{e^{(\lambda-2\mu)s^*} - 1}{\lambda - 2\mu} + V \frac{e^{-2\mu s^*}}{2\mu} \right),
\end{aligned}$$

where the first fraction is interpreted as s^* if $\lambda = 2\mu$. Put $\xi = \alpha|\Delta y_0|/V \leq 1$. Then this evaluates to

$$|\Delta P_+(0)| \leq 2VD^2 \frac{\xi^{2\mu/\lambda} - \xi}{\lambda - 2\mu} + \frac{VD^2}{\mu}\xi^{2\mu/\lambda}, \tag{24}$$

showing that $P_+(0)$ is Hölder continuous with respect to y_0, with exponent $2\mu/\lambda$ if $2\mu < \lambda$, any exponent less than 1 if $2\mu = \lambda$, and exponent 1 if $2\mu > \lambda$. Note that if $2\mu > \lambda$ then $P_+(0)$ is Lipschitz in y_0, with

$$|\Delta P_+(0)| \leq \frac{2\alpha D^2}{2\mu - \lambda}|\Delta y_0|,$$

by subdividing Δy_0 into N equal steps, adding the estimates (24) and taking the limit as $N \to \infty$.

The estimates can be sharpened and the hypotheses weakened (e.g. Hölder continuity of $\mathcal{A}$ and u), and comparing A at widely different points doesn't have an intrinsic meaning so a different approach (e.g. exponentially weighted spaces?) should give stronger results, but we shall develop the specific estimates we shall need at the time.

2.4 Uniform hyperbolicity for pseudo-orbits

Next we suppose $\|L_{y_0}^{-1}\| \leq K^{-1}$ for the trajectories of all initial conditions y_0 of $\dot{y} = u(y,t)$ for a vector field u with Lipschitz constant λ with respect to y, and wish to show that $\tilde{L}$ is invertible for all trajectories $\tilde{y}$ of a nearby vector field $\tilde{u}$, and $\|\tilde{L}^{-1}\|$ is at most a little larger than K^{-1}.

There are various approaches. A nice one (cf. [Pa]) is to construct approximate right and left inverses T and U in the sense that $\|I - \tilde{L}T\| = \varepsilon_T < 1$, $\|I - U\tilde{L}\| = \varepsilon_U < 1$, so that $\tilde{L}T$ and $U\tilde{L}$ are invertible and their inverses have norms at most $1/(1-\varepsilon_T)$, $1/(1-\varepsilon_U)$ respectively. Then $T(\tilde{L}T)^{-1}$ is a true right inverse to $\tilde{L}$ and $(U\tilde{L})^{-1}U$ is a true left inverse, so $\tilde{L}$ is invertible. Finally, one should show that T or U is bounded and then deduce $\|\tilde{L}^{-1}\| \leq \|T\|/(1-\varepsilon_T)$ or $\|U\|/(1-\varepsilon_U)$ respectively.

Even within this approach there are various possible choices for the approximate inverses, each of which has merits. One is to define T to be the operator with kernel $\bar{G}(t,s) = \tilde{X}(t,s)P_+(s)$ for $t \in (s, s+T)$, $-\tilde{X}(t,s)P_-(s)$ for $t \in (s-T, s)$, and zero elsewhere, where $\tilde{X}$ is the linear flow for the perturbed trajectory $\tilde{y}$, $P_\pm(s)$ are the projections for the unperturbed trajectory through $(\tilde{y}(s), s)$ and $T \in \mathbb{R}$ (not to be confused with the operator T) is chosen large enough that the unperturbed Green functions have decayed to a size significantly less than 1. Then for $\Delta u = \tilde{u} - u$ small enough, ε_T can be shown to be small and $\|T\|$ not much larger than K^{-1}. The key is to bound the difference between $\bar{G}(t,s)$ and the Green function for the unperturbed trajectory from s, in terms of $|\Delta u|$: the deviations can grow with t but if Δu is small enough they remain small up to time T, which is as far as we need to evaluate them. An advantage of this choice of operator T is that in the attracting (or repelling) case one can take the time T to infinity and obtain the exact Green function, since then P_+ (respectively P_-) $= I$. Nevertheless, estimating how large it is in not particularly simple.

For the left inverse U one might try to use kernel $\bar{G}(t,s) = P_+(t)\tilde{X}(t,s)$ for $t \in (s, s+T)$ and similar for $t \in (s-T, s)$. However, $U\tilde{L}$ is a mapping from C^1 to C^1 so we have to estimate the derivative too, but $P_+(t)$ might not be differentiable with respect to t (because it is evaluated on different trajectories of the unperturbed flow) and this requires us to use a smoother approximation. One option is to take $\bar{G}(t,s) = \frac{1}{2a}\int_{t-a}^{t+a} \tilde{X}(t,\sigma)P_+(\sigma)\tilde{X}(\sigma,s)\ d\sigma$ for $t \in (s, s+T)$, similar for $t \in (s-T, s)$ and zero elsewhere, with a of order τ. This is feasible, but we decided on a different choice.

Our choice is to take

$$T[f](t) = \int ds \frac{1}{2a}\int_{t-a}^{t+a} d\sigma\ \bar{G}_\sigma(t,s)f(s),$$

where $\bar{G}_\sigma$ is the Green function for $L_\sigma x(t) = \dot{x}(t) - A_\sigma(t)x(t)$ with $A_\sigma(t) = \mathcal{A}(y(t),t)$ along the unperturbed orbit y of $(\tilde{y}(\sigma),\sigma)$ but changed to $\mathcal{A}(\tilde{y}(t),t)$ when the difference between $\mathcal{A}(\tilde{y}(t),t)$ and $\mathcal{A}(y(t),t)$ is less than some small positive $\eta < K$ to be chosen (I should make A relax continuously to make sure $\bar{G}$ depends continuously on σ and to make ΔA continuous), and a is some duration of order τ. Note that it makes sense to integrate $\bar{G}_\sigma$ over σ because it depends continuously on σ (extend continuity of the splitting analysis!). We also take $U = T$. (I would have liked to make T use $\bar{G}_s$ instead of this average over $\bar{G}_\sigma$ with σ near t and thought I could get $\|T\| \leq 1/(K-\eta)$ by considering the dual action on C^{1*} but was not convinced).

When $\tilde{u}$ is close to u, $A_\sigma(t) = \mathcal{A}(\tilde{y}(t),t))$ for at least a long interval $|t-\sigma| \leq T$. Let us determine a sufficient T. The difference $\Delta y(t)$ between the perturbed and unperturbed trajectories starting at $(\tilde{y}(\sigma),\sigma)$ evolves by

$$\dot{\Delta y} = \tilde{u}(\tilde{y},t) - u(y,t) = \Delta u(\tilde{y},t) + (u(\tilde{y},t) - u(y,t)), \tag{25}$$

starting from $\Delta y(\sigma) = 0$, and the second term is at most $\lambda \Delta y$. So

$$|\Delta y(t)| \leq \int_\sigma^t ds\ e^{\lambda|s-\sigma|}|\Delta u(\tilde{y}(s),s)| \leq \frac{e^{\lambda|t-\sigma|}-1}{\lambda}|\Delta u|. \tag{26}$$

Dropping the -1 which is not worth saving, and taking Lipschitz constant α for $\mathcal{A}$ as in the previous subsection, we obtain

$$|\Delta A(t)| \leq \alpha|\Delta y(t)| \leq \frac{\alpha}{\lambda}e^{\lambda|t-\sigma|}|\Delta u|. \tag{27}$$

Thus $|\Delta A(t)| \leq \eta$ for all $|t-\sigma| \leq T$ if

$$e^{-\lambda T} = \frac{\alpha}{\lambda}\frac{|\Delta u|}{\eta}. \tag{28}$$

This estimate could be improved considerably if we specified more about the unperturbed system. For example, if with respect to a Riemannian metric $\langle , \rangle$ on Y we have $\lambda_- \leq \langle \delta y, u_y \delta y\rangle / \langle \delta y, \delta y\rangle \leq \lambda_+$ then $|\Delta y(t)| \leq \frac{e^{\lambda_+(t-\sigma)}-1}{\lambda_+}|\Delta u|$ for $t > \sigma$ and similar for

$t < \sigma$. Since λ_+ and $-\lambda_- \leq \lambda$, a gain is possible. Note if $\lambda_+ = 0$ then $\frac{e^{\lambda_+ t}-1}{\lambda_+}$ should be interpreted as t.

Having determined the time T by (28) we are now ready to bound the operator T. Firstly, for arbitrary $f \in C^0$ we bound $T[f]$ in C^0. For each σ, $\int ds\ \bar{G}_\sigma(t,s)f(s) \leq |f|/(K-\eta)$ because A was changed by at most η along an unperturbed trajectory. Thus averaging over an interval of σ produces $|T[f](t)| \leq |f|/(K-\eta)$. This estimate can be improved if $\lambda < K$ because then the change in $A(t)$ is much smaller than η for t in an interval about σ of length almost T, but in the interests of simplicity we will not take this into account here.

Next, we bound $T[f]$ in C^1. To take care of the jump in $\bar{G}_\sigma(t,s)$ at $s=t$, we write

$$T[f](t) = (\int_{-\infty}^{t} + \int_{t}^{+\infty})ds\ \frac{1}{2a}\int_{t-a}^{t+a} d\sigma\ \bar{G}_\sigma(t,s)f(s)$$

and now differentiate with respect to t to obtain $\tau\partial_t(T[f])(t) =$

$$\frac{\tau}{2a}\int(\bar{G}_{t+a}(t,s)-\bar{G}_{t-a}(t,s))f(s)\ ds + \frac{\tau}{2a}\int d\sigma\left(\int A_\sigma(t)\bar{G}_\sigma(t,s)f(s)\ ds + f(t)\right). \quad (29)$$

The second term is just the average over σ of $\tau\partial_t L_\sigma^{-1}[f](t)$ so is bounded by $|f|/(K-\eta)$ (because $\|L_\sigma^{-1}\| \leq 1/(K-\eta)$ as an operator from C^0 to C^1). In fact, if we choose τ a bit smaller than $\frac{1}{2}|A|^{-1}$ then we can reduce this bound a bit, thereby allowing to absorb the contribution of the first integral, but we won't do that here.

Note interchanges of order of integration and differentiation under the integral sign and with respect to limits are all OK.

To bound the first integral in (29), we use the same idea as in the proof of continuity of the splitting:

$$\int \Delta\bar{G}(t,s)f(s)\ ds = \int dr\ \bar{G}_{t+a}(t,r)\Delta A(r)\int ds\ \bar{G}_{t-a}(r,s)f(s). \quad (30)$$

Now $|\int ds\ \bar{G}_{t-a}(r,s)f(s)| \leq |f|/(K-\eta)$, $|\Delta A| \leq V$ and $\Delta A(r) = 0$ for $|r-t| \leq T-a$, so applying Theorem 2.5(iv) we obtain

$$|\int \Delta\bar{G}(t,s)f(s)\ ds| \leq \varepsilon|f|/(K-\eta), \quad (31)$$

where

$$\varepsilon = (T-a)e^{1-(K-\eta)(T-a)}V, \quad (32)$$

provided $(K-\eta)(T-a) \geq 1$, which is true if Δu is small enough.

Combining the bounds for the two terms of (29), we obtain

$$\tau|\partial_t T[f](t)| \leq (1+\frac{\tau}{2a}\varepsilon)|f|/(K-\eta). \quad (33)$$

So

$$\|T\| \leq (1+\frac{\tau}{2a}\varepsilon)/(K-\eta), \quad (34)$$

which is only slightly larger than K^{-1} if $\eta \ll K$ and Δu is small enough (to make T large and hence ε small).

To optimise the result, it is useful to choose η to depend on $|\Delta u|$ in such a way as to make the corrections in the numerator and denominator of roughly equal relative size. By combining (28) and (32), this is achieved approximately by taking $\eta \propto |\Delta u|^{K/(K+\lambda)}$. Specifically (32) says $\varepsilon \approx Ve^{-KT}$ (on a logarithmic scale of approximation), so $\varepsilon \frac{\tau}{2a} = \eta/K$ if $\eta \approx KV\frac{\tau}{2a}e^{-KT}$. But (28) says $\eta = \frac{\alpha}{\lambda}|\Delta u|e^{\lambda T}$ so eliminating T between these two equations yields

$$\eta \approx \left(\frac{KV\tau}{2a}\right)^{\frac{\lambda}{K+\lambda}} \left(\frac{\alpha}{\lambda}|\Delta u|\right)^{\frac{K}{K+\lambda}}.$$

Next we estimate $I - \tilde{L}T$.

$$(I - \tilde{L}T)[f](t) = f(t) - (\partial_t - \tilde{A}(t))(\int_{-\infty}^{t} + \int_{t}^{+\infty})ds\ \frac{1}{2a}\int_{t-a}^{t+a} d\sigma\ \bar{G}_\sigma(t,s)f(s). \tag{35}$$

This evaluates to

$$-\frac{1}{2a}\int ds\ \Delta G(t,s)f(s) + \frac{1}{2a}\int d\sigma \int ds\ \Delta A_\sigma(t)G_\sigma(t,s)f(s). \tag{36}$$

But $|\Delta A_\sigma(t)| = 0$ for $|t-\sigma| \le T$, so taking $|\Delta u|$ small enough that $T > a$, we have only the first term, which we bounded in (31), so

$$\|I - \tilde{L}T\| \le \frac{\varepsilon}{2a(K-\eta)}. \tag{37}$$

Then we estimate $I - T\tilde{L}$. This is an operator from C^1 to C^1 so we have to bound both its value acting on any C^1 function x and the value of its derivative.

$$(I - T\tilde{L})[x](t) = x(t) - \frac{1}{2a}\int_{t-a}^{t+a} d\sigma(\int_{-\infty}^{t} + \int_{t}^{+\infty})ds\ \bar{G}_\sigma(t,s)(\partial_s - \tilde{A}(s))x(s). \tag{38}$$

Integrating by parts transforms this to

$$\frac{1}{2a}\int d\sigma \int ds\ \bar{G}_\sigma(t,s)\Delta A_\sigma(s)x(s), \tag{39}$$

and $\Delta A_\sigma(s) = 0$ for $|s-\sigma| \le T$, hence for $|s-t| \le T-a$, so it can be bounded by $\varepsilon|x|$ (with ε as in (32)).

Next we bound the derivative of (39).

$$\begin{aligned}\partial_t \frac{1}{2a}\int_{t-a}^{t+a} d\sigma(\int_{-\infty}^{t} + \int_{t}^{+\infty})ds\ \bar{G}_\sigma(t,s)\Delta A_\sigma(s)x(s) = \qquad (40)\\ \frac{1}{2a}\int (\bar{G}_{t+a}(t,s)\Delta A_{t+a}(s) - \bar{G}_{t-a}(t,s)\Delta A_{t-a}(s))x(s)\ ds + \\ \frac{1}{2a}\int (\bar{G}_\sigma(t,t-) - \bar{G}_\sigma(t,t+))\Delta A_\sigma(t)x(t)\ d\sigma + \\ \frac{1}{2a}\int\int A_\sigma(t)\bar{G}_\sigma(t,s)\Delta A_\sigma(s)x(s)\ dsd\sigma.\end{aligned}$$

The second term is zero because $\Delta A_\sigma(t) = 0$ for $\sigma \in (t-a, t+a)$. The third term has $\Delta A_\sigma(s) = 0$ for $|s-t| \le T-a$, so is bounded by $\varepsilon |A||x|$. Similarly, each term of the first integral is bounded by $\frac{\varepsilon}{2a}$ (one could rewrite the first integrand to one term of this size and a correction of the squared size, saving a factor of nearly 2). Thus

$$\tau |\partial_t (I - T\tilde{L})[x](t)| \le (\frac{\tau}{a} + \tau|A|)\varepsilon |x| \tag{41}$$

(XXX it is a bit surprising that this does not depend on $|\dot{x}|$; is there a mistake?).

So finally

$$\|I - T\tilde{L}\| \le 2\varepsilon, \tag{42}$$

where we have chosen $a \ge \tau$.

The conclusion is that for $\varepsilon = V(T-a)e^{1-(K-\eta)(T-a)} < \min(\frac{1}{2}, 2aK)$ we have both $\|I - \tilde{L}T\|$ and $\|I - T\tilde{L}\| < 1$, so $\tilde{L}$ is invertible and

$$\|\tilde{L}^{-1}\|^{-1} \ge \frac{(1-2\varepsilon)(K-\eta)}{1+\varepsilon/2} \tag{43}$$

(using the bound on $\|I - T\tilde{L}\|$).

Choosing $\eta \propto |\Delta u|^{K/(K+\lambda)}$ we obtain $\|\tilde{L}^{-1}\|^{-1} \ge K - O(|\Delta u|^{K/(K+\lambda)})$.

Note one can also tackle this by reducing to discrete time. Also it would be better to use constructions that generalise naturally to manifolds.

Knowing now that the pseudo-orbit is uniformly hyperbolic, one can construct its true Green function $\tilde{G}$ by taking $\tilde{E}_-(t) = \lim_{s\to-\infty} \tilde{X}(t,s)E_-(s)$, which exists because the forwards dynamics applied to subspaces is contracting near the unperturbed E_- subspace, and $\tilde{E}_+(t) = \lim_{s\to+\infty} \tilde{X}(t,s)E_+(s)$, constructing complementary projections $\tilde{P}_\pm(t)$ to have these as ranges, and then letting $\tilde{G}(t,s) = \tilde{X}(t,s)\tilde{P}_+(s)$ for $t > s$, $-\tilde{X}(t,s)\tilde{P}_-(s)$ for $t < s$.

2.5 Attracting case

3 Normal hyperbolicity

4 Application to non-autonomous oscillator

5 Synchronisation

6 Discussion

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