The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

2292-1

School and Conference on Analytical and Computational Astrophysics

14 - 25 November, 2011

Self Organization of Magnetospheric Plasma Vortex: Equilibrium on a Foliated Phase Space

Zensho Yoshida
University of Tokyo, Graduate School of Frontier Sciences
Japan

Strada Costiera 11, 34151 Trieste, Italy - Tel.+39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it

Self-Organization of Magnetospheric Plasma Vortex: Equilibrium on a Foliated Phase Space

Z. Yoshida
Graduate School of Frontier Sciences, The University of Tokyo, Kashiwa, Chiba 277-8561, Japan.
(Dated: November 11, 2011)

Inhomogeneous magnetic field (most typically, a dipole magnetic field) gives rise to various interesting properties of plasmas which are degenerate in homogeneous (or zero) magnetic fields. Here we show that a rotating vortex with a steep density gradient (and very high beta value) is self-organized in the vicinity of a magnetic dipole. Its nontrivial structure (heterogeneity) yet maximizes the entropy on a relevant phase space that is a macroscopic leaf of scale hierarchy; we formulate the foliation by a Casimir invariant that is created by separating (quantizing) a microscopic action-angle variables. The Casimir invariant (action) measures the number of quasi-particles, and the corresponding chemical potential yields density clump in response to the heterogeneity of the energy level (frequency).

PACS numbers: 52.25.Dg, 05.20.Dd, 45.20.Jj, 52.35.We

Self-organization of a structure is, at its surface, an antithesis of the entropy *ansatz*. However, disorder can still develop at microscopic scale while a structure emerges on some macroscopic scale; it seems more common in various nonlinear systems that order and disorder are simultaneous [1], and such co-existence may be possible if the self-organization and the entropy principle [2] work on different scales. Therefore we have to write a theory of self-organization as a discourse on *scale hierarchy* [3].

Scale hierarchy is a popular keyword in various arguments on "structures"; a biological body is a typical example in which an evident hierarchical structure is programmed to establish, enabling effective consumption of energy and materials as well as emission of entropy and wasts. But the theory of a physical macro-system — a collective system of "simple" elements, like a gravitational system or a plasma— hinges on a different framework; a scale hierarchy is not "programed" to emerge, or structures are not subject to some functions; yet one can observe a more fundamental and elementary process of *creation* in nonlinear dynamics.

Here we study the self-organization of a magnetospheric plasma vortex [4]. By this analysis, we delineate a clear and distinct scale hierarchy in terms of Hamiltonian mechanics and foliation, and show how a structure (heterogeneity) can emerge while maximizing the entropy.

I. LABORATORY MAGNETOSPHERE RT-1

Magnetospheres are self-organized structures found commonly in the Universe. A dipole magnetic field sets the stage for charged particles to cause a variety of interesting phenomena. To explore the basic physics of plasma in a strongly inhomogeneous dipole magnetic field, we have constructed the RT-1 device [5, 6], on which we have demonstrated stable confinement of a very high beta ($\beta \approx 1$) hot electron ($T_e > 10$ keV) plasma with a long confinement time ($\tau_E \approx 0.5$ s) [7]. This system (most common in nature, but new in laboratories) will have wide applications in confining various single- or multi-species charged particles (for example, highly charged ions, antimatter, or possibly high-temperature fusion plasmas) in a compact space [8–19].

Here we describe the observation of "inward diffusion" of particles resulting in the "spontaneous confinement."

A. Laboratory Magnetosphere

The RT-1 device levitates a superconducting (high-Tc superconductor Bi-2223) ring magnet in a vacuum chamber and produces a magnetospheric configuration (Fig. 1) [20]. The field strength in the confinement region varies from 0.3 T to 0.03 T. The conductor is first cooled to 20 K in the maintenance chamber (located at the bottom of the plasma chamber), and, then, charged to 0.25 MA (the coil consists of 12 pancakes and has a total of 2160 turns). After detaching the current leads and coolant (He gas) transfer tubes, the ring is moved up to the mid-plane of the plasma chamber and is then levitated by a feedback-controlled magnet installed on the top of the device. Three-cord laser sensors measure the position of the levitated ring. The super-conducting operation can continue for 7 hours before the coil temperature increases to 30 K. Current decay is less than 1% after 7 hours.

The plasma is produced by injecting X-mode microwaves (8.2 GHz maximum 25 kW, and 2.45 GHz maximum 20 kW). Direct heating of ions by ICH is under development.

B. Inward Diffusion

So-called *inward diffusion* (or up-hill diffusion) of magnetized particles, often observed in planetary magnetospheres [21–23], is the objective of the present study. Laboratory magnetic dipole systems simulate similar behavior of particles [17–19].

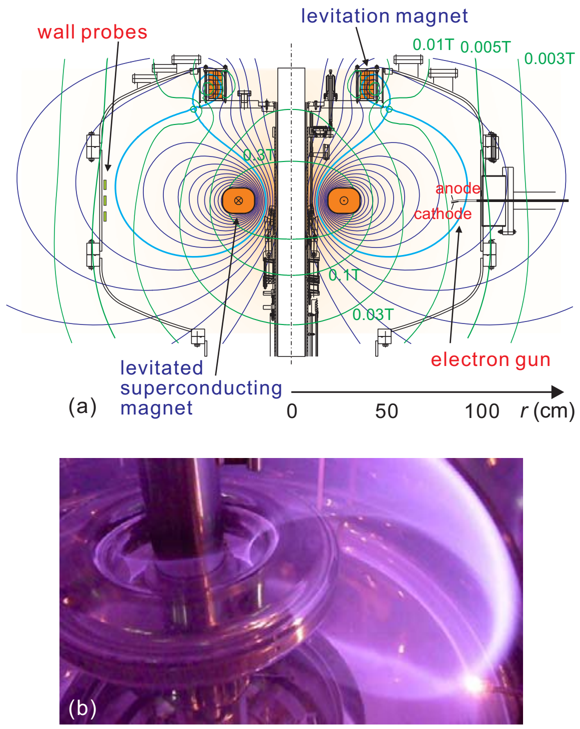


FIG. 1: The RT-1 magnetospheric plasma system. (a) A dipole magnetic field is produced by the levitating superconducting ring magnet. The magnetic field lines (contours of the flux function ψ) and the field-strength contours are shown. The field strength in the confinement region varies from 0.5 T to 0.01 T. (b) The magnetic surface is visualized by injecting electrons into hydrogen gas at a pressure $P_n = 1 \times 10^{-2}$ Pa from an electron gun located at $r = 0.70$ m on the mid-plane, producing weakly ionized plasma.

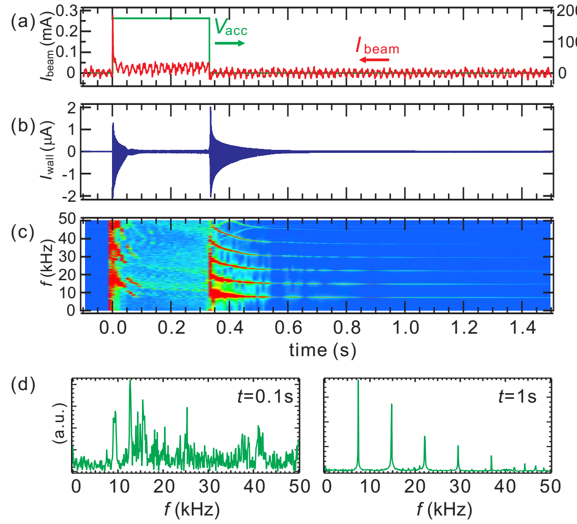


FIG. 2: The formation and sustainment of a magnetospheric electron plasma: (a) The acceleration voltage V_{acc} and the beam current I_{beam} of the electron gun. (b) The electrostatic fluctuation measured by a wall probe. (c) The evolution of the frequency spectrum of the fluctuation. (d) In the driven phase, the fluctuation has a broad spectrum ($t = 0.1$ s). In the confined phase, the small-amplitude fluctuation has a coherent peaked spectrum ($t = 1$ s).

A very clear evidence of *inward diffusion* is observed by producing a pure electron plasma. A single-species plasma has a strong internal electric field. Confinement is possible only if the electric field ($\boldsymbol{E}$) is balanced by an induction ($\boldsymbol{v} \times \boldsymbol{B}$) generated by a vortical motion ($\boldsymbol{v}$) in the magnetic field ($\boldsymbol{B}$). The flow and the electromagnetic field achieve stable coupling by self-organizing a vortex [24–27]: Galaxies are close cousins, in which rotation creates a balance between gravitational and centrifugal forces.

Figure 2 shows the typical formation process. Soon after the start of injection (injection point r =0.8 m, injection energy eV_{acc} = 175 eV, beam current $I_{beam} \sim 300\,\mu$A), a charged cloud is created, which repels the beam and diminishes the current to about 10^{-5} A; see Fig. 2 (a). Figure 2 (b) shows the electrostatic fluctuation measured by a wall probe (a small piece of metallic plate facing the plasma), and Fig. 2 (c) shows its frequency spectrum. The initial turbulent phase, in which the fluctuation reaches around 50% of the ambient electric field, quenches after about 0.05 s. As we show below, the turbulence drives particles inward. While the beam current is supplied, the fluctuation is rather strong and has a complex spectrum. When the beam current is stopped ($t = 0.32$ s), the plasma becomes turbulent and then relaxes into a quiescent state, in which the fluctuation level is less than 1%; through the transient turbulent phase, the *driven* state re-organizes into the *confined* state. By "triangulation," using an array of wall probes, we can estimate the location of the charged clump [28]. Electrons in the outer region (intersecting the electron gun and its mechanical support) are lost immediately after the beam is stopped. The remaining particles (typically 10^{-8} C which amounts to about 40 % of the total particles in the driven phase) move slowly inward and reside in a stronger field region. The quiescent phase continues for more than 300 s.

In Figs. 2 (c) and (d), we find a remarkable difference between the driven and confined states. In the confined state, the frequency spectrum is sharply localized around 10 kHz and its higher harmonics. The oscillations are highly coherent and propagate in the toroidal direction; the mode number of the dominant component is 1.

We observe a clear evidence of *inward diffusion*: the particles are pushed further and create an inward density gradient. The driver of the turbulence, Kelvin-Helmholtz (KH) instability, is active as long as the rotation has a shear, so a turbulence-free quiescent state is realized only in a rigidly rotating vortex. If KH instability tends to self-organize a *relaxed state* and stabilize itself, turbulence-induced diffusion will cast the charge distribution into a specific profile that rotates rigidly; this does in fact happen, as we observe in the experiment, and the relaxed state turns out to be not flat. The

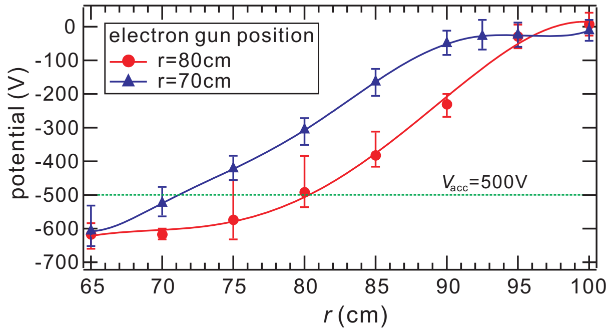


FIG. 3: The radial profile of the space potential ϕ_s measured by an emissive Langmuir probe. Two different gun positions are compared (circles: $r_{gun} = 80$ cm, triangles: $r_{gun} = 70$ cm). Electrons penetrate into the inner region ($r < r_{gun}$), where $-e\phi_s > V_{acc}$. The probe measurement can be performed only in the *driven phase*, because insertion of the probe damages the confinement.

angular frequency ω, the flux function ψ, and the electric potential ϕ are related by $\omega = \nabla\phi/\nabla\psi$. For ω to be homogeneous, the density n of the charged particles (by Poisson's equation, $n \propto \nabla^2\phi$) must have a profile such that $n \propto B$ [29]. The strength B of a dipole magnetic field is strongly inhomogeneous, so the density of the relaxed state must have a steep inward gradient. This is in marked contrast to the homogeneous-density equilibrium of a Penning-Malmberg trap, which also rotates rigidly in a homogeneous magnetic field [26].

Interestingly, these two different equilibria are unified because the factor B, which dominates the density distribution, is the Jacobian weight for the map from the action $\mu = E_\perp/\omega_c$ to the perpendicular energy $E_\perp = mv_\perp^2/2$ ($\omega_c = eB/m$, $v_\perp$: perpendicular velocity with respect to the magnetic field). If the velocity distribution is a function of the action μ (instead of the energy $E_\perp$, as predicted by the adiabatic invariance theorem), the velocity-space integral $dv_\perp^2/2$ operates as $(\omega_c/m)d\mu$ to yield the laboratory-frame density n, including the Jacobian $\omega_c \propto B$ [8, 30].

In a frame moving with the $\boldsymbol{E} \times \boldsymbol{B}$ drift velocity, the electric field vanishes, so particles are seemingly subject to no force. Hence, it might be thought that the density will become homogeneous. Indeed, it does so if $\boldsymbol{B}$ is homogeneous. However, particles are subject to magnetic force, and the equilibrium selects a different quantity, the particle number per unit magnetic flux tube, to homogenize [30, 31].

Because the action μ, which dictates the distorted phase space in which the distribution homogenizes, must be constant, particles are accelerated as they diffuse toward the larger-B region (so-called betatron acceleration). In fact, by probing the charged cloud with an emissive Langmuir probe, we find that the space potential is higher than the initial energy imparted to the electrons by the gun, so the particles must be accelerated to climb the potential hill. The energy can be supplied by the turbulent electric field that drives the diffusion. Figure 3 shows the space potentials (ϕ_s) measured at two different positions of the electron gun ($r_{gun} = 80$ and 70 cm). In both cases, the potential energy $-e\phi_s$ at the position of the electron gun agrees with the gun's acceleration voltage $V_{acc} = 500$ V, implying that the magnetosphere is fully charged. In the inner region ($r < r_{gun}$), $-e\phi_s$ increases beyond V_{acc}, clearly showing inward diffusion.

II. SEPARATION OF ACTION-ANGLE VARIABLES AND SCALE HIERARCHY

The inward diffusion and self-organization of a density clump near the dipole is the subject of present study. The inward diffusion is driven by some spontaneous fluctuations (symmetry breaking) that violates the constancy the the angular momentum; in a strong enough, symmetric magnetic field, the canonical angular momentum P_θ is dominated by the charge q multiple of the flux function ψ (Gauss' potential of the magnetic field), thus the conservation of $P_\theta \approx q\psi$ constrains the charged particle on a magnetic surface (level-set of ψ). Perturbed by a random-phase fluctuations, particles can diffuse across magnetic surfaces. Although the "diffusion" is normally a process of diminishing gradients, numerical experiments do exhibit preferential inward shifts through random motions of test particles [32, 33]. Detailed specification of the fluctuations or the microscopic motion of particles is not the subject of present discussion. The aim of this work is to construct a clear-cut description of the self-organized state, the goal of the diffusion in which the density has a steep gradient, but is consistent with the entropy principle.

We describe the scale hierarchy by a phase-space foliation, and explain the self-organization (creation of heterogeneity) by a distortion of the metric (invariant measure) dictating the "equipartition" —the Jacobian on the leaf of the macroscopic hierarchy measures the distortion. In a strongly inhomogeneous magnetic field (typically a dipole magnetic field), the phase-space metric of magnetized particles is distorted, producing reciprocal inhomogeneity in the laboratory-fame flat space; even if particles distribute homogeneously on some relevant phase space, its projection onto the laboratory frame yields peaked profile by the multiplication of the inhomogeneous Jacobian weight [34] (this idea was proposed in the pioneering work of A. Hasegawa [8, 31]; a comparison of the present theory will be given in the latter discussion). We will embody this scenario by invoking a Hamiltonian formalism of the magnetized charged particles and the Boltzmann distribution on a foliated phase space endowed with a *non-canonical* Poisson bracket [35].

A. Hamiltonian of charged particle

To analyze the effect of magnetization in an inhomogeneous magnetic field (specifically, a dipole magnetic field), we invoke the hierarchical Hamiltonian structure built by adiabatic invariants (actions). The Hamiltonian of a charged particle is a sum of the kinetic energy and the potential energy:

$$H = \frac{m}{2}v^2 + q\phi, \tag{1}$$

where $\boldsymbol{v} := (\boldsymbol{P} - q\boldsymbol{A})/m$ is the velocity, $\boldsymbol{P}$ is the canonical momentum, $(\phi, \boldsymbol{A})$ is the electromagnetic 4-potential, m is the mass, and q is the charge. In the present work, we may treat electrons and ions equally (in the latter discussions, we will neglect ϕ assuming charge neutrality). Denoting by $\boldsymbol{v}_\parallel$ and $\boldsymbol{v}_\perp$ the parallel and perpendicular (with respect to the local magnetic field) components of the velocity, we may write

$$H = \frac{m}{2}v_\perp^2 + \frac{m}{2}v_\parallel^2 + q\phi. \tag{2}$$

We will also denote $\boldsymbol{p} := m\boldsymbol{v}$, $\boldsymbol{p}_\parallel := m\boldsymbol{v}_\parallel$, and $\boldsymbol{p}_\perp := m\boldsymbol{v}_\perp$.

B. Creation of an action-angle pair by magnetization

In a strong magnetic field, $\boldsymbol{v}_\perp$ can be decomposed into a small-scale cyclotron motion $\boldsymbol{v}_c$ and a macroscopic guiding-center drift motion $\boldsymbol{v}_d$. The periodic cyclotron motion $\boldsymbol{v}_c$ can be *quantized* to write $(m/2)v_c^2 = \mu\omega_c(\boldsymbol{x})$ in terms of the magnetic moment μ and the cyclotron frequency $\omega_c(\boldsymbol{x})$; μ (adiabatic invariant) and $\vartheta_c := \omega_c t$ (the gyration phase) constitute an action-angle pair. In the standard interpretation, in analog of quantum theory, ω_c is the energy level, and μ is the "number" of quasi-particles (quantized periodic motions) in the corresponding energy level.

We consider an axisymmetric system with a poloidal magnetic field (no toroidal magnetic field): Let (ψ, ζ, θ) be a magnetic coordinate system such that $\boldsymbol{B} = \nabla\psi \times \nabla\theta$ (θ is the toroidal angle in which the system is assumed to be homogeneous) and $\boldsymbol{B} = \nabla\xi = B\nabla\zeta$ [36]. The macroscopic part of the perpendicular kinetic energy can be written as $mv_d^2/2 = (P_\theta - q\psi)^2/(2mr^2)$, where v_d is the drift velocity, P_θ is the angular momentum in the azimuthal direction and r is the radius from the geometric axis. The canonical variables are $\boldsymbol{z} = (\vartheta_c, \mu, \zeta, p_\parallel, \theta, P_\theta)$ [37]. The Hamiltonian is now

$$H_c = \mu\omega_c + \frac{1}{2m}\frac{(P_\theta - q\psi)^2}{r^2} + \frac{1}{2m}p_\parallel^2 + q\phi, \tag{3}$$

which describes the motion of the guiding center (or, the quasi-particle). Note that the energy of the cyclotron motion has been quantized in term of the frequency $\omega_c(\boldsymbol{x})$ and the action μ; the gyro-phase ϑ_c has been coarse grained (integrated to yield 2π).

C. Boltzmann distribution

The standard Boltzmann distribution function is derived when we assume that the Lebesgue measure $d^3v d^3x$ is an invariant measure and the Hamiltonian H is the determinant of the ensemble; maximizing the entropy

$$S = -\int f \log f d^3v d^3x \tag{4}$$

under the constraints on the total energy

$$E = \int H f d^3v d^3x \tag{5}$$

and the total particle number $N = \int f d^3v d^3x$, we obtain

$$f(\boldsymbol{x}, \boldsymbol{v}) = Z^{-1}e^{-\beta H}, \tag{6}$$

where Z is the normalization factor ($\log Z - 1$ is the Lagrange multiplier on N) and β is the inverse temperature (the Lagrange multiplier on E). The corresponding configuration-space density is

$$\rho(\boldsymbol{x}) = \int f d^3v \propto e^{-\beta q\phi}, \tag{7}$$

which becomes homogeneous if the charge neutrality condition applies ($\phi \equiv 0$).

Needless to say, the Boltzmann distribution or the corresponding configuration-space density, with an appropriate Jacobian multiplication [34], is independent of the choice of coordinates on the phase space. Moreover, the density is invariant no matter whether we quantize the cyclotron morion or not. Let us confirm this fact by direct calculation. The Boltzmann distribution of the quantized plasma is

$$\begin{aligned} f(\mu, v_d, v_\parallel; \boldsymbol{x}) &= Z^{-1}e^{-\beta H_c} \\ &= Z^{-1}e^{-\beta\left(\mu\omega_c(\boldsymbol{x}) + mv_d^2/2 + mv_\parallel^2/2 + q\phi(\boldsymbol{x})\right)}. \end{aligned} \tag{8}$$

The density is given by

$$\rho(\boldsymbol{x}) = \int f d^3v = \int f \frac{2\pi\omega_c}{m} d\mu dv_d dv_\parallel \propto e^{-\beta q\phi}, \tag{9}$$

which reproduces (7).

D. Equilibrium on macroscopic hierarchy

Now we formulate the *scale hierarchy* as a foliation of the phase space. The adiabatic invariance of the magnetic moment μ (or, the *number* of the quantized quasi-particles) poses a *topological constraint* on the motion of particles; it is this constraint that determines a macroscopic hierarchy and creates a structure there. To explain how the scale hierarchy is formulated, we start by the general (micro-macro total) formulation, and then separate the microscopic action-angle pair μ-ϑ_c; the *macroscopic phase space* is the remaining sub-manifold immersed in the general phase space, which we delineate as a leaf of foliation in terms of a *Casimir element* [35].

The Poisson bracket on the total phase space, span by the canonical variables $\boldsymbol{z} = (\vartheta_c, \mu; \zeta, p_\parallel; \theta, P_\theta)$, is defined as

$$\{F, G\} := \langle \mathcal{J}\partial_{\boldsymbol{z}} F, \partial_{\boldsymbol{z}} G\rangle,$$

where $\langle \boldsymbol{u}, \boldsymbol{v}\rangle := \int u^j v_j d^6 z$ is the inner-product and $\mathcal{J}$ is the canonical symplectic matrix (Poisson tensor):

$$\mathcal{J} := \begin{pmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{pmatrix}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{10}$$

Given the Hamiltonian H_c, the equation of motion is written as $dz^j/dt = \{H_c, z^j\}$. Notice that the quantization of the cyclotron motion has suppressed the change of μ. Liouville's theorem determines the invariant measure $d^6 z$, by which we obtain the Boltzmann distribution (8).

To define the macroscopic hierarchy, we "separate" the microscopic variables (ϑ_c, μ) by modifying the symplectic matrix as

$$\mathcal{J}_{nc} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{pmatrix}. \tag{11}$$

The Poisson bracket, determining the kinematics on the macroscopic hierarchy, is defined as $\{F, G\}_{nc} := \langle \mathcal{J}_{nc}\partial_{\boldsymbol{z}} F, \partial_{\boldsymbol{z}} G\rangle$. The macroscopic kinetic equation reads as $\partial_t f + \{H_c, f\}_{nc} = 0$, which reproduces the familiar drift-kinetic equation (see for example [38]).

The nullity of $\mathcal{J}_{nc}$ makes the Poisson bracket $\{\ ,\ \}_{nc}$ *non-canonical* [35], i.e., there is a non-trivial $C(\boldsymbol{z})$ such that $\{G, C\}_{nc} = 0$ for every G. We call $C(\boldsymbol{z})$ a Casimir element (since $\{H_c, C\} \equiv 0$, C is an invariant). Evidently, μ is a Casimir element (more generally, we may put $C = g(\mu)$ with any smooth function g). The level-set of $C(\boldsymbol{z}) = \mu$, a leaf of the Casimir foliation, identifies what we call the *macroscopic hierarchy*.

By Liouville's theorem applying to the Poisson bracket $\{\ ,\ \}_{nc}$, the invariant measure on the macroscopic hierarchy is $d^4 z = d^6 z/(2\pi d\mu)$, the modulo by the microscopic measure of the total phase-space measure. The most probable state (statistical equilibrium) on the macroscopic ensemble must maximize the entropy with respect to this invariant measure. We can perform the variational principle by the standard recipe of Lagrange multiplier —immersing the macroscopic hierarchy into the general phase space, and applying the constraint with a Lagrange multiplier: we maximize entropy $S = -\int f \log f\, d^6 z$ for a given particle number $N = \int f d^6 z$, a quasi-particle number $M = \int \mu f d^6 z$, and an energy $E = \int H_c f d^6 z$, to obtain a distribution function

$$f = f_\alpha := Z^{-1} e^{-(\beta H_c + \alpha\mu)}, \tag{12}$$

where α, β and $\log Z - 1$ are, respectively the Lagrange multipliers on M, E, and N. Evidently, $\{H_c, f\}_{nc} = 0$, i.e., this f is a stationary solution of the macroscopic kinetic equation. Giving grand-canonical interpretation for this distribution function, we may read α as the chemical potential of the quasi-particle [39].

The factor $e^{-\alpha\mu}$ in f_α yields a direct ω_c dependence of the configuration-space density:

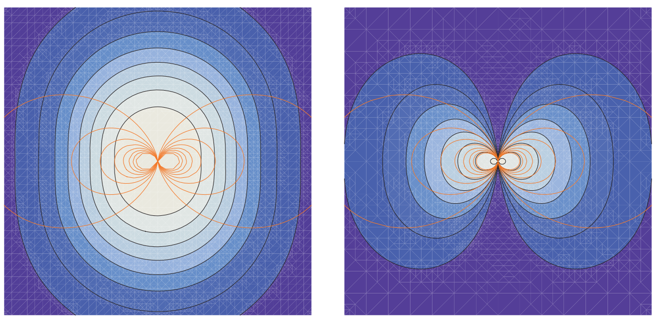
$$\rho = \int f_\alpha \frac{2\pi\omega_c}{m} d\mu dv_d dv_\parallel \propto \frac{\omega_c(\boldsymbol{x})}{\beta\omega_c(\boldsymbol{x}) + \alpha}, \tag{13}$$

which is compared with the density (9) evaluated for the Boltzmann distribution (here we put $\phi = 0$ assuming charge neutrality). Notice that the Jacobian $(2\pi\omega_c/m)d\mu$ multiplies on the macroscopic measure $d^4 z$, reflecting the distortion of the macroscopic phase space (Casimir leaf) due to the magnetic field. Figure 4-(A) shows the density distribution and the magnetic field lines.

E. Macro-scale action-angle pairs in an axisymmetric system

In an axisymmetric system, the quasi-particle motion is periodic in both the parallel and θ directions, creating macroscopic action-angle pairs: $J_\parallel$-$\vartheta_\parallel$ ($:= \sin^{-1}(\zeta/\ell_\parallel)$; $\ell_\parallel$ is the bounce orbit length) and P_θ ($\approx q\psi$)-θ (for a hierarchy of adiabatic invariants, see [40] and papers cited there). To give explicit expressions, we invoke the Hamiltonian H_c of the form of (3). Here we neglect the curvature effect [41]. We also neglect ϕ assuming charge neutrality. Then, the equation of the parallel motion reads as

$$\begin{cases} \dfrac{d\zeta}{dt} = \dfrac{\partial H_c}{\partial p_\parallel} = \dfrac{p_\parallel}{m}, \\[2ex] \dfrac{dp_\parallel}{dt} = -\dfrac{\partial H_c}{\partial \zeta} = -\mu\nabla_\parallel \omega_c, \end{cases} \tag{14}$$



(A) (B)

FIG. 4: Density distribution (contours) and the magnetic field lines (level-sets of ψ) in the neighborhood of a point dipole. (A) The equilibrium distribution on the leaf of μ-foliation. (B) The equilibrium distribution on the leaf of μ and $J_\parallel$-foliation.

which combine to yield

$$m\frac{d^2}{dt}\zeta = -\mu\nabla_\parallel \omega_c. \tag{15}$$

We approximate

$$\omega_c = \Omega_c(\psi) + \Omega_c''(\psi)\zeta^2/2$$

on each magnetic surface (level-set of ψ), where $\Omega_c(\psi)$ is the ω_c evaluated at $\zeta = 0$ (which is the minimum of ω_c on each magnetic surface), and $\Omega_c''(\psi) := d^2\omega_c/d\zeta^2|_\psi$. We will denote

$$L_\parallel(\psi) := \sqrt{2\Omega_c(\psi)/\Omega_c''(\psi)},$$

which scales the variation of ω_c along ζ. Integrating (15), we obtain the parallel periodic motion $\zeta = \ell_\parallel \sin\vartheta_\parallel$, $\vartheta_\parallel = \omega_b t$ with the bounce frequency

$$\omega_b = \sqrt{\frac{\Omega_c''(\psi)\mu}{m}} = \frac{v_\perp}{L_\parallel(\psi)}, \tag{16}$$

and bounce amplitude $\ell_\parallel = \sqrt{2E_\parallel/(m\omega_b^2)}$, where

$$E_\parallel := (mv_\parallel^2)/2|_{\zeta=0}.$$

Assuming $E_\parallel \approx E_\perp := \mu\Omega_c$, we estimate $\ell_\parallel \approx L_\parallel$. Defining

$$J_\parallel := \frac{1}{2\pi}\oint m v_\parallel d\zeta,$$

we may write $E_\parallel = J_\parallel\omega_b$, and $dv_\parallel = (\omega_b/mv_\parallel)dJ_\parallel = \sqrt{\omega_b/2mJ_\parallel}dJ_\parallel$. Using the relation $\omega_b/(mv_\parallel) = v_\perp/(L_\parallel m v_\parallel) \approx 1/(mL_\parallel)$, we may also write $dv_\parallel \approx (1/mL_\parallel)dJ_\parallel$.

By quantizing the parallel action-angle pair $J_\parallel$-$\vartheta_\parallel$, we may consider a further constrained equilibrium distribution function:

$$f_{\alpha,\gamma} = Z^{-1}e^{-(\beta H_c + \alpha\mu + \gamma J_\parallel)}, \tag{17}$$

which yields

$$\begin{aligned}\rho &= \int f_{\alpha,\gamma}\frac{2\pi\omega_c d\mu}{m}\frac{dJ_\parallel}{mL_\parallel(\psi)}dv_d \\ &\propto \frac{\omega_c(\boldsymbol{x})}{m^2}\int_0^\infty \frac{e^{-(\beta\omega_c+\alpha)\mu}d\mu}{\beta\sqrt{2\omega_c\mu/m} + \gamma L_\parallel(\psi)}.\end{aligned} \tag{18}$$

Through $L_\parallel(\psi)$, the density ρ has a dependence on ψ. We may estimate $L_\parallel(\psi) \sim \psi^{-1}$. Numerical integration of (18) gives a density profile as shown in Fig. 4-(B).

Finally, we solve the equation of motion in the azimuthal (θ) direction, and show that the self-organized clump of density is a "vortex". Averaging the cyclotron and bounce motions, the Hamiltonian becomes (neglecting ϕ and approximating $P_\theta \approx q\psi$)

$$H_{cb} = \mu\omega_c(\psi,\zeta) + J_\parallel\omega_b(\psi,\mu). \tag{19}$$

The governing equation of the canonical pair $q\psi$-θ is

$$\begin{cases} \dfrac{d}{dt}\theta = \dfrac{\partial H_{cb}}{\partial(q\psi)} = \dfrac{\mu}{q}\dfrac{\partial\omega_c}{\partial\psi} + \dfrac{J_\parallel}{q}\dfrac{\partial\omega_b}{\partial\psi} =: \omega_d, \\ \dfrac{d}{dt}(q\psi) = -\dfrac{\partial H_{cb}}{\partial\theta} = 0. \end{cases} \tag{20}$$

We thus find that a particle rotates with a frequency ω_d.

III. CONCLUDING REMARKS

We have derived a grand-canonical distribution function f_α (or $f_{\alpha,\gamma}$) on a macroscopic phase space that is separated from the microscopic action-angle variables (the action, then, reads as the number of the quasi-particles abstracting the microscopic variables). In embedding the macroscopic leaf into the laboratory-frame flat phase space, a Jacobian weight multiplies to yield an inhomogeneous density profile. Evidently, f_α (or $f_{\alpha,\gamma}$) is a particular solution of the stationary kinetic equation $\{H_c, f\}_{nc} = 0$. A general solution, freed from the maximum-entropy condition, may be written as $f = F(H_c, \mu, J_\parallel)$. Especially $f = F(\mu, J_\parallel)$ yields a density such that $\rho \propto \omega_c/L_\parallel$, which, in a dipole magnetic field, scales as $\propto r^{-4}$; this the density profile given by A. Hasegawa [8, 31], and is the asymptotic form of (18) in the limit $r \to \infty$ ($\omega_c \to 0$).

We end this lecture with comparing the chosen example of self-organization and the familiar narrative based on the integral fluid invariants like a *helicity* [42, 43]. The helicity is a Casimir element of the fluid/magneto-fluid equation (Appendix), and the "Taylor relaxed state" (or the *Beltrami field* such that $\nabla\times\boldsymbol{B} = \lambda\boldsymbol{B}$) is the minimum energy state on the "helicity leaf" immersed in an infinite-dimensional function space. We can construct a grand canonical distribution of (second-quantized) Beltrami fields (the Beltrami parameter λ is the chemical potential) [44].

Appendix (Hamiltonian form of MHD)

An ideal MHD plasma endowed with a Hamiltonian

$$\mathcal{H} := \int_\Omega \left\{ n\left[\frac{V^2}{2} + \epsilon(n)\right] + \frac{B^2}{2}\right\} d^3x,$$

and a *non-canonical* symplectic operator

$$\mathcal{J}(\boldsymbol{u}) := \begin{pmatrix} 0 & -\nabla\cdot & 0 \\ -\nabla & -n^{-1}(\nabla\times\boldsymbol{V})\times & n^{-1}(\nabla\times\circ)\times\boldsymbol{B} \\ 0 & \nabla\times\left[\circ\times n^{-1}\boldsymbol{B}\right] & 0 \end{pmatrix}.$$

The state vector is $\boldsymbol{u} := {}^t(n, \boldsymbol{V}, \boldsymbol{B})$, which is normalized in the standard Alfvén units (n is the density, $\boldsymbol{V}$ is the fluid velocity, $\boldsymbol{B}$ is the magnetic field, and $\epsilon(n)$ is the thermal energy density; here we assume a barotropic relation ($\partial[n\epsilon(n)]/\partial n = h(n)$ is the molar enthalpy, and $n\nabla h = \nabla p$ is the pressure force). We assume that $\Omega \subset \mathbb{R}^3$ is a smoothly bounded domain, and the vector fields $\boldsymbol{V}$ and $\boldsymbol{B}$ are confined in Ω, i.e., we impose boundary conditions (denoting by $\boldsymbol{n}$ the unit normal vector onto the boundary $\partial\Omega$) $\boldsymbol{n} \cdot \boldsymbol{V} = 0$ and $\boldsymbol{n} \cdot \boldsymbol{B} = 0$. When Ω is multiply connected, these boundary conditions are not sufficient to determine a unique solution, and we have to fix the flux.

The MHD equation is written in a Hamiltonian form

$$\partial_t \boldsymbol{u} = \mathcal{J}(\boldsymbol{u})\partial_{\boldsymbol{u}}\mathcal{H}(\boldsymbol{u}), \tag{21}$$

which is evidently equivalent to the well-known system of equations:

$$\begin{cases} \partial_t n = -\nabla \cdot (\boldsymbol{V} n), \\ \partial_t \boldsymbol{V} = -(\boldsymbol{V} \cdot \nabla)\boldsymbol{V} + n^{-1}\left[(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \nabla p\right], \\ \partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}). \end{cases}$$

The operator $\mathcal{J}(\boldsymbol{u})$ has three independent Casimir elements

$$C_1(\boldsymbol{u}) := \int_\Omega n \, d^3x. \tag{22}$$

$$C_2(\boldsymbol{u}) := \int_\Omega \boldsymbol{V} \cdot \boldsymbol{B} \, d^3x, \tag{23}$$

$$C_3(\boldsymbol{u}) := \frac{1}{2}\int_\Omega \boldsymbol{A} \cdot \boldsymbol{B} \, d^3x, \tag{24}$$

The equation of motion (21) is invariant against the transformation $\mathcal{H}(\boldsymbol{u}) \mapsto \mathcal{H}_\mu(\boldsymbol{u}) := \mathcal{H}(\boldsymbol{u}) - \sum \mu_j C_j(\boldsymbol{u})$ (each μ_j is a constant number); we have an equivalent representation of the equation of motion:

$$\partial_t \boldsymbol{u} = \mathcal{J}(\boldsymbol{u})\partial_{\boldsymbol{u}}\mathcal{H}_\mu(\boldsymbol{u}). \tag{25}$$

The transformed Hamiltonian $\mathcal{H}_\mu(\boldsymbol{u})$ is called an *energy-Casimir functional.* The *Beltrami field* is an equilibrium point of the energy-Casimir functional, i.e. the solution of

$$\partial_{\boldsymbol{u}}\left(\mathcal{H}(\boldsymbol{u}) - \sum_{j=1}^{3}\mu_j C_j(\boldsymbol{u})\right) = 0, \tag{26}$$

which reads as

$$V^2/2 + h - \mu_1 = 0, \tag{27}$$

$$n\boldsymbol{V} - \mu_2 \boldsymbol{B} = 0, \tag{28}$$

$$\nabla \times \boldsymbol{B} - \mu_3 \boldsymbol{B} - \mu_2 \nabla \times \boldsymbol{V} = 0, \tag{29}$$

In deriving (29), we have operated curl. The "Taylor relaxed state" is given by putting $\boldsymbol{V} = 0$ and $n = 1$ (or choosing only C_3 to define the energy-Casimir functional). See also [45] for a generalization to hall MHD system, in which much more nontrivial structures stem from richer Casimir elements associated with the non-canolicality of the symplectic operator.

[1] A. Hasegawa, Adv. Phys. **34**, 1 (1985).
[2] The notion of "entropy" should be generalized to dictate an open system. For example, entropy production rate is often deemed applicable; G. Nicolis and I. Prigogine, *Self-organization in nonequilibrium systems* (John Wiley & Sons, New York, 1977).
[3] Z. Yoshida, *Nonlinear Science —The Challenge of Complex Systems* (Springer-Verlag, Berlin-Heidelberg, 2010).
[4] The experimental work is a collaboration with the RT-1 project members. The theoretical work is a collaboration with Swadesh M. Mahajan.
[5] Z. Yoshida *et al.*, Plasma Fusion Res. **1**, 008 (2006).
[6] Y. Ogawa *et al.*, Plasma Fusion Res. **4**, 020 (2009).
[7] H. Saitoh *et al.*, Nucl. Fusion **51**, 063034 (2011).
[8] A. Hasegawa, Comments Plasma Phys. Contr. Fusion **1**, 147 (1987).
[9] A. Hasegawa *et al.*, Nucl. Fusion **30**, 2405 (1990).
[10] P. Zaveri *et al.*, Phys. Rev. Lett. **68**, 3295 (1992).
[11] Z. Yoshida *et al.*, in *Non-Neutral Plasma Physics III*, AIP Conf. Procs. No. **498** (AIP, New York, 1999), p. 397.
[12] T. S. Pedersen and A. H. Boozer, Phys. Rev. Lett. **88**, 205002 (2002).
[13] H. Saitoh *et al.*, Phys. Rev. Lett. **92**, 255005 (2004).
[14] D. T. Garnier *et al.*, Phys. Plasmas **13**, 056111 (2006).
[15] J. P. Kremer *et al.*, Phys. Rev. Lett. **97**, 095003 (2006).
[16] J. P. Marler and M. R. Stoneking, Phys. Rev. Lett. **100**, 155001 (2008).
[17] A. C. Boxer *et al.*, Nature Phys. **6**, 207 (2010).
[18] Z. Yoshida *et al.*, Phys. Rev. Lett. **104**, 235004 (2010).
[19] H. Saitoh *et al.*, Phys. Plasmas **17**, 112111 (2010).
[20] Z. Yoshida *et al.*, Fusion Sci. Tech. **51**, 29 (2007).
[21] M. Schulz and L. J. Lanzerotti, *Particle Diffusion in the Radiation Belts* (Springer, New York, 1974).
[22] D. H. Brautigam and J. M. Albert, J. Geophys. Res. **105**, 291 (2000).
[23] Y. Chen, G. D. Reeves, and R. H. W. Friedel, Nature Phys. **3**, 614 (2007).
[24] J. H. Malmberg and J. S. deGrassie, Phys. Rev. Lett. **35**, 577 (1975).
[25] J. H. Malmberg and C. F. Driscoll, Phys. Rev. Lett. **44**, 654 (1980).
[26] D. H. E. Dubin and T. M. O'Neil, Rev. Mod. Phys. **71**, 87 (1999).
[27] T. M. O'Neil, Phys. Today **52**, 24 (1999).
[28] H. Saitoh *et al.*, Plasma Fusion Res. **4**, 054 (2009).
[29] The rotation velocity may be approximated by the $\boldsymbol{E}\times\boldsymbol{B}$ drift velocity $r(\nabla\phi \cdot \nabla\psi)/|\nabla\psi|^2\boldsymbol{e}_\theta$. If the vortex rotates with a uniform angular frequency ω_0, we may put $\phi = \omega_0\psi + \phi_\parallel$, where $\nabla\phi_\parallel \cdot \nabla\psi = 0$. In the confinement region, $\nabla \times \boldsymbol{B} = 0$, which reads as $\mathcal{L}\psi =$

$r\partial_r(r^{-1}\partial_r\psi) + \partial_z^2\psi = 0$. If $\phi_\parallel = 0$, we observe $\nabla^2\phi = \omega_0\nabla^2\psi = \omega_0(\mathcal{L}\psi + 2r^{-1}\partial_r\psi) = 2\omega_0 B_z$. Poisson's equation then yields $n = 2\varepsilon_0\omega_0 B_z/e$. A finite parallel potential $\phi_\parallel$ confines particles near the equatorial region, and also modifies the foregoing n. In the limit of zero temperature, n is localized in the vicinity of the potential-minimum, so the electron cloud becomes a *thin disk*, in which we may approximate $n \propto B = B_z$. Finite-temperature solutions will be discussed elsewhere.

[30] At low temperature, the density of the relaxed state is localized in a thin disk [29]. Then we may write $n = \omega_c \int F(\mu)d\mu$, where $F(\mu)d\mu$ is the number of particles in a unit *magnetic flux tube* with an infinitesimal length, the thickness of the disk.

[31] A. Hasegawa, Phys. Scr. **T116**, 72 (2005).

[32] T.J. Birmingham, T.G. Northrop, and C.-G. Falthammar, Phys. Fluids **10**, 2389 (1967).

[33] M. Walt, Space Sci. Rev. **12**, 446 (1971).

[34] A density f (on an n-dimensional space) is an extensive quantity (n-form), which differs from an intensive quantity (scalar, or 0-form). A diffusion equation for f, with respect to an invariant measure $dx^1 \wedge \cdots \wedge dx^n$, is written as $\partial_t f = d(\mathcal{D}\delta f)$, where $\mathcal{D}$ is a diffusion coefficient, d is the exterior derivative (gradient) and $\delta = (-1)^{n+1} {}^*d^*$ is the codifferential (* is the Hodge star); a diffusion equation for a scalar ϕ is such that $\partial_t\phi = \delta(\mathcal{D}d\phi)$. Thus, the diffusion of f is a process of flattening ${}^*f$, while that of ϕ simply flattens ϕ. If we write $f = f(y^1, \cdots, y^n)$, ${}^*f = f(y^1, \cdots, y^n)D(y^1, \cdots, y^n)/D(x^1, \cdots, x^n)$.

[35] We endow the phase space X with a Lie-Poisson algebra; Let X be a phase space which has a finite dimension $2n$. For arbitrary smooth functions $a(\boldsymbol{z})$ and $b(\boldsymbol{z})$, a Poisson bracket is written as $\{a(\boldsymbol{z}), b(\boldsymbol{z})\} = \langle \partial_{\boldsymbol{z}} a(\boldsymbol{z}), \mathcal{J}\partial_{\boldsymbol{z}} b(\boldsymbol{z})\rangle$, where $\langle a, b\rangle$ is the inner product of X, and $\mathcal{J}$ is an antisymmetric linear map $X \to X$ (may be inhomogeneous on X). If the bracket $\{a, b\}$ satisfies Jacob's relations, it defines a Lie-Poisson algebra. A Casimir element $C(\boldsymbol{z})$ is such that $\{a, C\} = 0$ for all a, which constitutes the *center* of the Lie-Poisson algebra, In this Letter, we say that the Hamiltonian system is *non-canonical* when $\mathrm{Ker}(\mathcal{J}) = \mathrm{Coker}(\mathcal{J})$ contains non-trivial (non-zero) elements. If $\dim(\mathrm{Ker}(\mathcal{J})) = 2\nu < 2n$, there are 2ν Casimir elements (Darboux's theorem). For a review of non-canonical Hamiltonian systems (including infinite-dimensional systems), see P. J. Morrison, Rev. Mod. Phys. **70**, 467 (1998).

[36] The flux function (Gauss' potential) ψ is equal to rA_θ (A_θ is the azimuthal component of the vector potential $\boldsymbol{A}$), which is, for a point dipole of magnetic moment M, $\psi(r, z) = Mr^2(r^2+z^2)^{-3/2}$. On the other hand, the scalar potential is $\xi = Mz(r^2 + z^2)^{-3/2}$. The magnetic field strength is $B = M\sqrt{(r^4 + 5r^2z^2 + 4z^4)/(r^2 + z^2)^5}$.

[37] These variables do not span the configuration space, so the spatial dependence of ω_c will be written as $\omega_c(\boldsymbol{x})$ with invoking the natural coordinates $\boldsymbol{x} = (r, \theta, z)$. In the latter discussion, P_θ will be approximated by $q\psi$, then, ψ plays the role of a configuration-space coordinate.

[38] K. Nishikawa and M. Wakatani, *Plasma Physics*, 3rd ed. (Springer-Verlag, Berlin-Heidelberg, 2000), Sec. 4.3.

[39] We can also derive (12) by an *energy-Casimir function*. With a Casimir element μ, we can transform the Hamiltonian as $H_c \mapsto H_\alpha := H_c + \alpha\mu$ (α is an arbitrary constant) without changing the macroscopic dynamics; H_α is called an energy-Casimir function. The Boltzmann distribution with respect to H_α is equivalent to (12).

[40] A.J. Lichtenberg and M.A. Lieberman, *Regular and Chaotic Dynamics*, 2nd ed. (Springer-Verlag, New York, 1992), Sec. 2.3b.

[41] If $v_\parallel/R$ (R: curvature) is comparable to the bounce-frequency (16), we have to take into account the centrifugal force; see for example [38].

[42] L. Woltjer, Proc. Natl. Acad. Sci. USA **44**, 489 (1958).

[43] J.B. Taylor, Phys. Rev. Lett. **33**, 1139 (1974); Rev. Mod. Phys. **58**, 741 (1986).

[44] N. Ito and Z. Yoshida, Phys. Rev. E **53**, 5200 (1996).

[45] S.M. Mahajan and Z. Yoshida, Phys. Rev. Lett. **81**, 4863 (1998).