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The MHD Model
(magnetic flux, conservation laws, discontinuities, model problems)

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Outline

- **The ideal MHD equations:** postulating the basic equations, assumptions made, scale independence, what is a physical model?
- **Magnetic flux:** flux tubes, flux conservation
- **Conservation laws:** conservation form of the equations, conserved quantities
- **Discontinuities:** shocks and jump conditions, *boundary conditions* for interface plasmas
- **Model problems:** laboratory & astrophysical models
- **Applicability** issues

Based on…

"Principles of Magnetohydrodynamics:
with applications to laboratory and astrophysical plasmas",

J.P. Goedbloed and S. Poedts

Cambridge University Press, 2004.

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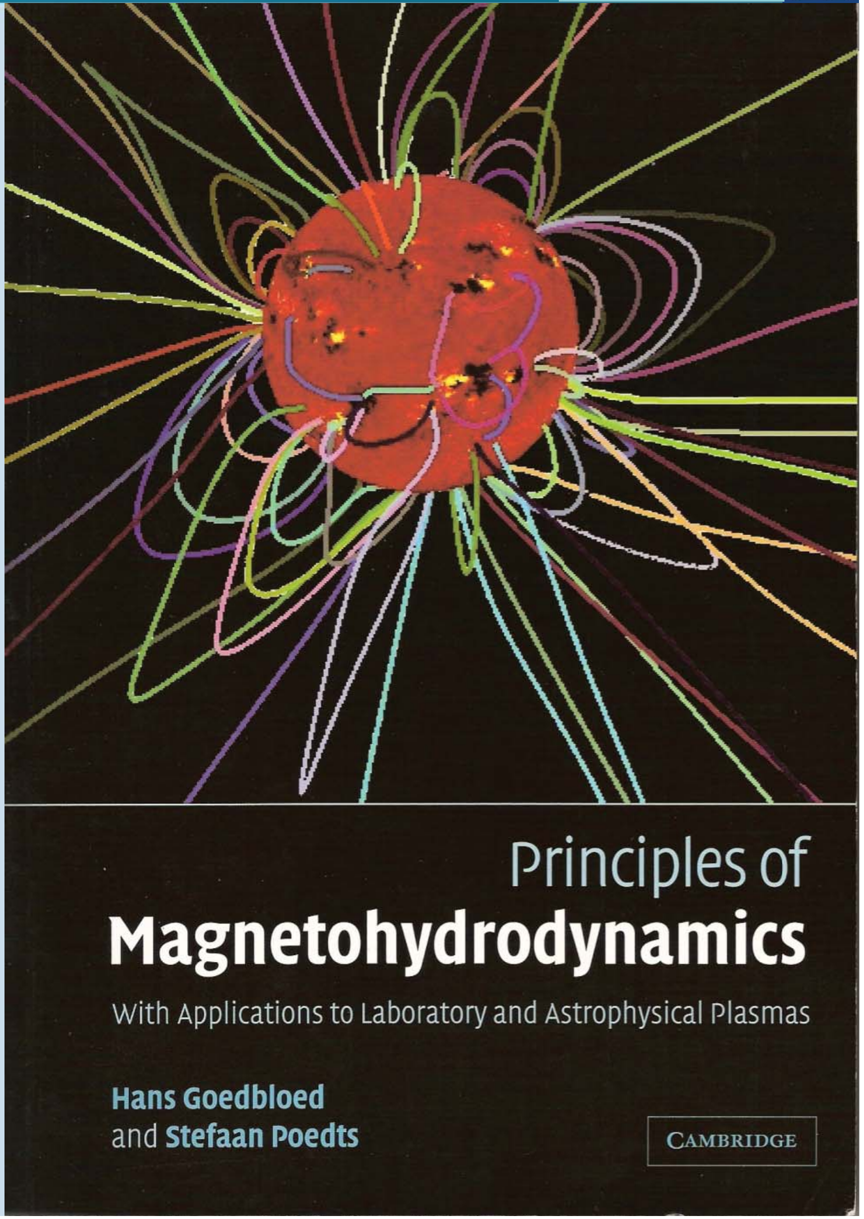


Postulating the basic equations

MHD equations can be introduced by

- *averaging the kinetic equations* by moment expansion and closure through transport theory
- just *posing them as postulates* for a hypothetical medium called ‘plasma’ and use physical arguments and mathematical criteria to justify the result.
  [There is nothing suspicious about posing the basic equations. That is what is actually done with all basic equations in physics.]

In the second approach, since *the MHD equations describe the motion of a conducting fluid interacting with a magnetic field*, we need to *combine Maxwell’s equations with the equations of gas dynamics* and provide *equations describing the interaction*.

The ideal MHD equations

- **Maxwell's equations** describe evolution of electric field $\mathbf{E}(\mathbf{r}, t)$ and magnetic field $\mathbf{B}(\mathbf{r}, t)$ in response to current density $\mathbf{j}(\mathbf{r}, t)$ and space charge $\tau(\mathbf{r}, t)$:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,, \qquad \textit{(Faraday)} \qquad (1)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\,, \qquad c \equiv (\epsilon_0 \mu_0)^{-1/2}\,, \qquad \textit{('Ampère')} \qquad (2)$$

$$\nabla \cdot \mathbf{E} = \frac{\tau}{\epsilon_0}\,, \qquad \textit{(Poisson)} \qquad (3)$$

$$\nabla \cdot \mathbf{B} = 0\,. \qquad \textit{(no monopoles)} \qquad (4)$$

- **Gas dynamics equations** describe evolution of density $\rho(\mathbf{r}, t)$ and pressure $p(\mathbf{r}, t)$:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \mathbf{v} \equiv \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \qquad \textit{(mass conservation)} \qquad (5)$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} + \gamma p \nabla \cdot \mathbf{v} \equiv \frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0\,, \qquad \textit{(entropy conservation)} \qquad (6)$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

is the *Lagrangian time-derivative* (moving with the fluid).

The ideal MHD equations

- **Coupling between system described by $\{\mathrm{E}, \mathrm{B}\}$ and system described by $\{\rho, p\}$** comes about through equations involving the velocity $\mathbf{v}(\mathbf{r}, t)$ of the fluid:
  'Newton's' *equation of motion for a fluid element* describes the acceleration of a fluid element by pressure gradient, gravity, and electromagnetic contributions,

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{F} \equiv -\nabla p + \rho \mathbf{g} + \mathbf{j} \times \mathbf{B} + \tau \mathbf{E} \,; \qquad \textit{(momentum conservation)} \tag{7}$$

  'Ohm's' law (for a perfectly conducting moving fluid) expresses that *the electric field E' in a co-moving frame vanishes*,

$$\mathbf{E}' \equiv \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \,. \qquad \textit{('Ohm')} \tag{8}$$

- Equations (1)–(8) are complete, but inconsistent for *non-relativistic velocities*:

$$v \ll c \,. \tag{9}$$

$\Rightarrow$ We need to consider **pre-Maxwell equations**.

Consequences of pre-Maxwell

1. *Maxwell's displacement current negligible* [$\mathcal{O}(v^2/c^2)$] for non-relativistic velocities:

$$\frac{1}{c^2}\left|\frac{\partial \mathbf{E}}{\partial t}\right| \sim \frac{v^2}{c^2}\frac{B}{l_0} \quad \ll \quad \mu_0|\mathbf{j}| \approx |\nabla \times \mathbf{B}| \sim \frac{B}{l_0} \qquad \text{[using Eq. (8)]}\,,$$

indicating length scales by l_0 and time scales by t_0, so that $v \sim l_0/t_0$.

$\Rightarrow$ Recover original Ampère's law:

$$\mathbf{j} = \frac{1}{\mu_0}\nabla \times \mathbf{B}\,. \tag{10}$$

2. *Electrostatic acceleration is also negligible* [$\mathcal{O}(v^2/c^2)$]:

$$\tau|\mathbf{E}| \sim \frac{v^2}{c^2}\frac{B^2}{\mu_0 l_0} \quad \ll \quad |\mathbf{j} \times \mathbf{B}| \sim \frac{B^2}{\mu_0 l_0} \qquad \text{[using Eqs. (3), (8), (10)]}\,.$$

$\Rightarrow$ Space charge effects may be ignored and Poisson's law (3) can be dropped.

3. *Electric field then becomes a secondary quantity,* determined from Eq. (8):

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}\,. \tag{11}$$

$\Rightarrow$ For non-relativistic MHD, $|\mathbf{E}| \sim |\mathbf{v}||\mathbf{B}|$, i.e. $\mathcal{O}(v/c)$ smaller than for EM waves.

Basic MHD equations

- Exploiting these approximations, and eliminating $\mathbf{E}$ and $\mathbf{j}$ through Eqs. (10) and (11), the basic equations of ideal MHD are recovered *in their most compact form*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \tag{12}$$

$$\rho(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}) + \nabla p - \rho \mathbf{g} - \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = 0\,, \tag{13}$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0\,, \tag{14}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,. \tag{15}$$

$\Rightarrow$ *Set of eight nonlinear partial differential equations (PDEs) for the eight variables* $\rho(\mathbf{r}, t)$, $\mathbf{v}(\mathbf{r}, t)$, $p(\mathbf{r}, t)$, *and* $\mathbf{B}(\mathbf{r}, t)$.

- The magnetic field equation (15)(b) is to be considered as a initial condition: once satisfied, it remains satisfied for all later times by virtue of Eq. (15)(a).

Scale independence

- The MHD equations (12)–(15) can be made dimensionless by means of a choice for *the units of length, mass, and time*, based on typical magnitudes l_0 for length scale, ρ_0 for plasma density, and B_0 for magnetic field at some representative position. The unit of time then follows by exploiting *the Alfvén speed*:

$$v_0 \equiv v_{A,0} \equiv \frac{B_0}{\sqrt{\mu_0 \rho_0}} \quad \Rightarrow \quad t_0 \equiv \frac{l_0}{v_0} \,. \tag{24}$$

- By means of this basic triplet l_0, B_0, t_0 (and derived quantities ρ_0 and v_0), we create dimensionless independent variables and associated differential operators:

$$\bar{l} \equiv l/l_0 \,, \quad \bar{t} \equiv t/t_0 \qquad \Rightarrow \qquad \bar{\nabla} \equiv l_0 \nabla \,, \quad \partial/\partial \bar{t} \equiv t_0 \, \partial/\partial t \,, \tag{25}$$

and dimensionless dependent variables:

$$\bar{\rho} \equiv \rho/\rho_0 \,, \quad \bar{\mathbf{v}} \equiv \mathbf{v}/v_0 \,, \quad \bar{p} \equiv p/(\rho_0 v_0^2) \,, \quad \bar{\mathbf{B}} \equiv \mathbf{B}/B_0 \,, \quad \bar{\mathbf{g}} \equiv (l_0/v_0^2)\, \mathbf{g} \,. \tag{26}$$

- Barred equations are now identical to unbarred ones (except that μ_0 is eliminated).

  $\Rightarrow$ *Ideal MHD equations independent of size of the plasma (l_0), magnitude of the magnetic field (B_0), and density (ρ_0), i.e. time scale (t_0).*

Scales of actual plasmas

	l_0 (m)	B_0 (T)	t_0 (s)
tokamak	20	3	3×10^{-6}
magnetosphere Earth	4×10^{7}	3×10^{-5}	6
solar coronal loop	10^{8}	3×10^{-2}	15
magnetosphere neutron star	10^{6}	10^{8} *	10^{-2}
accretion disc YSO	1.5×10^{9}	10^{-4}	7×10^{5}
accretion disc AGN	4×10^{18}	10^{-4}	2×10^{12}
galactic plasma	10^{21} $(= 10^{5}$ ly$)$	10^{-8}	10^{15} $(= 3 \times 10^{7}$ y$)$

* Some recently discovered pulsars, called magnetars, have record magnetic fields of 10^{11} T : the plasma Universe is ever expanding!

Note Tokamak: 1 min $(60\,\mathrm{s})$ $\Rightarrow$ 20×10^{6} crossing times ,
Coronal loop: 1 month $(2.6 \times 10^{6}\,\mathrm{s})$ $\Rightarrow$ 2×10^{5} " .

A crucial question

Do the MHD equations (12)–(15) provide a complete model for plasma dynamics?

Answer: **NO!**

Two most essential elements of a scientific model are still missing, viz.

1. What is the *physical problem* we want to solve?
2. How does this translate into *conditions on the solutions of the PDEs?*

This brings in the space and time constraints of the *boundary conditions* and *initial data*. Initial data just amount to prescribing arbitrary functions

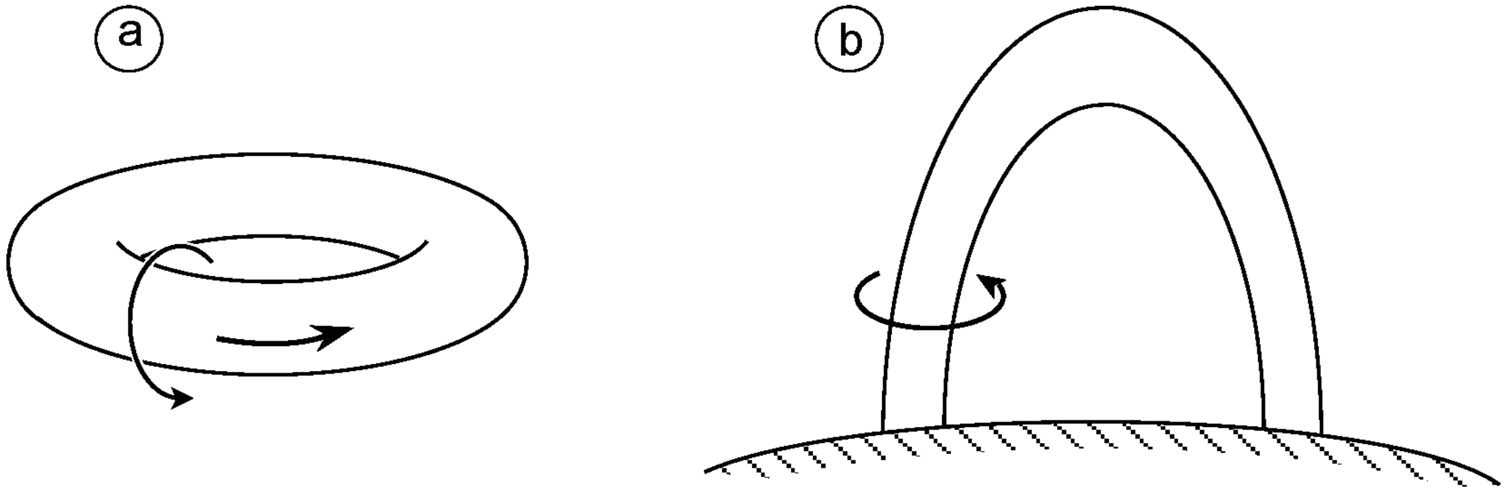
$$\rho_i(\mathbf{r})\,[\,\equiv \rho(\mathbf{r},t\!=\!0)\,]\,,\quad \mathbf{v}_i(\mathbf{r})\,,\quad p_i(\mathbf{r})\,,\quad \mathbf{B}_i(\mathbf{r})\quad \text{on domain of interest}\,. \tag{27}$$

Boundary conditions is a much more involved issue since it implies specification of a **magnetic confinement geometry**.

$\Rightarrow$ *magnetic flux tubes* (Sec.4.2), *conservation laws* (Sec.4.3), *discontinuities* (Sec.4.4), *formulation of model problems for laboratory and astrophysical plasmas* (Sec.4.5).

Flux tubes

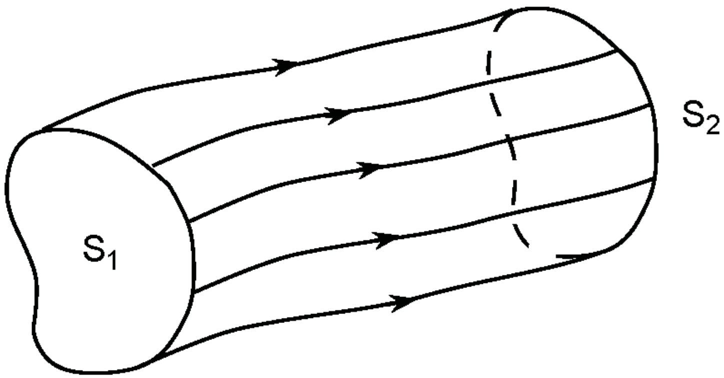
- Magnetic flux tubes are the basic magnetic structures that determine which *boundary conditions* may be posed on the MHD equations.



- Two different kinds of flux tubes:
  (a) closed onto itself, like in thermonuclear *tokamak* confinement machines,
  (b) connecting onto a medium of vastly different physical characteristics so that the flux tube may be considered as finite and separated from the other medium by suitable jump conditions, like in *coronal flux tubes*.

Magnetic flux

- Magnetic fields confining plasmas are essentially *tubular structures*: The magnetic field equation
$$\nabla \cdot \mathbf{B} = 0 \tag{28}$$
is not compatible with spherical symmetry. Instead, magnetic flux tubes become the essential constituents.



- Gauss' theorem:
$$\iiint_V \nabla \cdot \mathbf{B}\, d\tau = \oiint \mathbf{B} \cdot \mathbf{n}\, d\sigma = -\iint_{S_1} \mathbf{B}_1 \cdot \mathbf{n}_1\, d\sigma_1 + \iint_{S_2} \mathbf{B}_2 \cdot \mathbf{n}_2\, d\sigma_2 = 0\,,$$
Magnetic flux of all field lines through surface element $d\sigma_1$ is the same as through arbitrary other element $d\sigma_2$ intersecting that field line bundle.
$$\Rightarrow \qquad \Psi \equiv \iint_S \mathbf{B} \cdot \mathbf{n}\, d\sigma \quad \text{is well defined} \tag{29}$$
(does not depend on how S is taken). Also true for smaller subdividing flux tubes!

Conservation form

- The MHD equations can be brought in conservation form:

$$\frac{\partial}{\partial t}(\cdots) + \nabla \cdot (\cdots) = 0\,. \tag{30}$$

This yields: *conservation laws*, *jump conditions*, *and powerful numerical algorithms!*

- By intricate vector algebra, one obtains the *conservation form of the ideal MHD equations* (suppressing gravity): $\Downarrow$ From now on, putting $\mu_0 \to 1$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \tag{31}$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot [\,\rho \mathbf{v}\mathbf{v} + (p + \tfrac{1}{2}B^2)\,\mathbf{I} - \mathbf{B}\mathbf{B}\,] = 0\,, \qquad p = (\gamma - 1)\rho e\,, \tag{32}$$

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho e + \tfrac{1}{2}B^2) + \nabla \cdot [(\tfrac{1}{2}\rho v^2 + \rho e + p + B^2)\mathbf{v} - \mathbf{v} \cdot \mathbf{B}\mathbf{B}] = 0\,, \tag{33}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}) = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,. \tag{34}$$

It remains to analyze the meaning of the different terms.

Conservation

- Defining

  – *momentum density*: $$\boldsymbol{\pi} \equiv \rho \mathbf{v} \,, \tag{35}$$

  – *stress tensor*: $$\mathbf{T} \equiv \rho \mathbf{v}\mathbf{v} + (p + \tfrac{1}{2}B^2)\,\mathbf{I} - \mathbf{B}\mathbf{B} \,, \tag{36}$$

  – *total energy density*: $$\mathcal{H} \equiv \tfrac{1}{2}\rho v^2 + \tfrac{1}{\gamma-1}\,p + \tfrac{1}{2}B^2 \,, \tag{37}$$

  – *energy flow*: $$\mathbf{U} \equiv (\tfrac{1}{2}\rho v^2 + \tfrac{\gamma}{\gamma-1}\,p)\mathbf{v} + B^2\mathbf{v} - \mathbf{v}\cdot\mathbf{B}\,\mathbf{B} \,, \tag{38}$$

  – *(no name)*: $$\mathbf{Y} \equiv \mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v} \,, \tag{39}$$

yields

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\boldsymbol{\pi} = 0 \qquad \textit{(conservation of mass),} \tag{40}$$

$$\frac{\partial \boldsymbol{\pi}}{\partial t} + \nabla\cdot\mathbf{T} = 0 \qquad \textit{(conservation of momentum),} \tag{41}$$

$$\frac{\partial \mathcal{H}}{\partial t} + \nabla\cdot\mathbf{U} = 0 \qquad \textit{(conservation of energy),} \tag{42}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla\cdot\mathbf{Y} = 0 \qquad \textit{(conservation of magnetic flux).} \tag{43}$$

Effect of gravity on conservation

Conservation laws, gravity included

- Including gravity, momentum and energy equation are:

$$\frac{\partial \boldsymbol{\pi}}{\partial t} + \nabla \cdot \mathbf{T} = -\rho \nabla \Phi \qquad \textit{(momentum)}, \tag{44}$$

$$\frac{\partial \mathcal{H}}{\partial t} + \nabla \cdot \mathbf{U} = -\rho \mathbf{v} \cdot \nabla \Phi \qquad \textit{(energy)}. \tag{45}$$

$$\tag{46}$$

$\Rightarrow$ work done by gravitational force

- include gravitational potential energy: $\mathcal{H}_g \equiv \mathcal{H} + \rho\Phi$ and rewrite to

$$\frac{\partial \mathcal{H}_g}{\partial t} + \nabla \cdot [\mathbf{U} + \rho \mathbf{v} \Phi] = \rho \frac{\partial \Phi}{\partial t} \qquad \textit{(energy)} \tag{47}$$

Global conservation laws

- Defining
  - *total mass*: $$M \equiv \int \rho \, d\tau \,, \tag{48}$$
  - *total momentum*: $$\mathbf{\Pi} \equiv \int \boldsymbol{\pi} \, d\tau \,, \tag{49}$$
  - *total energy*: $$H \equiv \int \mathcal{H} \, d\tau \,, \tag{50}$$
  - *total magnetic flux*: $$\Psi \equiv \int \mathbf{B} \cdot \tilde{\mathbf{n}} \, d\tilde{\sigma} \,, \tag{51}$$

  gives, by the application of the right BCs (see later):

$$\dot{M} = \int \dot{\rho} \, d\tau = -\int \nabla \cdot \boldsymbol{\pi} \, d\tau \stackrel{\text{Gauss}}{=} -\oint \boldsymbol{\pi} \cdot \mathbf{n} \, d\sigma = 0 \,, \tag{52}$$

$$\mathbf{F} = \dot{\mathbf{\Pi}} = \int \dot{\boldsymbol{\pi}} \, d\tau = -\int \nabla \cdot \mathbf{T} \, d\tau \stackrel{\text{Gauss}}{=} -\oint (p + \tfrac{1}{2} B^2) \, \mathbf{n} \, d\sigma \,, \tag{53}$$

$$\dot{H} = \int \dot{\mathcal{H}} \, d\tau = -\int \nabla \cdot \mathbf{U} \, d\tau \stackrel{\text{Gauss}}{=} -\oint \mathbf{U} \cdot \mathbf{n} \, d\sigma = 0 \,, \tag{54}$$

$$\dot{\Psi} = \int \dot{\mathbf{B}} \cdot \tilde{\mathbf{n}} \, d\tilde{\sigma} = \int \nabla \times (\mathbf{v} \times \mathbf{B}) \cdot \tilde{\mathbf{n}} \, d\tilde{\sigma} \stackrel{\text{Stokes!}}{=} \oint \mathbf{v} \times \mathbf{B} \cdot \, d\boldsymbol{l} = 0 \,. \tag{55}$$

$\Rightarrow$ *Total mass, momentum, energy, and flux conserved:* **the system is closed!**

Jump conditions

Extending the MHD model

- The BCs for *plasmas surrounded by a solid wall:*

$$\mathbf{n}_w \cdot \mathbf{v} = 0 \qquad \text{(on } W) \quad \Rightarrow \quad \textit{no flow accross the wall,}$$

$$\mathbf{n}_w \cdot \mathbf{B} = 0 \qquad \text{(on } W) \quad \Rightarrow \quad \textit{magnetic field lines do not intersect the wall.}$$

  Under these conditions, conservation laws apply and the system is closed.

- For many applications (both in the laboratory and in astrophysics) this is not enough. One also needs BCs (jump conditions) for *plasmas with an internal boundary* where the magnitudes of the plasma variables 'jump'.

  Example: at the photospheric boundary the density changes $\sim 10^{-9}$.

- Such a boundary is a special case of a *shock,* i.e. an irreversible (entropy-increasing) transition. In gas dynamics, the *Rankine–Hugoniot relations* relate the variables of the subsonic flow downstream the shock with those of the supersonic flow upstream.

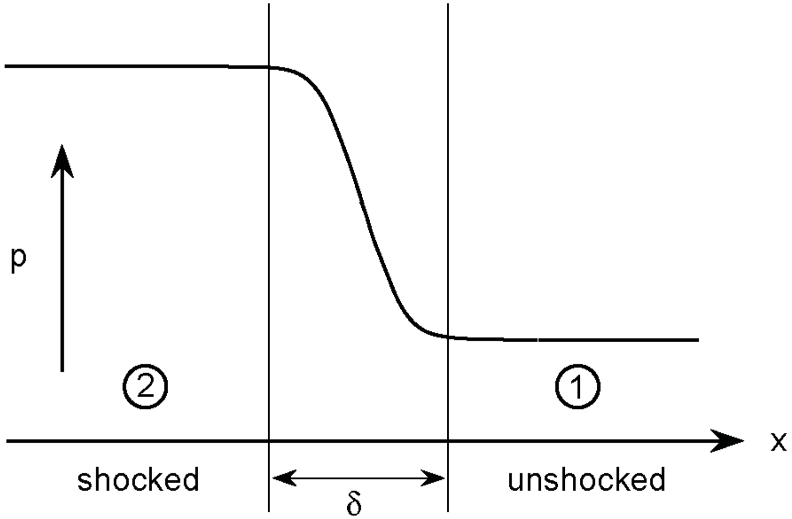
  We will generalize these relations to MHD, but only to get the right form of the jump conditions, not to analyze transonic flows (subject for a much later chapter).

Shock formation (1/2)

- Excite sound waves in a 1D compressible gas (HD): the local perturbations travel with the sound speed $c \equiv \sqrt{\gamma p/\rho}$.

  $\Rightarrow$ Trajectories in the x–t plane *(characteristics):* $dx/dt = \pm c$.

- Now suddenly increase the pressure, so that p changes in a thin layer of width δ :

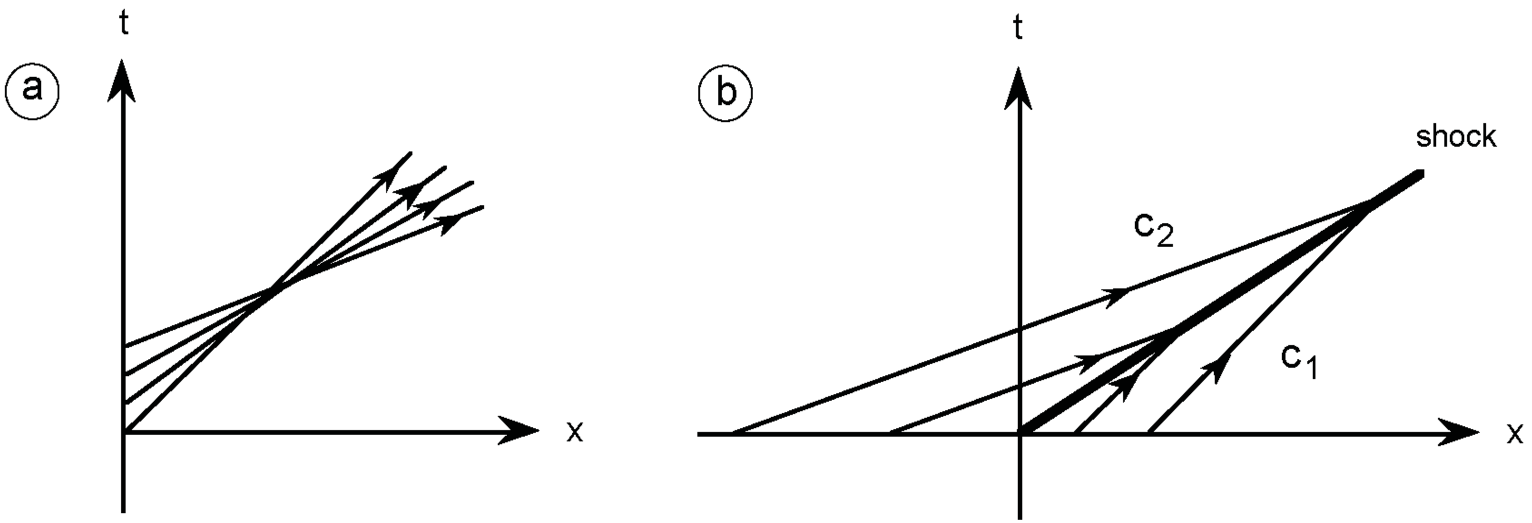


  $\Rightarrow$ 'Converging' characteristics in the x–t plane.

  $\Rightarrow$ Information from different space-time points accumulates, gradients build up until steady state reached where dissipation and nonlinearities balance $\Rightarrow$ **shock.**

Shock formation (2/2)

- Wihout the non-ideal and nonlinear effects, the characteristics would cross (a). With those effects, in the limit $\delta \to 0$, the characteristics meet at the shock front (b).



$\Rightarrow$ *Moving shock front separates two ideal regions.*

- Neglecting the thickness of the shock (not the shock itself of course), all there remains is to derive jump relations across the infinitesimal layer.

$\Rightarrow$ *Limiting cases of the conservation laws at shock fronts.*

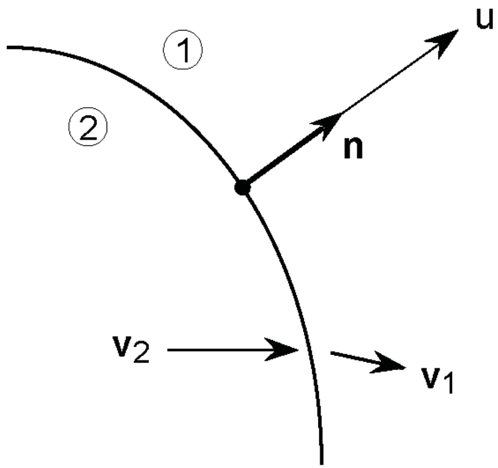
Deriving jump conditions

Procedure to derive the jump conditions

Integrate conservation equations across shock from ① (undisturbed) to ② (shocked).

- Only contribution from gradient normal to the front:

$$\lim_{\delta\to 0}\int_1^2 \nabla f\, dl = -\lim_{\delta\to 0}\mathbf{n}\int_1^2 \frac{\partial f}{\partial l}\, dl = \mathbf{n}(f_1 - f_2) \equiv \mathbf{n}\,[\![f]\!]\,. \tag{56}$$



- In frame moving with the shock at normal speed u :

$$\left(\frac{\mathrm{D}f}{\mathrm{D}t}\right)_{\rm shock} = \frac{\partial f}{\partial t} - u\frac{\partial f}{\partial l} \text{ finite} \ll \frac{\partial f}{\partial t} \approx u\frac{\partial f}{\partial l} \sim \infty$$

$$\Rightarrow \quad \lim_{\delta\to 0}\int_1^2 \frac{\partial f}{\partial t}\, dl = u\lim_{\delta\to 0}\int_1^2 \frac{\partial f}{\partial l}\, dl = -u\,[\![f]\!]\,. \tag{57}$$

- Hence, *jump conditions follow from the conservation laws by simply substituting*

$$\nabla f \to \mathbf{n}\,[\![f]\!]\,, \qquad \partial f/\partial t \to -u\,[\![f]\!]\,. \tag{58}$$

MHD jump conditions…

- Conservation of mass,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \Rightarrow \quad -u\,[\![\rho]\!] + \mathbf{n} \cdot [\![\rho \mathbf{v}]\!] = 0\,. \tag{59}$$

- Conservation of momentum,

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot [\,\rho \mathbf{v}\mathbf{v} + (p + \tfrac{1}{2}B^2)\,\mathsf{I} - \mathbf{B}\mathbf{B}\,] = 0$$

$$\Rightarrow \quad -u\,[\![\rho \mathbf{v}]\!] + \mathbf{n} \cdot [\![\rho \mathbf{v}\mathbf{v} + (p + \tfrac{1}{2}B^2)\,\mathsf{I} - \mathbf{B}\mathbf{B}]\!] = 0\,. \tag{60}$$

- Conservation ot total energy,

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho e + \tfrac{1}{2}B^2) + \nabla \cdot [(\tfrac{1}{2}\rho v^2 + \rho e + p + B^2)\mathbf{v} - \mathbf{v} \cdot \mathbf{B}\mathbf{B}] = 0$$

$$\Rightarrow \quad -u\,[\![\tfrac{1}{2}\rho v^2 + \tfrac{1}{\gamma-1}p + \tfrac{1}{2}B^2]\!] + \mathbf{n} \cdot [\![(\tfrac{1}{2}\rho v^2 + \tfrac{\gamma}{\gamma-1}p + B^2)\mathbf{v} - \mathbf{v} \cdot \mathbf{B}\mathbf{B}]\!] = 0\,. \tag{61}$$

- Conservation of magnetic flux,

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}) = 0\,, \qquad \nabla \cdot \mathbf{B} = 0$$

$$\Rightarrow \quad -u\,[\![\mathbf{B}]\!] + \mathbf{n} \cdot [\![\mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}]\!] = 0\,, \qquad \mathbf{n} \cdot [\![\mathbf{B}]\!] = 0\,. \tag{62}$$

… in the shock frame

MHD jump conditions in the shock frame

- Simplify jump conditions by *transforming to co-moving shock frame,* where relative plasma velocity is $\mathbf{v}' \equiv \mathbf{v} - u\mathbf{n}$, and split vectors in tangential and normal to shock:

$$[\![\rho v_n']\!] = 0\,, \qquad \textit{(mass)} \qquad (63)$$

$$[\![\rho {v_n'}^2 + p + \tfrac{1}{2}B_t^2]\!] = 0\,, \qquad \textit{(normal momentum)} \qquad (64)$$

$$\rho v_n'\,[\![\mathbf{v}_t']\!] = B_n\,[\![\mathbf{B}_t]\!]\,, \qquad \textit{(tangential momentum)} \qquad (65)$$

$$\rho v_n'\,[\![\tfrac{1}{2}({v_n'}^2 + {v_t'}^2) + (\tfrac{\gamma}{\gamma-1}p + B_t^2)/\rho]\!] = B_n\,[\![\mathbf{v}_t' \cdot \mathbf{B}_t]\!]\,, \qquad \textit{(energy)} \qquad (66)$$

$$[\![B_n]\!] = 0\,, \qquad \textit{(normal flux)} \qquad (67)$$

$$\rho v_n'\,[\![\mathbf{B}_t/\rho]\!] = B_n[\![\mathbf{v}_t']\!]\,. \qquad \textit{(tangential flux)} \qquad (68)$$

$\Rightarrow$ 6 relations for the 6 jumps $[\![\rho]\!]$, $[\![v_n]\!]$, $[\![\mathbf{v}_t]\!]$, $[\![p]\!]$, $[\![B_n]\!]$, $[\![\mathbf{B}_t]\!]$.

- *Do not use entropy conservation law* since shock is entropy-increasing transition:

$$\text{not} \quad \frac{\partial}{\partial t}(\rho S) + \nabla \cdot (\rho S \mathbf{v}) = 0 \quad \Rightarrow \quad \rho v_n'\,[\![S]\!] = 0\,, \quad \text{but} \quad [\![S]\!] \equiv [\![\rho^{-\gamma} p]\!] \leq 0\,. \qquad (69)$$

$\Rightarrow$ This is the only remnant of the dissipative processes in the thin layer.

Different discontinuities

$\Rightarrow$ Two classes of discontinuities:

(1) *Boundary conditions for moving plasma-plasma interfaces,* where there is no flow accross the discontinuity ($v'_n = 0$) $\quad\Rightarrow$ will continue with this here.

(2) *Jump conditions for shocks* ($v'_n \neq 0$) $\Rightarrow$ leave for advanced MHD lectures.

BCs at co-moving interfaces

- When $v'_n = 0\,,$ jump conditions (63)–(68) reduce to:

$$[\![p + \tfrac{1}{2}B_t^2]\!] = 0\,, \qquad \textit{(normal momentum)} \qquad (70)$$

$$B_n\,[\![\mathbf{B}_t]\!] = 0\,, \qquad \textit{(tangential momentum)} \qquad (71)$$

$$B_n\,[\![\mathbf{v}'_t \cdot \mathbf{B}_t]\!] = 0\,, \qquad \textit{(energy)} \qquad (72)$$

$$[\![B_n]\!] = 0\,, \qquad \textit{(normal flux)} \qquad (73)$$

$$B_n\,[\![\mathbf{v}'_t]\!] = 0\,. \qquad \textit{(tangential flux)} \qquad (74)$$

- Two possibilities, depending on whether $\mathbf{B}$ intersects the interface or not:

  (a) *Contact discontinuities* when $B_n \neq 0\,,$

  (b) *Tangential discontinuities* if $B_n = 0\,.$

Contact discontinuities

(a) Contact discontinuities

- For co-moving interfaces with an intersecting magnetic field, $B_n \neq 0\,,$ the jump conditions (70)–(74) only admit a jump of the density (or temperature, or entropy) whereas all other quantities should be continuous:

$$\begin{aligned} &\text{– jumping:} && [\![\rho]\!] \neq 0\,, \\ &\text{– continuous:} && v_n' = 0\,, \quad [\![\mathbf{v}_t']\!] = 0\,, \quad [\![p]\!] = 0\,, \quad [\![B_n]\!] = 0\,, \quad [\![\mathbf{B}_t]\!] = 0\,. \end{aligned} \tag{75}$$

Examples: photospheric footpoints of coronal loops where density jumps,
'divertor' tokamak plasmas with $\mathbf{B}$ intersecting boundary.

- These BCs are most typical for astrophysical plasmas, *modelling plasmas with very different properties of the different spatial regions involved* (e.g. close to a star and far away): difficult! Computing waves in such systems usually requires extreme resolutions to follow the disparate time scales in the problem.

Tangential discontinuities

(b) Tangential discontinuities

- For co-moving interfaces with purely tangential magnetic field, $B_n = 0$, the jump conditions (70)–(74) are much less restrictive:

$$\begin{aligned} &\text{– jumping:} \quad [\![\rho]\!] \neq 0\,, \quad [\![\mathbf{v}'_t]\!] \neq 0\,, \quad [\![p]\!] \neq 0\,, \quad [\![\mathbf{B}_t]\!] \neq 0\,, \\ &\text{– continuous:} \quad v'_n = 0\,, \quad B_n = 0\,, \quad [\![p + \tfrac{1}{2}B_t^2]\!] = 0\,. \end{aligned} \tag{76}$$

Examples: tokamak plasma separated from wall by tenuous plasma (or 'vacuum'), dayside magnetosphere where IMF meets Earth's dipole.

- *Plasma–plasma interface BCs* by transforming back to lab frame, $v_n - u \equiv v'_n = 0$:

$$\mathbf{n} \cdot \mathbf{B} = 0 \qquad (\mathbf{B} \parallel \text{interface})\,, \tag{77}$$

$$\mathbf{n} \cdot [\![\mathbf{v}]\!] = 0 \qquad (\text{normal velocity continuous})\,, \tag{78}$$

$$[\![p + \tfrac{1}{2}B^2]\!] = 0 \qquad (\text{total pressure continuous})\,. \tag{79}$$

- Jumps tangential components, $[\![\mathbf{B}_t]\!]$ & $[\![\mathbf{v}_t]\!]$, due to *surface current & surface vorticity:*

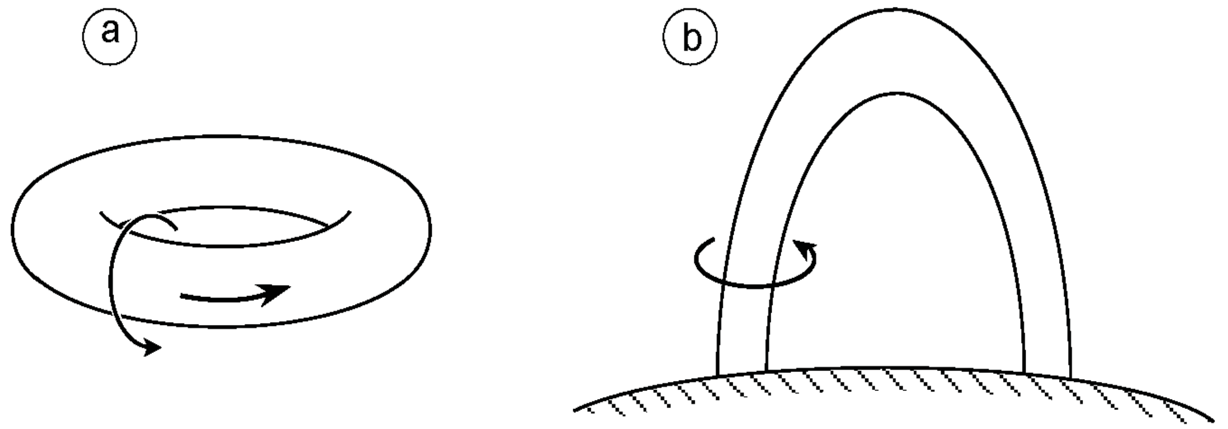
$$\mathbf{j} = \nabla \times \mathbf{B} \quad \Rightarrow \quad \mathbf{j}^\star \equiv \lim_{\delta \to 0, |\mathbf{j}| \to \infty} (\delta\, \mathbf{j}) = \mathbf{n} \times [\![\mathbf{B}]\!]\,, \tag{80}$$

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{v} \quad \Rightarrow \quad \boldsymbol{\omega}^\star \equiv \lim_{\delta \to 0, |\boldsymbol{\omega}| \to \infty} (\delta\, \boldsymbol{\omega}) = \mathbf{n} \times [\![\mathbf{v}]\!]\,. \tag{81}$$

Model problems

- We are now prepared to formulate **complete models for plasma dynamics** $\equiv$ **MHD equations** $+$ **specification of magnetic geometries** $\Rightarrow$ **appropriate BCs.**

- For example, recall two generic magnetic structures: (a) tokamak; (b) coronal loop.

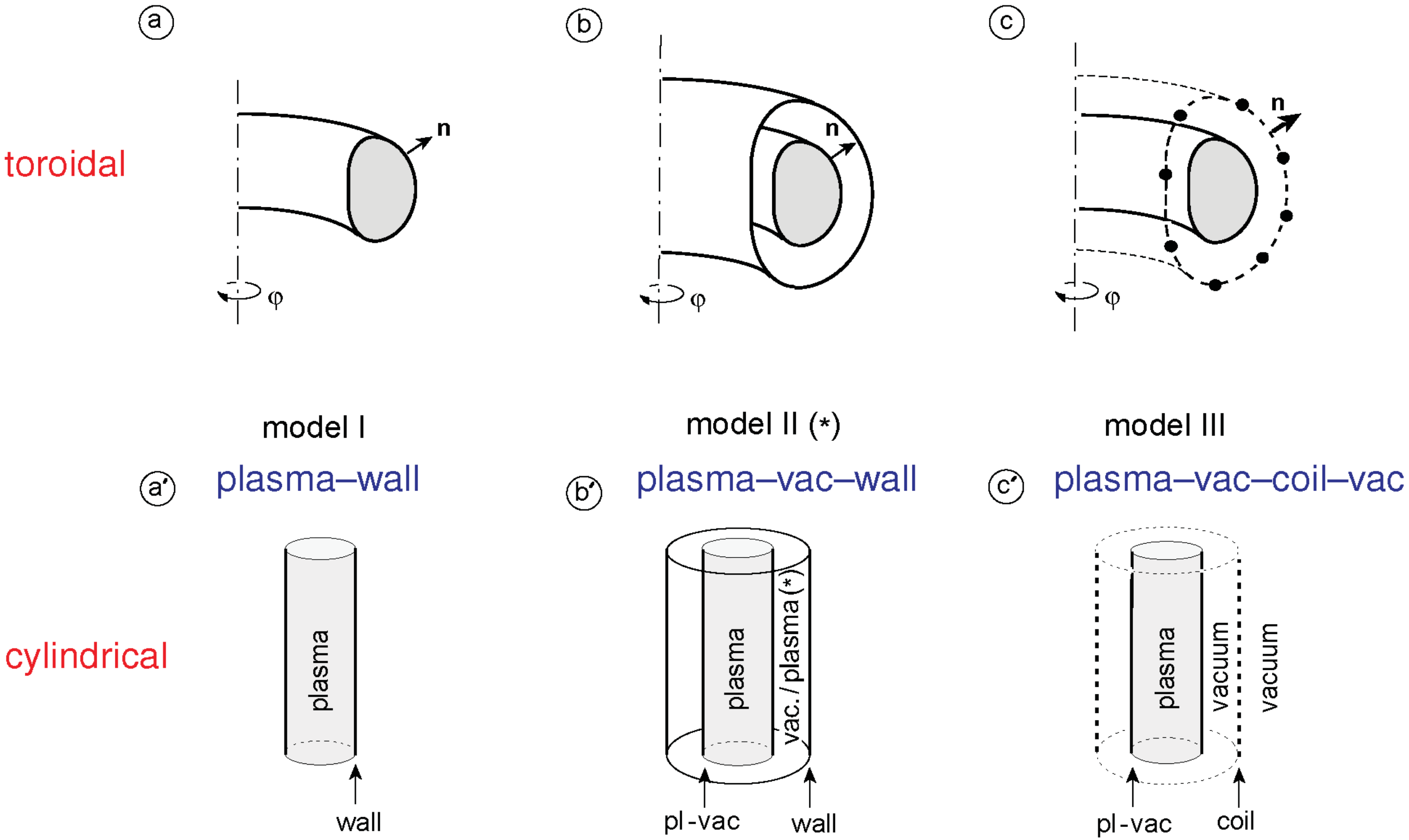


- Generalize this to **six model problems,** separated in two classes:

  $\Rightarrow$ **Models I–III (laboratory plasmas) with tangential discontinuities;**

  $\Rightarrow$ **Models IV–VI (astrophysical plasmas) with contact discontinuities.**

Laboratory plasma models



Model I

Model I: plasma confined inside rigid wall

- Model I: axisymmetric (2D) plasma contained in a ‘donut’-shaped vessel (tokamak) which *confines the magnetic structure to a finite volume.* Vessel + external coils need to be firmly fixed to the laboratory floor since magnetic forces are huge.

  ⇒ Plasma–wall, impenetrable wall needs not be conducting (remember why?).

  ⇒ Boundary conditions are

$$\mathbf{n} \cdot \mathbf{B} = 0 \qquad \textit{(at the wall)}\,, \tag{82}$$

$$\mathbf{n} \cdot \mathbf{v} = 0 \qquad \textit{(at the wall)}\,. \tag{83}$$

  ⇒ just two BCs for 8 variables!

- These BCs guarantee conservation of mass, momentum, energy and magnetic flux: *the system is closed off from the outside world.*

- Most widely used simplification: cylindrical version (1D) with symmetry in θ and z.

  ⇒ Non-trivial problem only in the radial direction, therefore: one-dimensional.

Model II

Model II: plasma-vacuum system inside rigid wall

- Model II: as I, but plasma separated from wall by vacuum (tokamak with a ‘limiter’).

  $\Rightarrow$ Plasma–vacuum–wall, wall now perfectly conducting (since vacuum in front).

- Vacuum has no density, velocity, current, only $\hat{\mathbf{B}}$ $\Rightarrow$ pre-Maxwell dynamics:

$$\nabla \times \hat{\mathbf{B}} = 0\,, \qquad \nabla \cdot \hat{\mathbf{B}} = 0\,, \tag{84}$$

$$\nabla \times \hat{\mathbf{E}} = -\frac{\partial \hat{\mathbf{B}}}{\partial t}\,, \qquad \nabla \cdot \hat{\mathbf{E}} = 0\,. \tag{85}$$

BC at exterior interface (only on $\hat{\mathbf{B}}$, consistent with $\hat{\mathbf{E}}_t = 0$):

$$\mathbf{n} \cdot \hat{\mathbf{B}} = 0 \qquad \textit{(at conducting wall)}\,. \tag{86}$$

- BCs at interior interface ($\mathbf{B}$ not pointing into vacuum and total pressure balance):

$$\mathbf{n} \cdot \mathbf{B} = \mathbf{n} \cdot \hat{\mathbf{B}} = 0 \qquad \textit{(at plasma–vacuum interface)}\,, \tag{87}$$

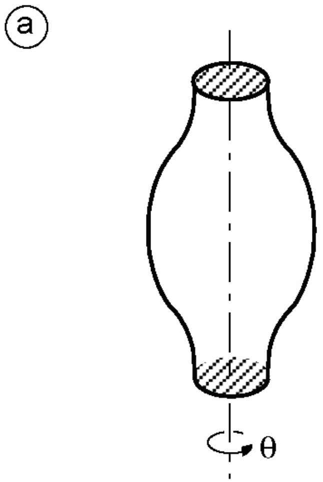
$$[\![p + \tfrac{1}{2} B^2]\!] = 0 \qquad \textit{(at plasma–vacuum interface)}\,. \tag{88}$$

  $\Rightarrow$ Consequence (not a BC) is jump in $\mathbf{B}_t$, i.e. skin current:

$$\mathbf{j}^\star = \mathbf{n} \times [\![\mathbf{B}]\!] \qquad \textit{(at plasma–vacuum interface)}\,. \tag{89}$$

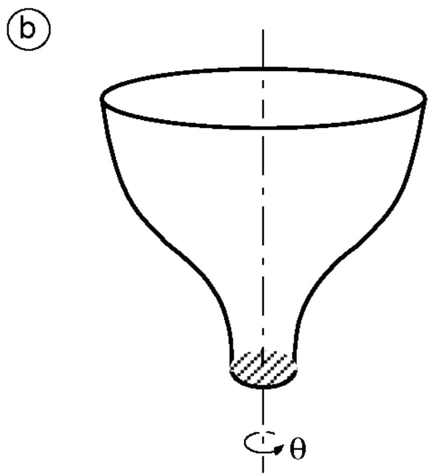
Astrophysical plasmas

2D



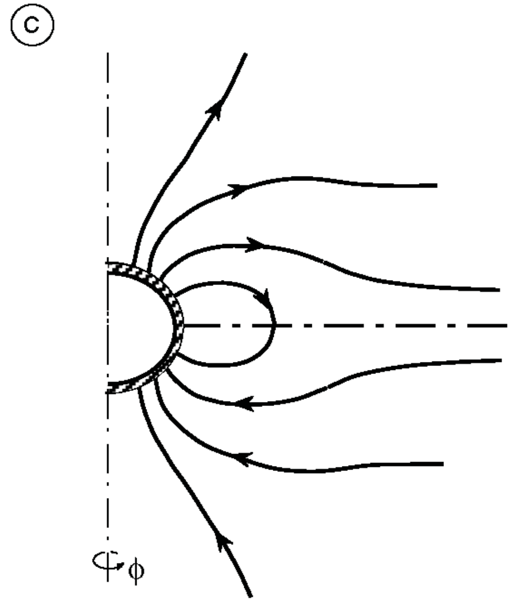
model IV

closed loop



model V

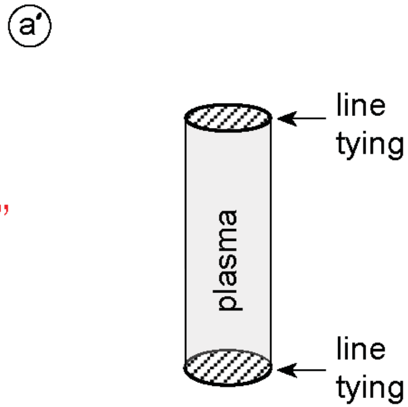
open loop

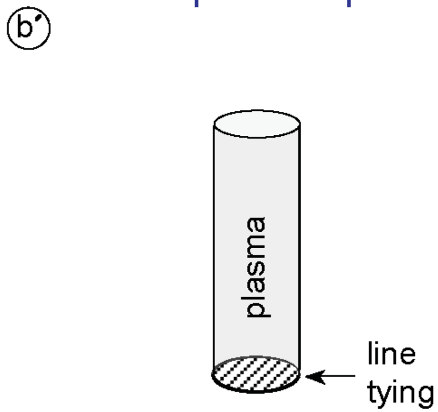


model VI

stellar wind

“1D”





Model IV

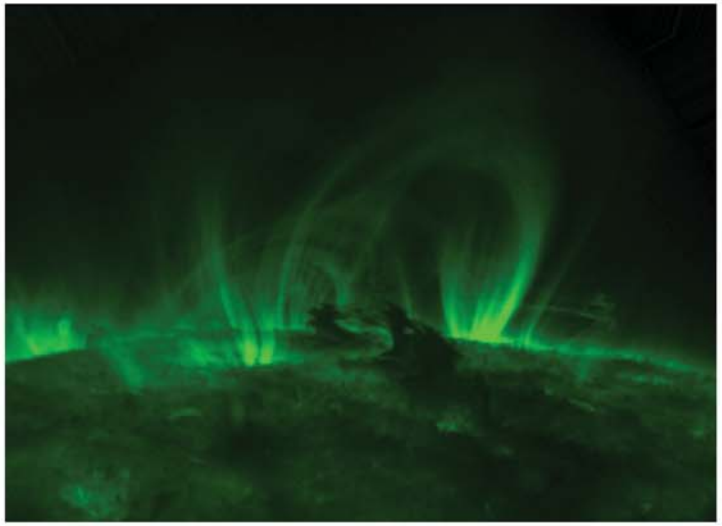
Model IV: ‘closed’ coronal magnetic loop

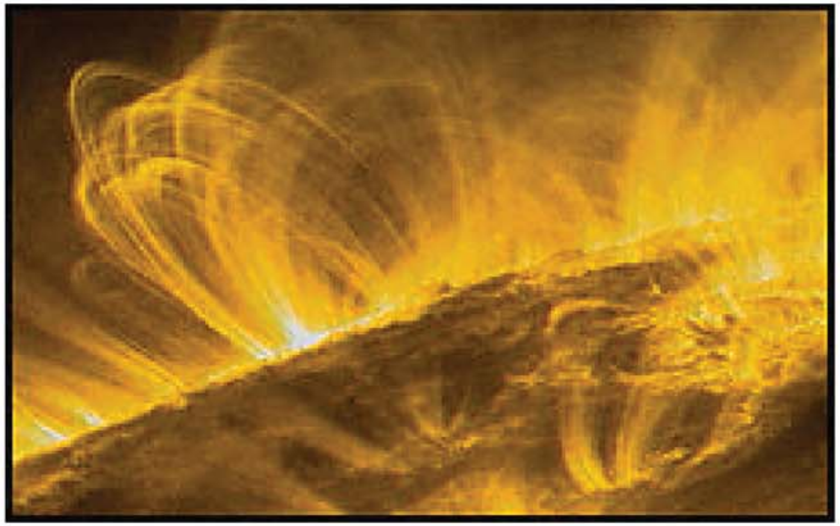
- In model IV, the field lines of finite plasma column (coronal loop) are line-tied on both sides to plasma of such high density (photosphere) that it is effectively immobile.

  $\Rightarrow$ Line-tying boundary conditions:

$$\mathbf{v} = 0 \qquad \textit{(at photospheric end planes)}\,. \tag{92}$$

  $\Rightarrow$ Applies to waves in solar coronal flux tubes, no back-reaction on photosphere:





- In this model, loops are straightened out to 2D configuration (depending on r and z). Also neglecting fanning out of field lines $\Rightarrow$ quasi-1D (finite length cylinder).

Model V

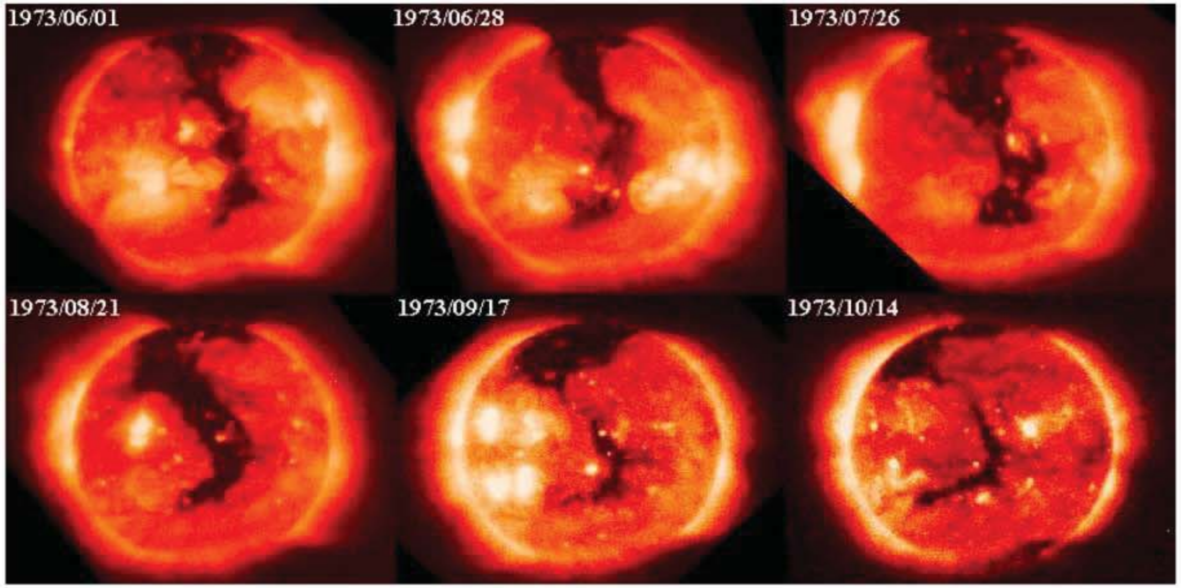
Model V: open coronal magnetic loop

- In model V, the magnetic field lines of a semi-infinite plasma column are line-tied on one side to a massive plasma.

  $\Rightarrow$ Line-tying boundary condition:

$$\mathbf{v} = 0 \qquad \textit{(at photospheric end plane)}\,.$$

  $\Rightarrow$ Applies to dynamics in coronal holes, where (fast) solar wind escapes freely:



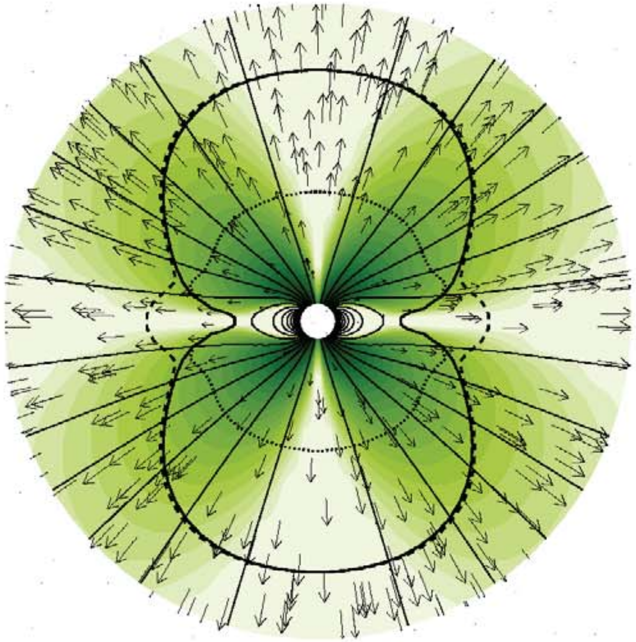
- *Truly open* variants of models IV & V: photospheric excitation ($\mathbf{v}(t) \neq 0$ prescribed).

Model VI

Model VI: Stellar wind

- In model VI, a plasma is ejected from photosphere of a star and accelerated along the open magnetic field lines into outer space.

  ⇒ Combines closed & open loops (models IV & V), line-tied at dense photosphere, but stress on outflow rather than waves (requires more advanced discussion).



- Output from an actual simulation with the Versatile Advection code: 2D (axisymm.) magnetized wind with 'wind' and 'dead' zone. Sun at the center, field lines drawn, velocity vectors, density coloring. Dotted, drawn, dashed: slow, Alfvén, fast critical surfaces.

  [Keppens & Goedbloed, Ap. J. **530**, 1036 (2000)]

Basic assumptions/applicability

Microscopic definition:

Plasma is a quasi-neutral gas of charged and neutral particles which exhibits collective behaviour (Chen).

(a) Long-range collective interactions dominate over binary collisions with neutrals

(b) Length scales large enough that quasi-neutrality ($n_e \approx Z n_i$) holds

(c) Sufficiently many particles in a Debye sphere (statistics)

Collective behavior

Conditions:

(a) $\tau \ll \tau_n \equiv \dfrac{1}{n_n \sigma v_{\rm th}}$

tokamak: $\tau \ll 2.4 \times 10^6\,\mathrm{s}$

corona: $\tau \ll 2 \times 10^{20}\,\mathrm{s}$;

(b) $\lambda \gg \lambda_D \equiv \sqrt{\dfrac{\epsilon_0 kT}{e^2 n}}$

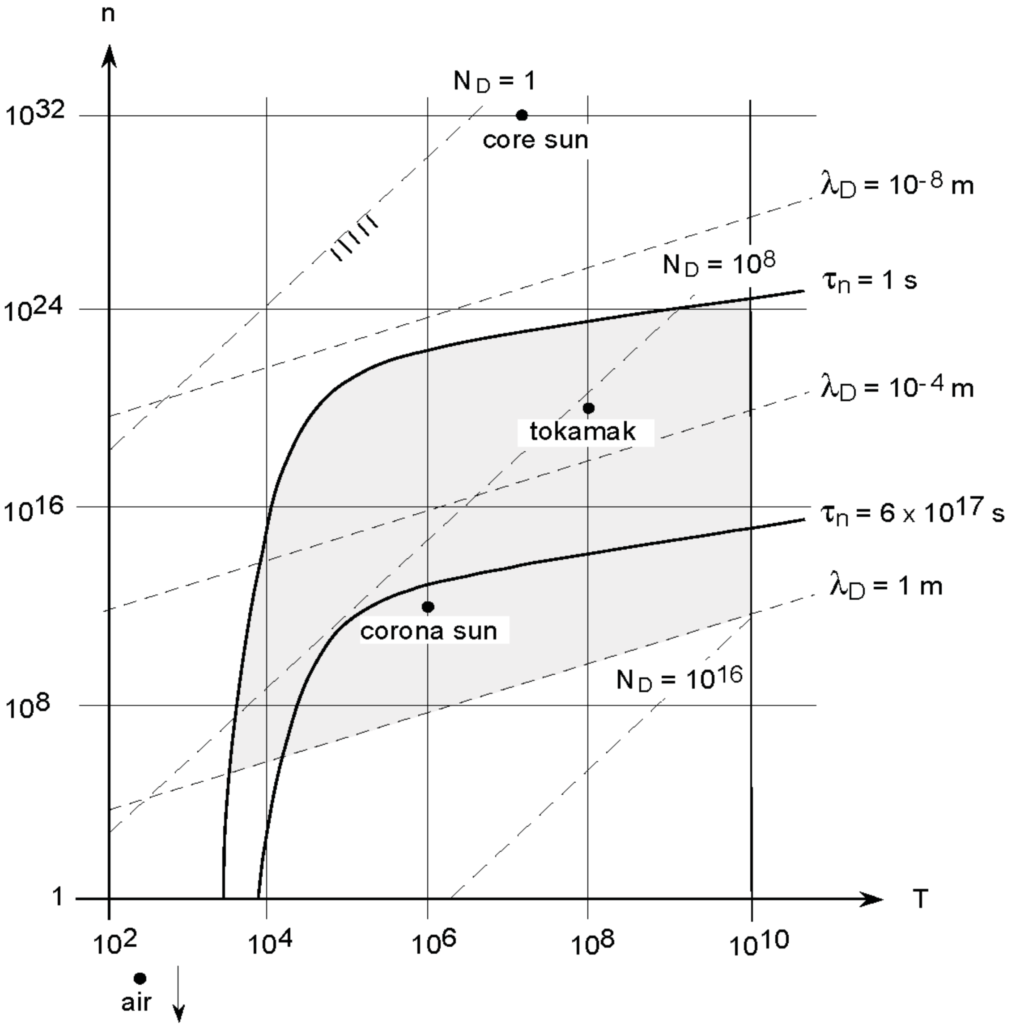
tokamak: $\lambda_D = 7\times10^{-5}\,\mathrm{m}$

corona: $\lambda_D = 0.07\,\mathrm{m}$;

(c) $N_D \equiv \frac{4}{3}\pi\lambda_D^3\, n \gg 1$

tokamak: $N_D = 1.4 \times 10^8$

corona: $N_D = 1.4 \times 10^9$.



Basic assumptions/applicability

So far, only the electric field appeared. (LOCAL)

Macroscopic definition:

For a valid macroscopic model of magnetized plasma dynamical configurations, size, duration, density, and magnetic field strength should be large enough to establish fluid behavior and to average out the microscopic phenomena (i.e. collective plasma oscillations and cyclotron motions of electrons and ions).

Now, the magnetic field enters: (GLOBAL !)

(a) $\tau \gg \Omega_i^{-1} \sim B^{-1}$ (time scale longer than inverse cyclotron frequency);

(b) $\lambda \gg R_i \quad \sim B^{-1}$ (length scale larger than cyclotron radius).

$\Rightarrow$ MHD $\equiv$ magnetohydrodynamics

Basic assumptions/applicability

- **Macroscopic behaviour**, i.e.

  $l_0 >>$ typical microscopic length scales (R_i, R_e, λ_D, ...)

- **One single fluid**

  e.g. cold plasmas: not fully ionised → multi-fluid

- **Electrically quasi-neutral**, i.e.

  $|Zn_i - n_e| << n_i, n_e \leftrightarrow \lambda_D << l_0$

- **Non-relativistic speeds**:

  $v << c$

- **Thermodynamic equilibrium** with distribution function close to Maxwellian, i.e.

  $t_0 >>$ *collision times*, $l_0 >>$ *mean free path length*

- **Equations written in inertial system**

  (e.g. rotation: Coriolis and centrifugal forces to be added)

Next: MHD waves & instabilities

… after the coffee break!