The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

2292-12

School and Conference on Analytical and Computational Astrophysics

14 - 25November, 2011

MHD Waves and Instabilities

Stefaan Poedts

K.U. Leuven, Centre for Plasma Astrophysics, Leuven Belgium

MHD waves and instabilities

Stefaan Poedts

School and Conference on Analytical and Computational Astrophysics,
Abdus Salam ICTP, Trieste, Italy, 14-27 November 2011

Motivation

- plasma **WAVES and INSTABILITIES** play an important role. . .
  - in the *dynamics* of plasma perturbations
  - in *energy conversion and transport*
  - in the *heating & acceleration* of plasma

- characteristics (ν, λ, amplitude. . .) are determined by the ambient plasma

$\Rightarrow$ can be exploited as a *diagnostic tool* for plasma parameters, *e.g.*

- wave generation, propagation, and dissipation in a confined plasma
  - $\Rightarrow$ *helioseismology* *(e.g. Gough '83)*
  - $\Rightarrow$ *MHD spectroscopy* *(e.g. Goedbloed et al. '93)*
- interaction of external waves with (magnetic) plasma structures
  - $\Rightarrow$ *sunspot seismology* *(e.g. Thomas et al. '82, Bogdan '91)*
  - $\Rightarrow$ *AR / coronal seismology* *(e.g. Nakariakov et al. 2000)*

Outline

- **Ideal MHD waves:** different representations and reductions of the linearized MHD equations, obtaining the three main waves, dispersion diagrams
- **Phase and group diagrams:** propagation of plane waves and wave packets, asymptotic properties
- **Effects of inhomogeneity**
  - **Continuous spectra**
  - **Instability**
- **Resistive MHD spectrum**

Cf. sound waves

- Perturb the gas dynamic equations ($\mathbf{B} = 0$),

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \tag{1}$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) + \nabla p = 0\,, \tag{2}$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0\,, \tag{3}$$

about infinite, homogeneous gas at rest,

$$\begin{aligned} \rho(\mathbf{r},t) &= \rho_0 + \rho_1(\mathbf{r},t) \qquad (\text{where } |\rho_1| \ll \rho_0 = \text{const})\,,\\ p(\mathbf{r},t) &= p_0 + p_1(\mathbf{r},t) \qquad (\text{where } |p_1| \ll p_0 = \text{const})\,,\\ \mathbf{v}(\mathbf{r},t) &= \phantom{p_0 + {}} \mathbf{v}_1(\mathbf{r},t) \qquad (\text{since } \mathbf{v}_0 = 0)\,. \end{aligned} \tag{4}$$

$\Rightarrow$ *Linearised equations of gas dynamics:*

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0\,, \tag{5}$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + \nabla p_1 = 0\,, \tag{6}$$

$$\frac{\partial p_1}{\partial t} + \gamma p_0 \nabla \cdot \mathbf{v}_1 = 0\,. \tag{7}$$

Cf. sound waves

Wave equation

- Equation for ρ_1 does not couple to the other equations: drop. Remaining equations give *wave equation for sound waves*:

$$\frac{\partial^2 \mathbf{v}_1}{\partial t^2} - c^2 \, \nabla\nabla \cdot \mathbf{v}_1 = 0 \,, \tag{8}$$

  where

$$c \equiv \sqrt{\gamma p_0 / \rho_0} \tag{9}$$

  is *the velocity of sound* of the background medium.

- Plane wave solutions

$$\mathbf{v}_1(\mathbf{r}, t) = \sum_{\mathbf{k}} \hat{\mathbf{v}}_{\mathbf{k}} \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{10}$$

  turn the wave equation (8) into an algebraic equation:

$$\left(\omega^2 \mathsf{I} - c^2 \, \mathbf{k}\mathbf{k} \right) \cdot \hat{\mathbf{v}} = 0 \,. \tag{11}$$

- For $\mathbf{k} = k \, \mathbf{e}_z$, the solution is:

$$\omega = \pm k \, c \,, \qquad \hat{v}_x = \hat{v}_y = 0 \,, \quad \hat{v}_z \text{ arbitrary} \,, \tag{12}$$

  $\Rightarrow$ *Sound waves* propagating to the right ($+$) and to the left ($-$): *compressible* ($\nabla \cdot \mathbf{v} \neq 0$) and *longitudinal* ($\mathbf{v} \parallel \mathbf{k}$) waves.

Cf. sound waves

Counting

- There are also other solutions:

$$\omega^2 = 0\,, \qquad \hat{v}_x\,, \hat{v}_y \text{ arbitrary}\,, \qquad \hat{v}_z = 0\,, \tag{13}$$

  $\Rightarrow$ *incompressible transverse* ($\mathbf{v}_1 \perp \mathbf{k}$) *translations.* They do not represent interesting physics, but simply establish completeness of the velocity representation.

- Problem: 1st order system (5)–(7) for ρ_1, $\mathbf{v}_1$, p_1 has 5 degrees of freedom, whereas 2nd order system (8) for $\mathbf{v}_1$ appears to have 6 degrees of freedom ($\partial^2/\partial t^2 \to -\omega^2$). However, the 2nd order system actually only has 4 degrees of freedom, since ω^2 does not double the number of translations (13). Spurious doubling of the eigenvalue $\omega = 0$ happened when we applied the operator $\partial/\partial t$ to Eq. (6) to eliminate p_1.

- Hence, we *lost one degree of freedom* in the reduction to the wave equation in terms of $\mathbf{v}_1$ alone. This happened when we dropped Eq. (5) for ρ_1. Inserting $\mathbf{v}_1 = 0$ in the original system gives the signature of this lost mode:

$$\omega\hat{\rho} = 0 \quad \Rightarrow \quad \omega = 0\,, \quad \hat{\rho} \text{ arbitrary}\,, \quad \text{but } \hat{\mathbf{v}} = 0 \text{ and } \hat{p} = 0\,. \tag{14}$$

  $\Rightarrow$ *entropy wave*: perturbation of the density and, hence, of the entropy $S \equiv p\rho^{-\gamma}$. Like the translations (13), this mode does not represent important physics but is needed to account for the degrees of freedom of the different representations.

MHD waves

- Similar analysis for MHD in terms of ρ , $\mathbf{v}$, $e \left(\equiv \frac{1}{\gamma-1} p/\rho \right)$, and $\mathbf{B}$:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \,, \tag{15}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + (\gamma - 1) \nabla (\rho e) + (\nabla \mathbf{B}) \cdot \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{B} = 0 \,, \tag{16}$$

$$\frac{\partial e}{\partial t} + \mathbf{v} \cdot \nabla e + (\gamma - 1) e \nabla \cdot \mathbf{v} = 0 \,, \tag{17}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{B} + \mathbf{B} \nabla \cdot \mathbf{v} - \mathbf{B} \cdot \nabla \mathbf{v} = 0 \,, \qquad \nabla \cdot \mathbf{B} = 0 \,, \tag{18}$$

- *Linearise about plasma at rest,* $\mathbf{v}_0 = 0 \,, \rho_0 \,, e_0 \,, \mathbf{B}_0 = \text{const}$:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0 \,, \tag{19}$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + (\gamma - 1)(e_0 \nabla \rho_1 + \rho_0 \nabla e_1) + (\nabla \mathbf{B}_1) \cdot \mathbf{B}_0 - \mathbf{B}_0 \cdot \nabla \mathbf{B}_1 = 0 \,, \tag{20}$$

$$\frac{\partial e_1}{\partial t} + (\gamma - 1) e_0 \nabla \cdot \mathbf{v}_1 = 0 \,, \tag{21}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} + \mathbf{B}_0 \nabla \cdot \mathbf{v}_1 - \mathbf{B}_0 \cdot \nabla \mathbf{v}_1 = 0 \,, \qquad \nabla \cdot \mathbf{B}_1 = 0 \,. \tag{22}$$

MHD waves

Transformation

- *Sound* and vectorial *Alfvén speed*,

$$c \equiv \sqrt{\frac{\gamma p_0}{\rho_0}}, \qquad \mathbf{b} \equiv \frac{\mathbf{B}_0}{\sqrt{\rho_0}}, \tag{23}$$

and dimensionless variables,

$$\tilde{\rho} \equiv \frac{\rho_1}{\gamma\,\rho_0}, \qquad \tilde{\mathbf{v}} \equiv \frac{\mathbf{v}_1}{c}, \qquad \tilde{e} \equiv \frac{e_1}{\gamma\, e_0}, \qquad \tilde{\mathbf{B}} \equiv \frac{\mathbf{B}_1}{c\sqrt{\rho_0}}, \tag{24}$$

$\Rightarrow$ *linearised MHD equations* with coefficients c and $\mathbf{b}$:

$$\gamma \frac{\partial \tilde{\rho}}{\partial t} + c\,\nabla \cdot \tilde{\mathbf{v}} = 0\,, \tag{25}$$

$$\frac{\partial \tilde{\mathbf{v}}}{\partial t} + c\,\nabla\tilde{\rho} + c\,\nabla\tilde{e} + (\nabla\tilde{\mathbf{B}})\cdot\mathbf{b} - \mathbf{b}\cdot\nabla\tilde{\mathbf{B}} = 0\,, \tag{26}$$

$$\frac{\gamma}{\gamma - 1}\frac{\partial \tilde{e}}{\partial t} + c\,\nabla \cdot \tilde{\mathbf{v}} = 0\,, \tag{27}$$

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} + \mathbf{b}\,\nabla\cdot\tilde{\mathbf{v}} - \mathbf{b}\cdot\nabla\tilde{\mathbf{v}} = 0\,, \qquad \nabla\cdot\tilde{\mathbf{B}} = 0\,. \tag{28}$$

MHD waves

Symmetry

- *Plane wave solutions*, with $\mathbf{b}$ and $\mathbf{k}$ arbitrary now:

$$\tilde{\rho} = \tilde{\rho}(\mathbf{r}, t) = \hat{\rho}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\,, \quad \text{etc.} \tag{29}$$

yields an algebraic system of eigenvalue equations:

$$\begin{aligned}
c\,\mathbf{k}\cdot\hat{\mathbf{v}} &= \gamma\,\omega\,\hat{\rho}\,, \\
\mathbf{k}\,c\,\hat{\rho} \qquad + \mathbf{k}\,c\,\hat{e} + (\mathbf{k}\mathbf{b}\cdot - \mathbf{k}\cdot\mathbf{b})\,\hat{\mathbf{B}} &= \omega\,\hat{\mathbf{v}}\,, \\
c\,\mathbf{k}\cdot\hat{\mathbf{v}} &= \tfrac{\gamma}{\gamma-1}\,\omega\,\hat{e}\,, \\
(\mathbf{b}\mathbf{k}\cdot - \mathbf{b}\cdot\mathbf{k})\,\hat{\mathbf{v}} &= \omega\,\hat{\mathbf{B}}\,, \qquad \mathbf{k}\cdot\hat{\mathbf{B}} = 0\,.
\end{aligned} \tag{30}$$

$\Rightarrow$ **Symmetric eigenvalue problem!** (The equations for $\hat{\rho}$, $\hat{\mathbf{v}}$, $\hat{e}$, and $\hat{\mathbf{B}}$ appear to know about each other.) .

- The symmetry of the linearized system is closely related to an analogous property of the original nonlinear equations: *the nonlinear ideal MHD equations are symmetric hyperbolic partial differential equations*.

MHD waves

Matrix eigenvalue problem

- Choose $\mathbf{b} = (0, 0, b)\,, \quad \mathbf{k} = (k_\perp, 0, k_\parallel)$:

$$\begin{pmatrix} 0 & k_\perp c & 0 & k_\parallel c & 0 & 0 & 0 & 0 \\ k_\perp c & 0 & 0 & 0 & k_\perp c & -k_\parallel b & 0 & k_\perp b \\ 0 & 0 & 0 & 0 & 0 & 0 & -k_\parallel b & 0 \\ k_\parallel c & 0 & 0 & 0 & k_\parallel c & 0 & 0 & 0 \\ 0 & k_\perp c & 0 & k_\parallel c & 0 & 0 & 0 & 0 \\ 0 & -k_\parallel b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k_\parallel b & 0 & 0 & 0 & 0 & 0 \\ 0 & k_\perp b & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat\rho \\ \hat v_x \\ \hat v_y \\ \hat v_z \\ \hat e \\ \hat B_x \\ \hat B_y \\ \hat B_z \end{pmatrix} = \omega \begin{pmatrix} \gamma\,\hat\rho \\ \hat v_x \\ \hat v_y \\ \hat v_z \\ \frac{\gamma}{\gamma-1}\,\hat e \\ \hat B_x \\ \hat B_y \\ \hat B_z \end{pmatrix} . \tag{31}$$

$\Rightarrow$ **Another representation of the symmetry of linearized MHD equations.**

- New features of MHD waves compared to sound: occurrence of Alfvén speed b and anisotropy expressed by the two components $k_\parallel$ and $k_\perp$ of the wave vector. We could compute the dispersion equation from the determinant and study the associated waves, but we prefer again to exploit the much simpler velocity representation.

MHD wave equation

- Ignoring the magnetic field constraint $\mathbf{k}\cdot\hat{\mathbf{B}}=0$ in the 8×8 eigenvalue problem (31) would yield *one spurious eigenvalue $\omega=0$*. This may be seen by operating with the projector $\mathbf{k}\cdot$ onto Eq. (30)(d), which gives $\omega\,\mathbf{k}\cdot\hat{\mathbf{B}}=0$.
- Like in the gas dynamics problem, a *genuine but unimportant marginal entropy mode* is obtained for $\omega=0$ with $\hat{\mathbf{v}}=0$, $\hat{p}=0$, and $\hat{\mathbf{B}}=0$:

$$\omega=0\,,\qquad \hat{p}=\hat{e}+\hat{\rho}=0\,,\qquad \hat{S}=\gamma\hat{e}=-\gamma\hat{\rho}\neq 0\,. \tag{32}$$

- Both of these marginal modes are eliminated by exploiting *the velocity representation.* The perturbations ρ_1, e_1, $\mathbf{B}_1$ are expressed in terms of $\mathbf{v}_1$ by means of Eqs. (19), (21), and (22), and substituted into the momentum equation (20). This yields the **MHD wave equation for a homogeneous medium**:

$$\frac{\partial^2\mathbf{v}_1}{\partial t^2}-\Big[\,(\mathbf{b}\cdot\nabla)^2\,\mathbf{I}+(b^2+c^2)\,\nabla\,\nabla-\mathbf{b}\cdot\nabla\,(\nabla\,\mathbf{b}+\mathbf{b}\,\nabla)\,\Big]\cdot\mathbf{v}_1=0\,. \tag{33}$$

The sound wave equation (8) is obtained for the special case $\mathbf{b}=0$.

MHD wave equation (cont'd)

- Inserting plane wave solutions gives the required eigenvalue equation:

$$\left\{ \left[\omega^2 - (\mathbf{k}\cdot\mathbf{b})^2 \right] \mathsf{I} - (b^2 + c^2)\,\mathbf{k}\,\mathbf{k} + \mathbf{k}\cdot\mathbf{b}\,(\mathbf{k}\,\mathbf{b} + \mathbf{b}\,\mathbf{k}) \right\} \cdot \hat{\mathbf{v}} = 0\,, \tag{34}$$

or, in components:

$$\begin{pmatrix} -k_\perp^2(b^2+c^2) - k_\parallel^2 b^2 & 0 & -k_\perp k_\parallel c^2 \\ 0 & -k_\parallel^2 b^2 & 0 \\ -k_\perp k_\parallel c^2 & 0 & -k_\parallel^2 c^2 \end{pmatrix} \begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{pmatrix} = -\omega^2 \begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{pmatrix}. \tag{35}$$

Hence, a 3×3 symmetric matrix equation is obtained in terms of the variable $\hat{\mathbf{v}}$, with *quadratic eigenvalue* ω^2, corresponding to the original 6×6 representation with eigenvalue ω (resulting from elimination of the two marginal modes).

- Determinant yields the **dispersion equation:**

$$\det = \omega\,(\omega^2 - k_\parallel^2 b^2)\,\left[\omega^4 - k^2(b^2+c^2)\,\omega^2 + k_\parallel^2 k^2 b^2 c^2\right] = 0 \tag{36}$$

(where we have artificially included a factor ω for the marginal entropy wave).

MHD waves

Roots

1) Entropy waves:

$$\omega = \omega_E \equiv 0\,, \tag{37}$$

$$\hat{\mathbf{v}} = \hat{\mathbf{B}} = 0\,, \quad \hat{p} = 0\,, \quad \text{but} \quad \hat{s} \neq 0\,. \tag{38}$$

$\Rightarrow$ just perturbation of thermodynamic variables.

2) Alfvén waves:

$$\omega^2 = \omega_A^2 \equiv k_\parallel^2 b^2 \quad \rightarrow \quad \omega = \pm\omega_A\,, \tag{39}$$

$$\hat{v}_x = \hat{v}_z = \hat{B}_x = \hat{B}_z = \hat{s} = \hat{p} = 0\,, \quad \hat{B}_y = -\hat{v}_y \neq 0\,. \tag{40}$$

$\Rightarrow$ transverse $\hat{\mathbf{v}}$ and $\hat{\mathbf{B}}$ so that field lines follow the flow.

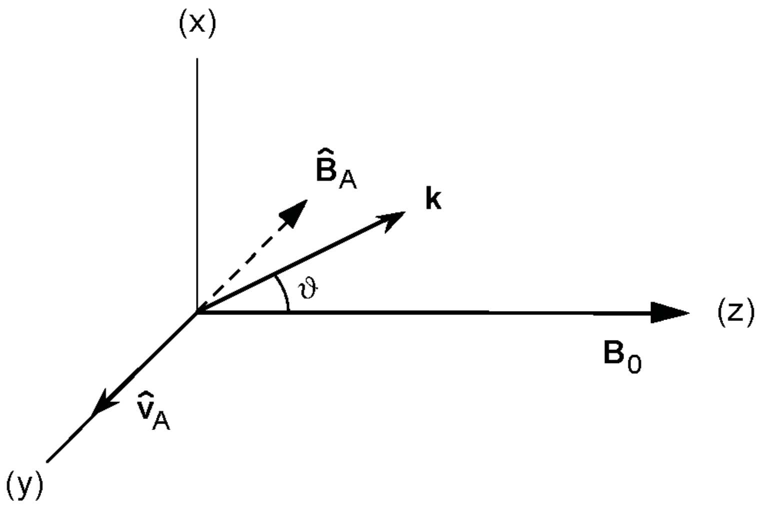
3) Fast (+) and Slow (−) magnetoacoustic waves:

$$\omega^2 = \omega_{s,f}^2 \equiv \tfrac{1}{2}k^2(b^2 + c^2)\left[\,1 \pm \sqrt{1 - \frac{4k_\parallel^2 b^2 c^2}{k^2(b^2+c^2)^2}}\,\right] \quad \rightarrow \quad \omega = \begin{cases} \pm\omega_s \\ \pm\omega_f \end{cases} \tag{41}$$

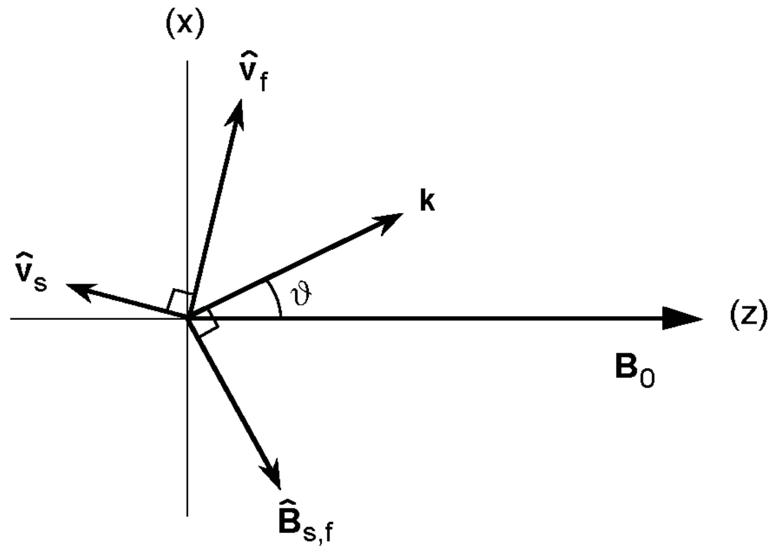
$$\hat{v}_y = \hat{B}_y = \hat{s} = 0\,, \quad \text{but} \quad \hat{v}_x\,, \hat{v}_z\,, \hat{p}\,, \hat{B}_x\,, \hat{B}_z \neq 0\,, \tag{42}$$

$\Rightarrow$ perturbations $\hat{\mathbf{v}}$ and $\hat{\mathbf{B}}$ in the plane through $\mathbf{k}$ and $\mathbf{B}_0$.

Eigenfunctions



Alfvén waves



Magnetosonic waves

- Note: **the eigenfunctions are mutually orthogonal:**

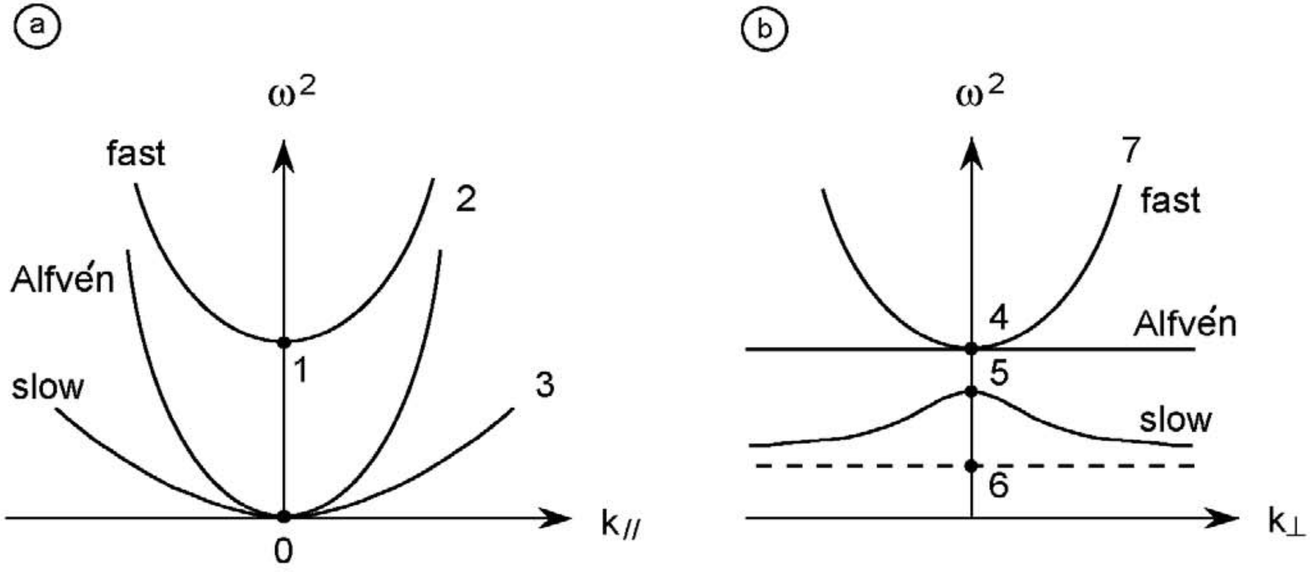
$$\hat{\mathbf{v}}_s \perp \hat{\mathbf{v}}_A \perp \hat{\mathbf{v}}_f \,. \tag{43}$$

$\Rightarrow$ Arbitrary velocity field may be decomposed at all times (e.g. at $t = 0$) in the three MHD waves: **the initial value problem is a well-posed problem**.

MHD waves

Dispersion diagrams (schematic)

[exact diagrams in book: Fig. 5.3, scaling $\bar{\omega} \equiv (l/b)\,\omega\,,\ \bar{k} \equiv k\,l$]



- Note: $\omega^2(k_\parallel = 0) = 0$ for Alfvén and slow waves $\Rightarrow$ potential onset of *instability.*
- Asymptotics of $\omega^2(k_\perp \to \infty)$ characterizes *local* behavior of the three waves:

$$\begin{cases} \partial\omega/\partial k_\perp > 0\,, & \omega_f^2 \to \infty \qquad \text{for fast waves,} \\ \partial\omega/\partial k_\perp = 0\,, & \omega_A^2 \to k_\parallel^2 b^2 \qquad \text{for Alfvén waves,} \\ \partial\omega/\partial k_\perp < 0\,, & \omega_s^2 \to k_\parallel^2 \dfrac{b^2 c^2}{b^2 + c^2} \qquad \text{for slow waves.} \end{cases} \tag{44}$$

Phase and group diagrams

Phase and group velocity

Dispersion equation $\omega = \omega(\mathbf{k}) \;\Rightarrow$ two fundamental concepts:

1. A *single plane wave* propagates in the direction of $\mathbf{k}$ with the **phase velocity**

$$\mathbf{v}_{\rm ph} \equiv \frac{\omega}{k}\,\mathbf{n}\,, \qquad \mathbf{n} \equiv \mathbf{k}/k = (\sin\vartheta, 0, \cos\vartheta)\,; \tag{45}$$

$\Rightarrow$ MHD waves are non-dispersive (only depend on angle ϑ, not on $|\mathbf{k}|$):

$$(\mathbf{v}_{\rm ph})_A \;\equiv\; b\cos\vartheta\,\mathbf{n}\,, \tag{46}$$

$$(\mathbf{v}_{\rm ph})_{s,f} \;\equiv\; \sqrt{\tfrac{1}{2}(b^2+c^2)}\sqrt{1\pm\sqrt{1-\sigma\cos^2\vartheta}}\,\mathbf{n}\,, \quad \sigma \equiv \frac{4b^2c^2}{(b^2+c^2)^2}\,. \tag{47}$$

2. A *wave packet* propagates with the **group velocity**

$$\mathbf{v}_{\rm gr} \equiv \frac{\partial\omega}{\partial\mathbf{k}} \quad \Big[\equiv \frac{\partial\omega}{\partial k_x}\mathbf{e}_x + \frac{\partial\omega}{\partial k_y}\mathbf{e}_y + \frac{\partial\omega}{\partial k_z}\mathbf{e}_z\Big]\,; \tag{48}$$

$\Rightarrow$ MHD caustics in directions $\mathbf{b}$, and mix of $\mathbf{n}$ and $\mathbf{t}$ ($\perp\,\mathbf{n}$):

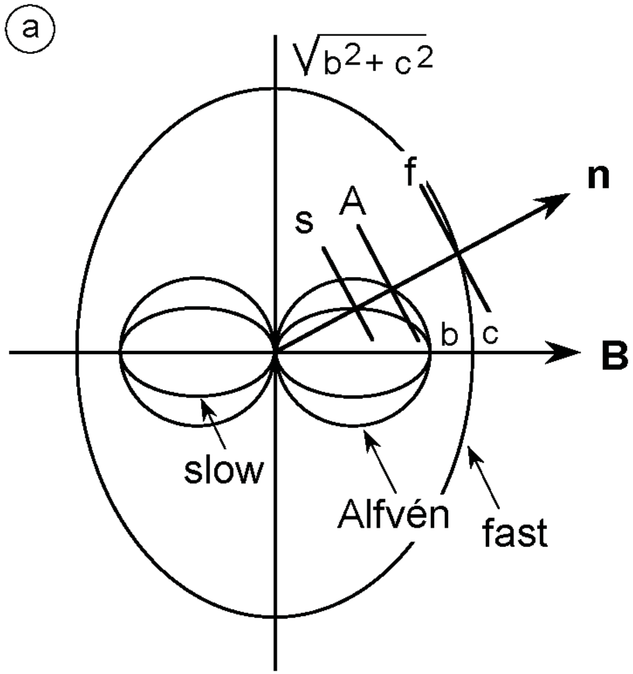
$$(\mathbf{v}_{\rm gr})_A \;=\; \mathbf{b}\,, \tag{49}$$

$$(\mathbf{v}_{\rm gr})_{s,f} \;=\; (v_{\rm ph})_{s,f}\left[\mathbf{n} \pm \frac{\sigma\sin\vartheta\cos\vartheta}{2\sqrt{1-\sigma\cos^2\vartheta}\,\big[1\pm\sqrt{1-\sigma\cos^2\vartheta}\,\big]}\,\mathbf{t}\right]. \tag{50}$$

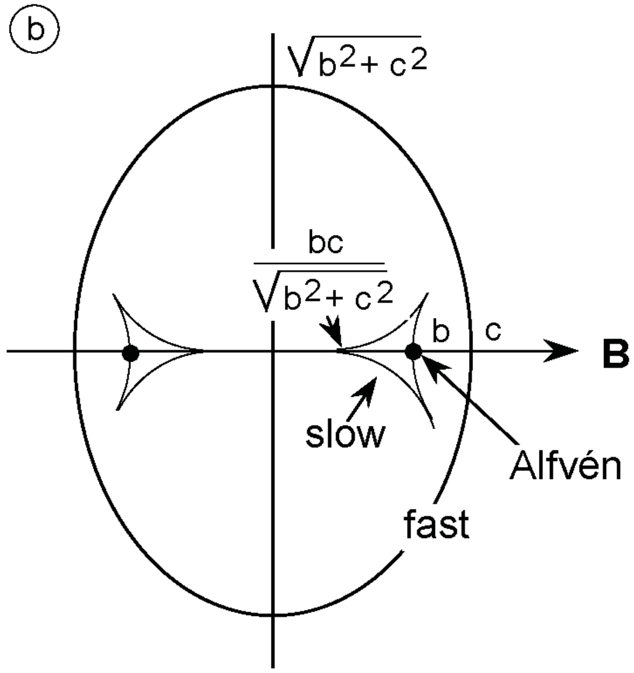
Phase and group diagrams

Friedrichs diagrams (schematic)

[exact diagrams in book: Fig. 5.5, parameter $c/b = \frac{1}{2}\gamma\beta\,,\ \beta \equiv 2p/B^2$]



Phase diagram
(plane waves)



Group diagram
(point disturbances)

Phase and group diagrams

Summary

- [*Entropy waves:* non-propagating density / entropy perturbations;]
- *Alfvén waves:* incompressible velocity perturbations $\perp$ plane of $\mathbf{k}$ & $\mathbf{B}$, preferably propagating $\parallel$ $\mathbf{B}$;
- *Fast magnetoacoustic waves:* compressible velocity perturbations in the plane of $\mathbf{k}$ & $\mathbf{B}$, generalization of sound waves with contributions of the magnetic pressure, propagating in all directions but fastest $\perp$ $\mathbf{B}$;
- *Slow magnetoacoustic waves:* compressible velocity perturbations in plane of $\mathbf{k}$ & $\mathbf{B}$, kind of sound waves with impeded propagation $\perp$ $\mathbf{B}$ (orthogonal to fast modes).

Connection with next subject

Group diagram has a much wider applicability than just wave propagation in infinite homogeneous plasmas: Construction of wave packet involves contributions of large $\mathbf{k}$ (small wavelengths) so that the **concept of group velocity is essentially a local one**. It returns in *non-linear MHD of inhomogeneous plasmas*, where the associated concept of **characteristics** describes the propagation of initial data information through the plasma.

Example: point perturbation triggers MHD waves in uniform plasma (friedrichs.qt)

Finite plasmas

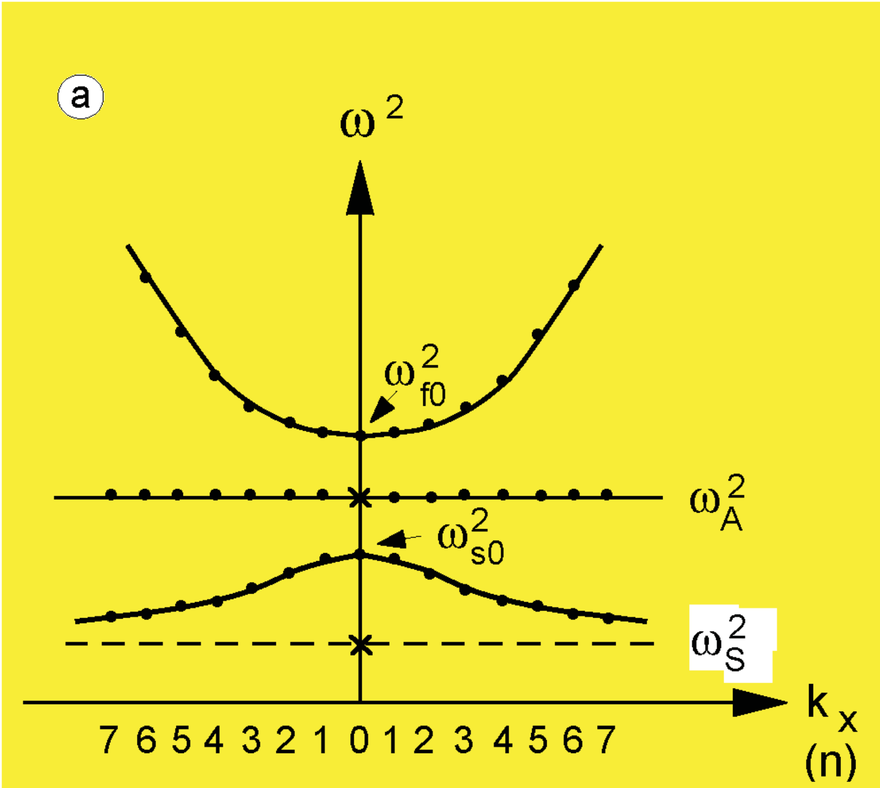
Finite homogeneous plasma slab

- equilibrium: $\mathbf{B}_0 = B_0 \mathbf{e}_z$
  - with $\rho_0\,, p_0\,, B_0 = \text{const}$
  - enclosed by plates at $x = \pm a$
- normal modes: $\sim \exp(-i\omega\, t)$

  $\Rightarrow$ *eigenvalueproblem*
- plane wave solutions $\sim \exp(\vec{k} \cdot \vec{x})$

  $\Rightarrow$ $k_x = \frac{\pi}{a} n$ is quantized

$\Rightarrow$ **three MHD waves: FMW, AW, SMW**



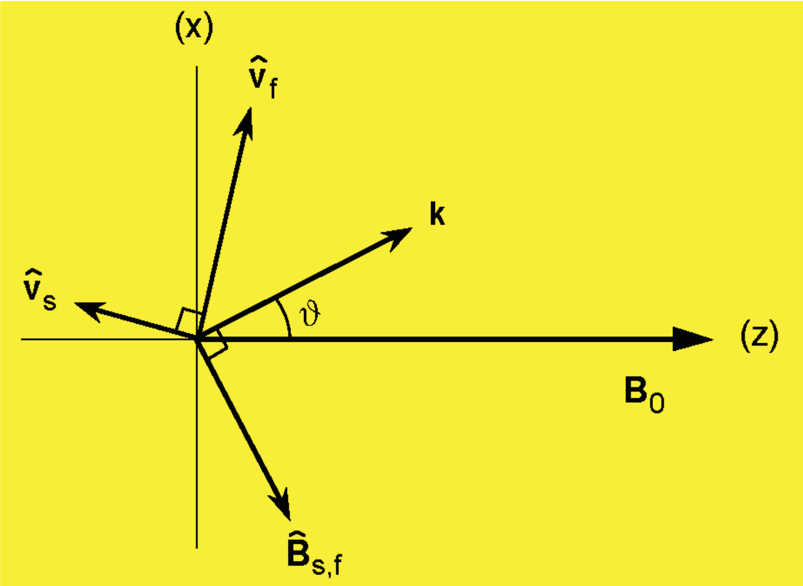
Dispersion diagram $\omega^2 = \omega^2(k_x)$ for k_y and k_z fixed

Finite plasmas

- *the eigenfunctions are mutually orthogonal*:

$$\hat{\mathbf{v}}_s \perp \hat{\mathbf{v}}_A \perp \hat{\mathbf{v}}_f$$

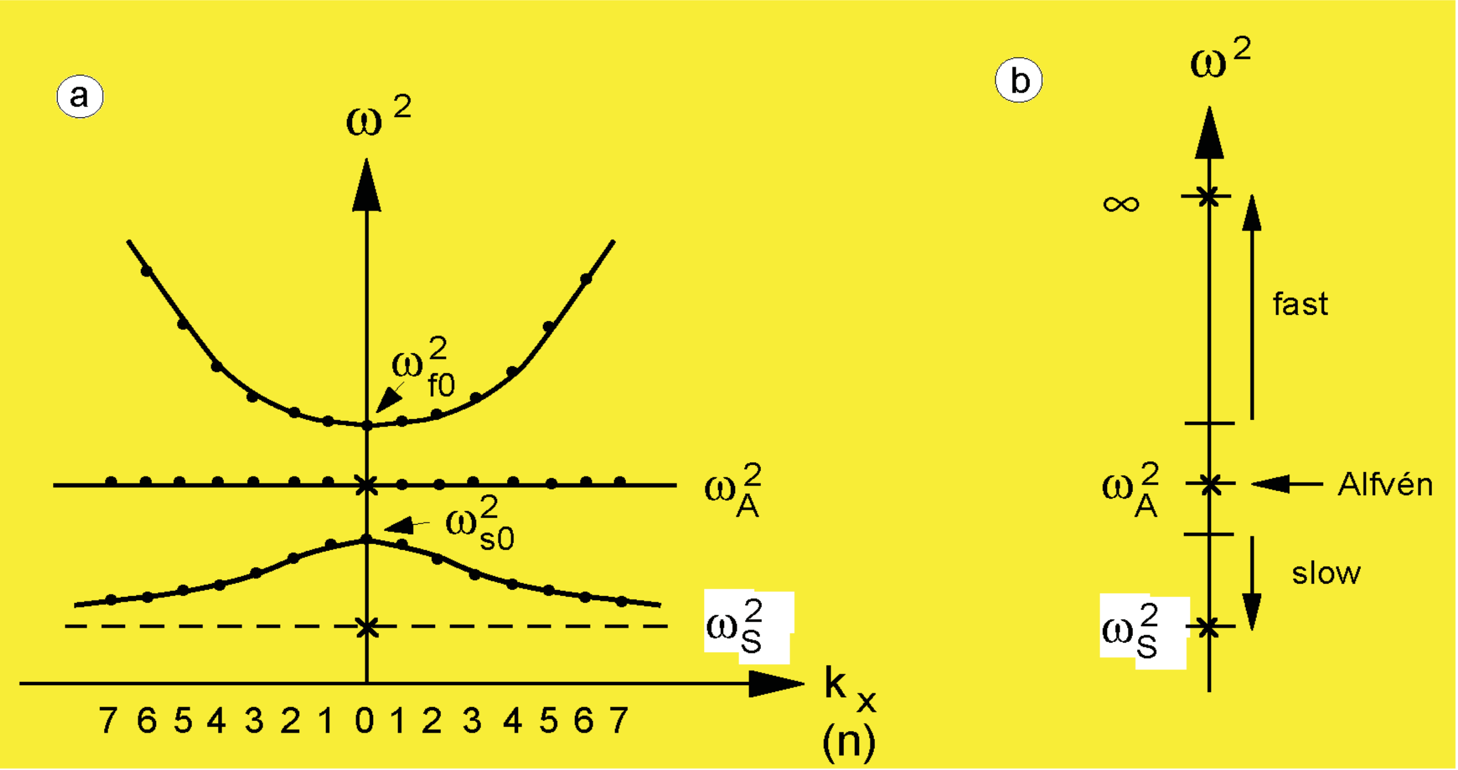
$\Rightarrow$ arbitrary velocity field may be decomposed in the three waves!



- **Remark**: for $\theta = 0$ the FMW is polarized almost perpendicular to $\vec{B}_0$ but *in* the $(\vec{k}, \vec{B}_0)$-plane

$\Rightarrow$ corresponds to the direction *normal to the magnetic flux surfaces* in the inhomogeneous plasmas discussed below

Finite plasmas



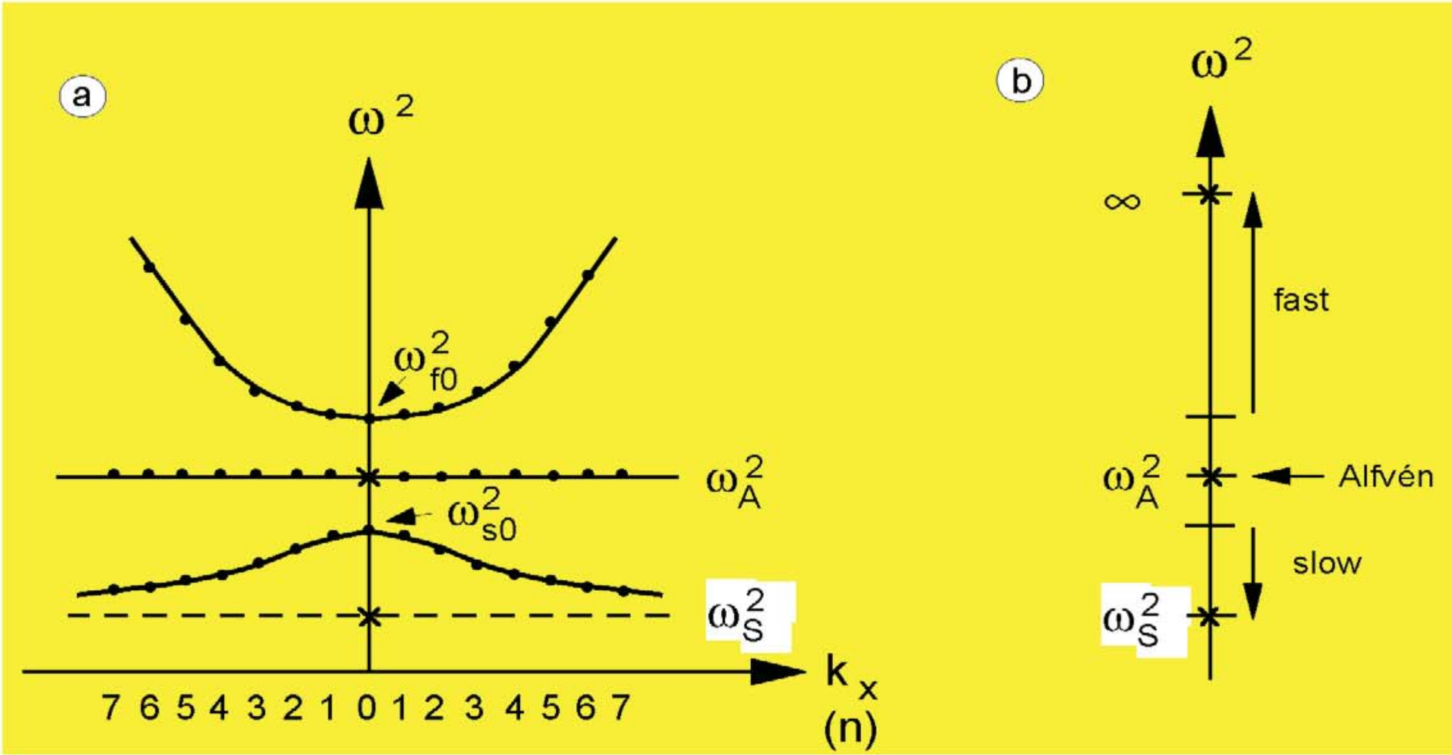
(a) Dispersion diagram $\omega^2 = \omega^2(k_x)$ for k_y and k_z fixed; (b) Corresponding structure of the spectrum.

- *the eigenfrequencies are well-ordered*:

$$0 \leq \omega_s^2 \leq \omega_{s0}^2 \leq \omega_A^2 \leq \omega_{f0}^2 \leq \omega_f^2 < \infty$$

$\Rightarrow$ crucial for spectral theory of MHD waves!

Finite plasmas

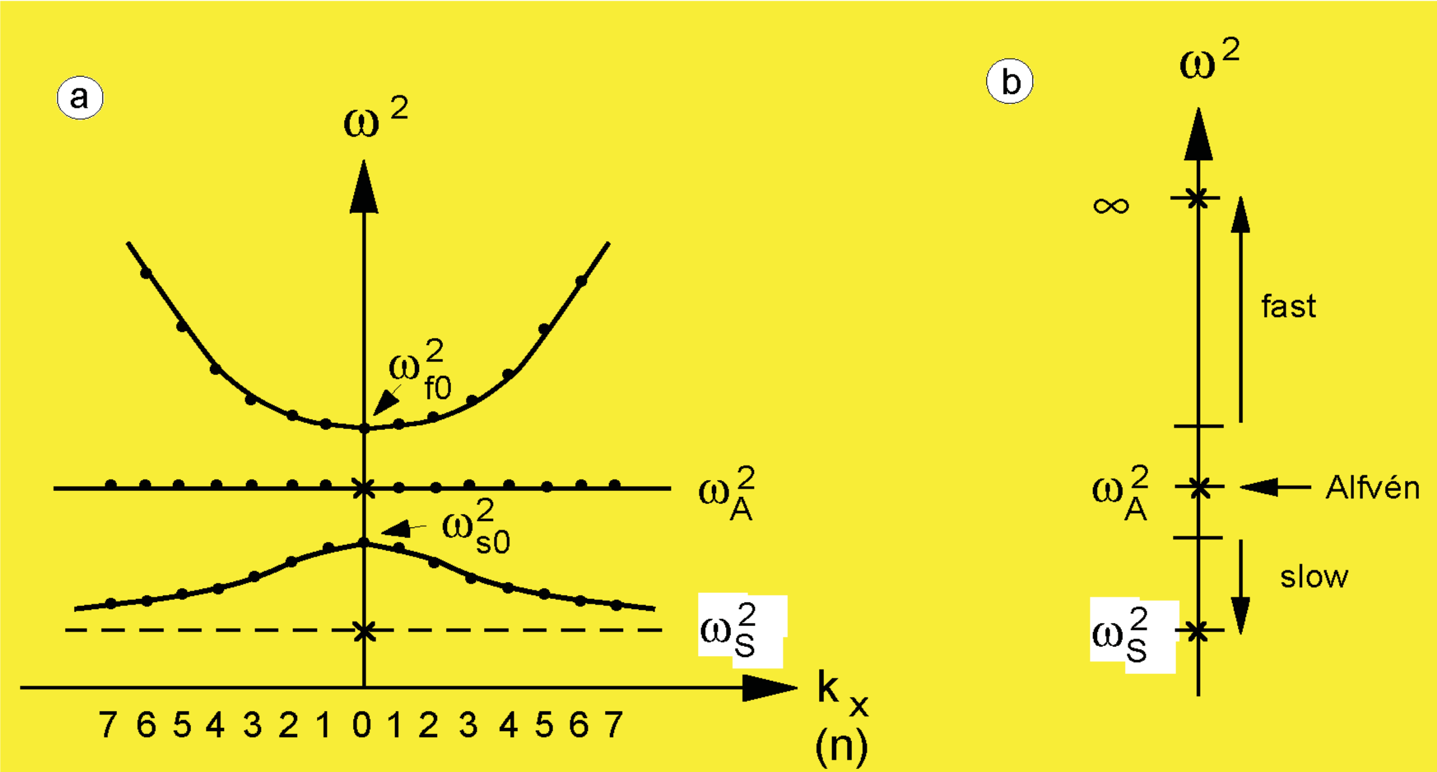


(a) Dispersion diagram $\omega^2 = \omega^2(k_x)$ for k_y and k_z fixed; (b) Corresponding structure of the spectrum.

- discrete eigenvalues of the fast subspectrum monotonically increase, so that

$\omega_F^2 \equiv \lim_{k_x \to \infty} \omega_f^2 \approx \lim_{k_x \to \infty} k_x^2(b^2 + c^2) = \infty$ is a formal *cluster point*

Finite plasmas

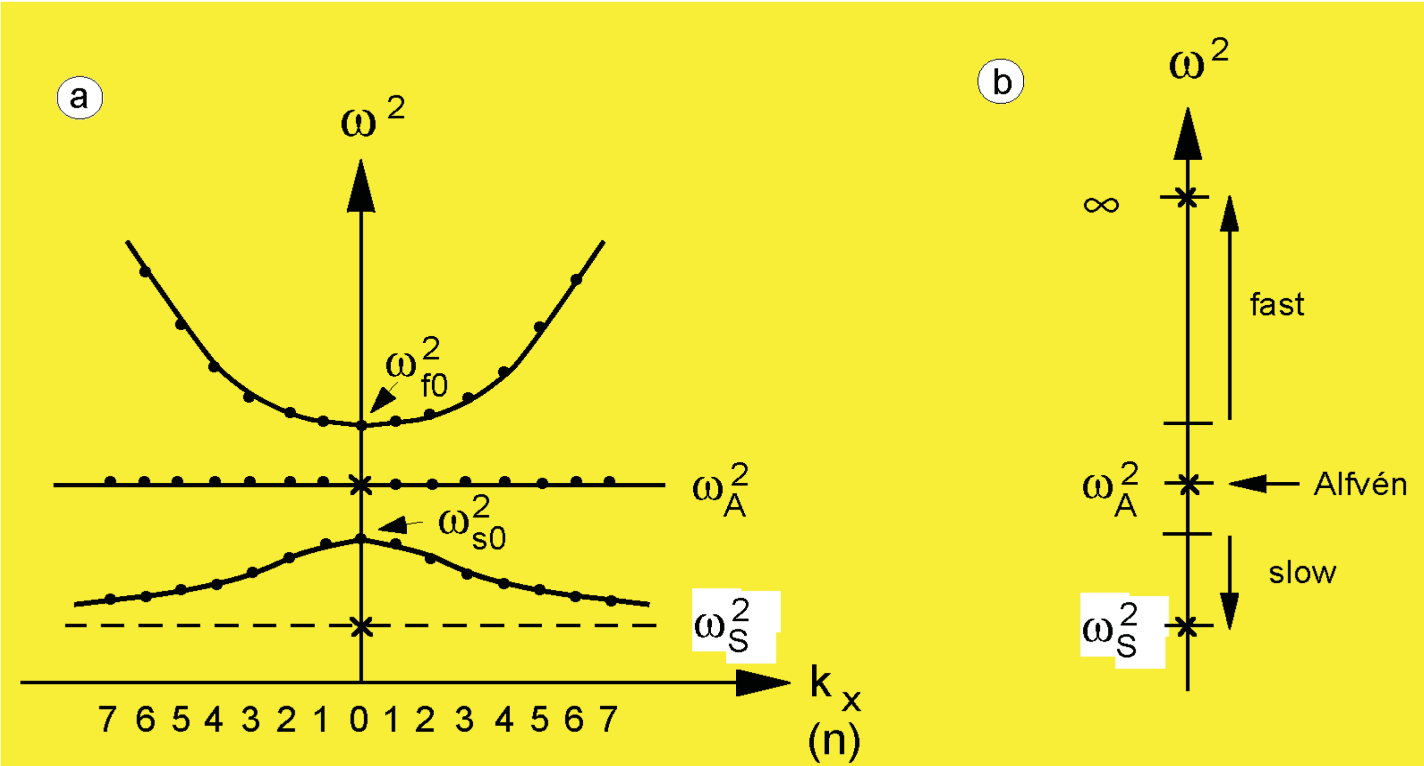


(a) Dispersion diagram $\omega^2 = \omega^2(k_x)$ for k_y and k_z fixed; (b) Corresponding structure of the spectrum.

- The eigenvalues ω_a^2 of the Alfvén subspectrum are *infinitely degenerate*, so that

$$\omega_A^2 \equiv \lim_{k_x \to \infty} \omega_a^2 = \omega_a^2 = k_\parallel^2 b^2$$

Finite plasmas



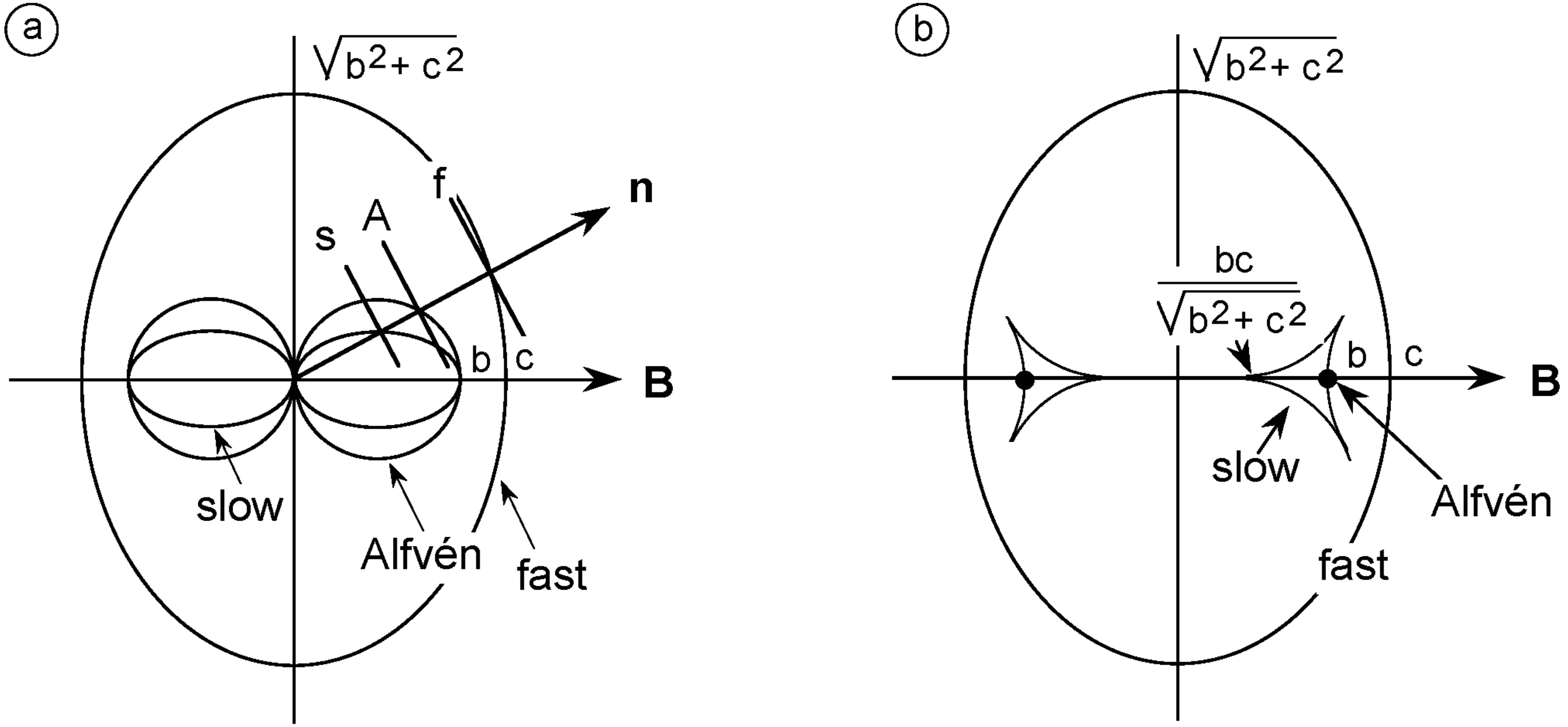
(a) Dispersion diagram $\omega^2 = \omega^2(k_x)$ for k_y and k_z fixed; (b) Corresponding structure of the spectrum.

- slow wave subspectrum monotonically decreases with a *cluster point* at

$$\omega_S^2 \equiv \lim_{k_x \to \infty} \omega_s^2 = k_\parallel^2 \frac{b^2 c^2}{b^2 + c^2}$$

Finite plasmas

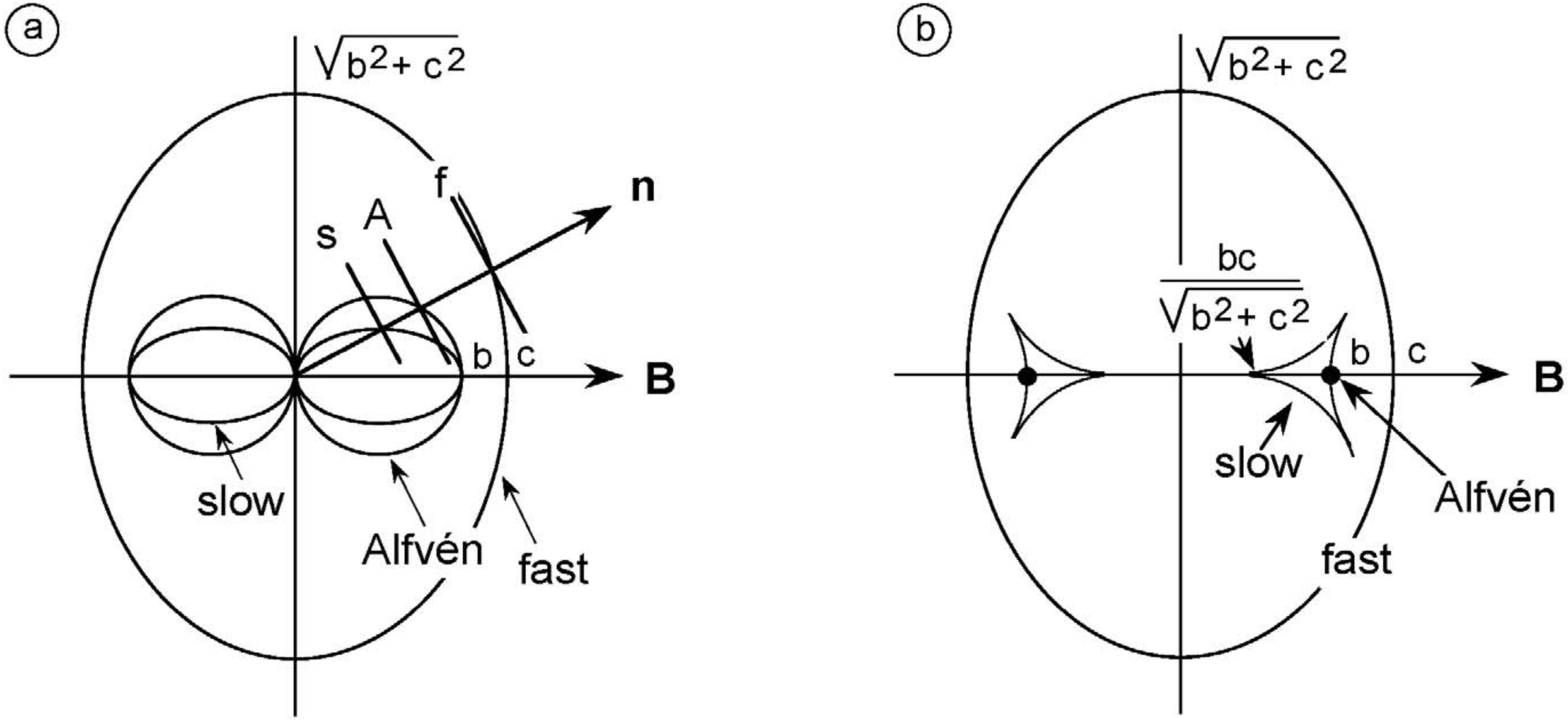
- three MHD waves exhibit a strong anisotropy depending on the direction of the wave vector $\mathbf{k}$ with respect to the magnetic field $\mathbf{B}_0$



Friedrichs diagrams: Schematic representation of (a) reciprocal normal surface (or phase diagram) and (b) ray surface (or group diagram) of the MHD waves ($b < c$).

Finite plasmas

⇒ in the corona the FMWs are the only waves that are able to **transfer energy across the magnetic surfaces**



Friedrichs diagrams: Schematic representation of (a) reciprocal normal surface (or phase diagram) and (b) ray surface (or group diagram) of the MHD waves ($b < c$).

Inhomogeneity

Finite inhomogeneous plasma slab

- $\mathbf{B}_0 = B_{0y}(x)\,\mathbf{e}_y + B_{0z}(x)\,\mathbf{e}_z\,, \quad \rho_0 = \rho_0(x)\,, \quad p_0 = p_0(x)$

- influence of inhomogeneity on the spectrum of MHD waves?

  $\Rightarrow$ different $\mathbf{k}$'s couple $\Rightarrow$ *wave transformations can occur*

  (e.g. fast wave character in one place, Alfvén character in another)

  $\Rightarrow$ two new phenomena, viz. *instabilities* and *continuous spectra*

- wave or spectral equation can be written in terms of

$$\xi \equiv \mathbf{e}_x \cdot \boldsymbol{\xi} = \xi_x, \quad \eta \equiv i\mathbf{e}_\perp \cdot \boldsymbol{\xi}, \quad \zeta \equiv i\mathbf{e}_\parallel \cdot \boldsymbol{\xi}$$

$\Rightarrow$ eliminate η and ζ with 2nd and 3rd component (algebraic in η and ζ):

$$\boxed{\frac{d}{dx}\frac{N}{D}\frac{d\xi}{dx} + [\,\rho(\omega^2 - f^2 b^2)\,]\,\xi = 0}$$

(Hain, Lust, Goedbloed equation)

Inhomogeneity

- the coefficient factor N/D of the ODE plays an important role in the analysis

$\Rightarrow$ may be written in terms of the four ω^2's introduced for homogeneous plasmas:

$$\frac{N}{D} = \rho(b^2+c^2)\,\frac{[\,\omega^2-\omega_A^2(x)\,]\,[\,\omega^2-\omega_S^2(x)\,]}{[\,\omega^2-\omega_{s0}^2(x)\,]\,[\,\omega^2-\omega_{f0}^2(x)\,]}$$

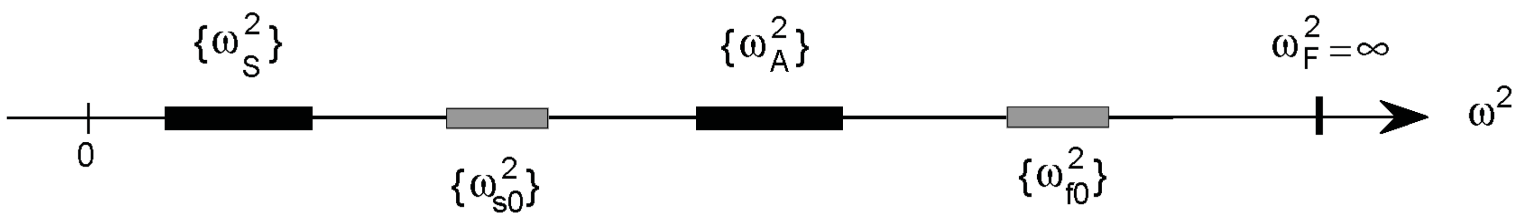
where

$$\omega_A^2(x) \equiv f^2 b^2 \equiv F^2/\rho\,, \qquad \omega_S^2(x) \equiv f^2\frac{b^2c^2}{b^2+c^2} \equiv \frac{\gamma p}{\gamma p + B^2}F^2/\rho\,,$$

$$\omega_{s0,f0}^2(x) \equiv \tfrac{1}{2}k_0^2(b^2+c^2)\left[\,1 \pm \sqrt{1-\frac{4f^2b^2c^2}{k_0^2(b^2+c^2)^2}}\,\right]$$

$\Rightarrow$ only **two continuous spectra** (2 apparent singularities)

$\Rightarrow$ the four finite ‘limiting frequencies’ now *spread out to a continuous range* :



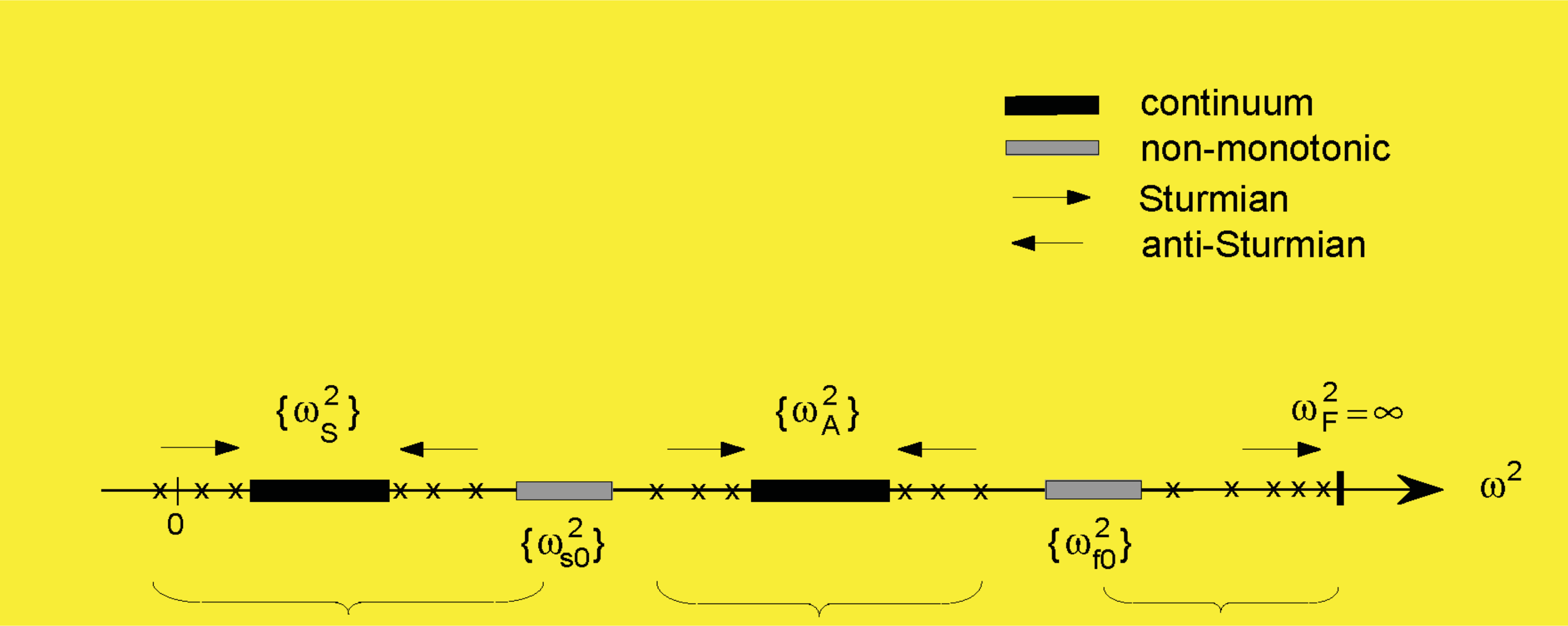
Inhomogeneity

- logarithmic contribution in ξ-component

$\Rightarrow$ but the dominant *(non-square integrable)* part of the eigenfunctions:

$$\xi_A \approx 0\,, \qquad \eta_A \approx \mathcal{P}\,\frac{1}{x - x_A(\omega^2)} + \lambda(\omega^2)\,\delta(x - x_A(\omega^2))\,, \qquad \zeta_A \approx 0\,,$$

$$\xi_S \approx 0\,, \qquad \eta_S \approx 0\,, \qquad \zeta_S \approx \mathcal{P}\,\frac{1}{x - x_S(\omega^2)} + \lambda(\omega^2)\,\delta(x - x_S(\omega^2))\,,$$

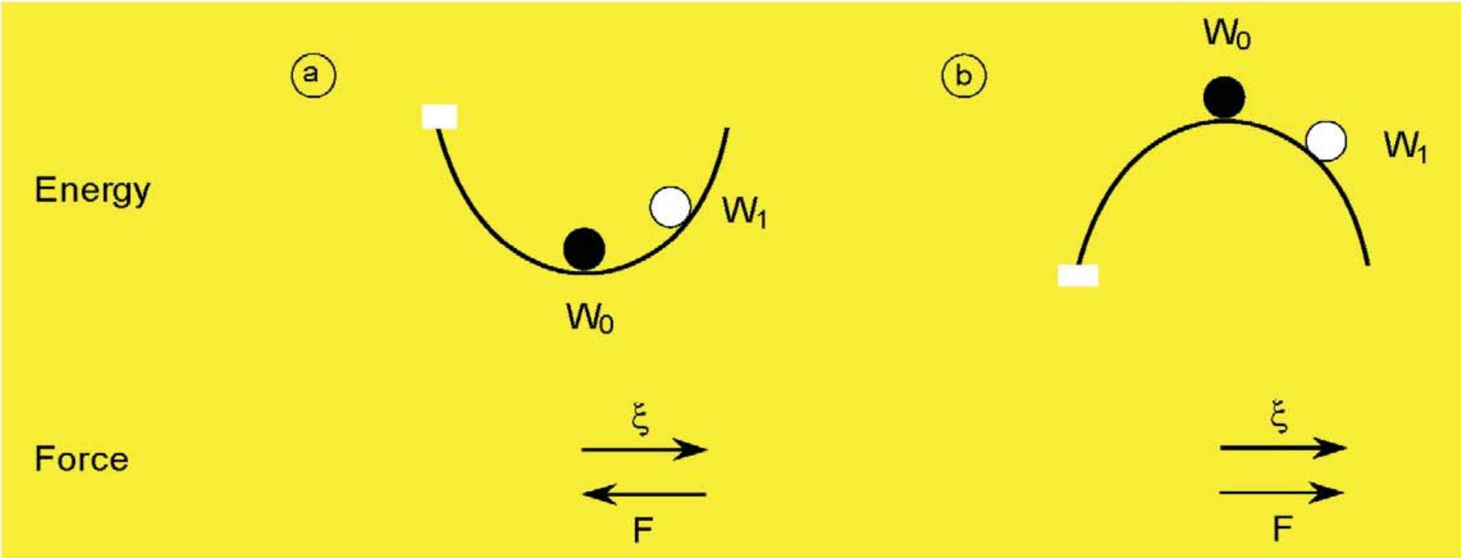


Schematic structure of the spectrum of an inhomogeneous plasma with gravity.

Instability

Two viewpoints

- *How does one know whether a dynamical system is stable or not?*



stable unstable

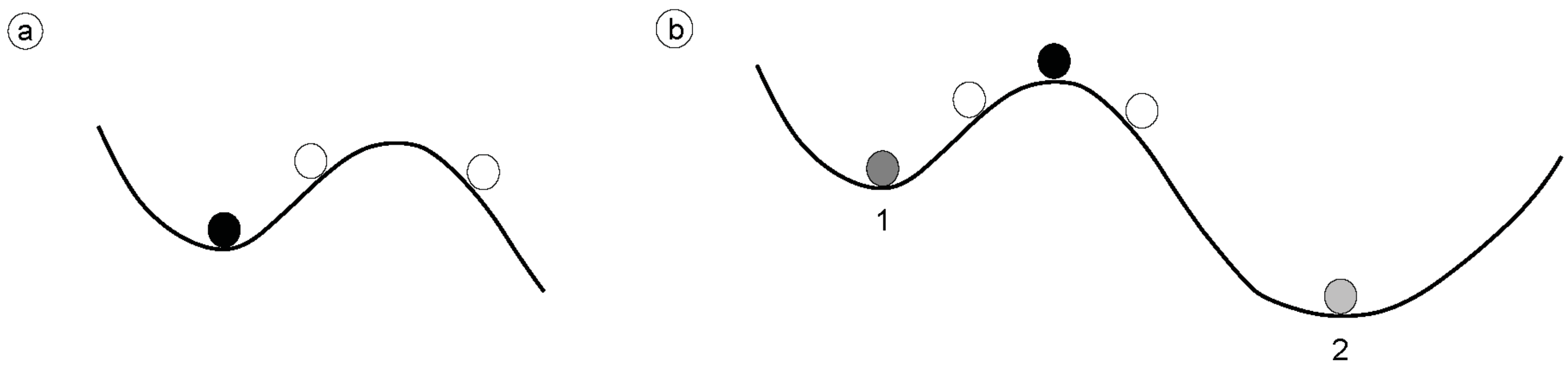
- Method: split the non-linear problem in *static equilibrium* (no flow) and small (linear) *time-dependent perturbations*.
- Two approaches: exploiting variational principles involving *quadratic forms* (energy), or solving *the partial differential equations themselves* (forces).

Aside: nonlinear stability

- Distinct from linear stability, involves *finite amplitude displacements:*

  (a) system can be linearly stable, nonlinearly unstable;

  (b) system can be linearly unstable, nonlinearly stable (e.g. evolving towards the equilibrium states 1 or 2).



- Quite relevant for topic of magnetic confinement, but too complicated at this stage.

Ideal MHD spectrum

- Consider *normal modes:*

$$\boldsymbol{\xi}(\mathbf{r}, t) = \hat{\boldsymbol{\xi}}(\mathbf{r})\, \mathrm{e}^{-\mathrm{i}\omega t} \,. \tag{24}$$

  $\Rightarrow$ Equation of motion becomes eigenvalue problem:

$$\mathbf{F}(\hat{\boldsymbol{\xi}}) = -\rho\omega^2 \hat{\boldsymbol{\xi}} \,. \tag{25}$$

- For given equilibrium, collection of eigenvalues $\{\omega^2\}$ is *spectrum of ideal MHD.*

  $\Rightarrow$ Generally both discrete and continuous ('improper') eigenvalues.

- The operator $\rho^{-1}\mathbf{F}$ is *self-adjoint* (for fixed boundary).

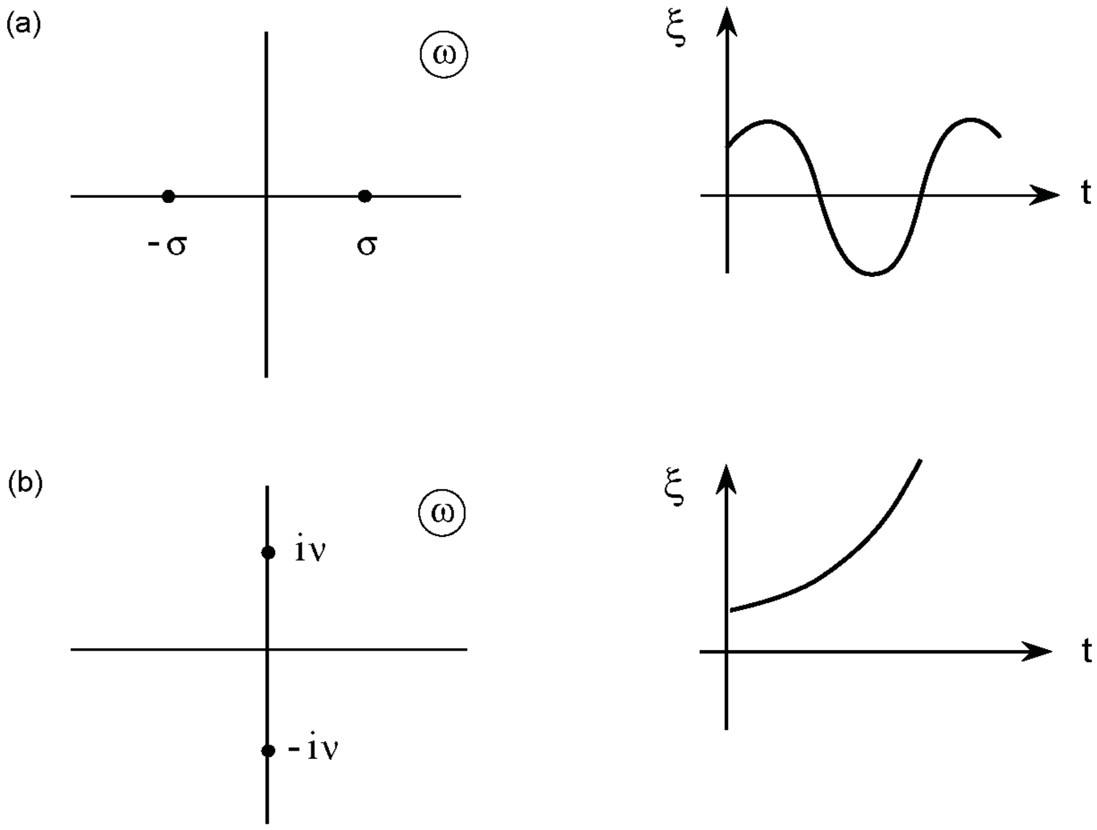
  $\Rightarrow$ The eigenvalues ω^2 are real.

  $\Rightarrow$ Same mathematical structure as for quantum mechanics!

Instability

- Since ω^2 real, ω itself either real or purely imaginary

  $\Rightarrow$ In ideal MHD, only *stable waves* ($\omega^2 > 0$) òr *exponential instabilities* ($\omega^2 < 0$):



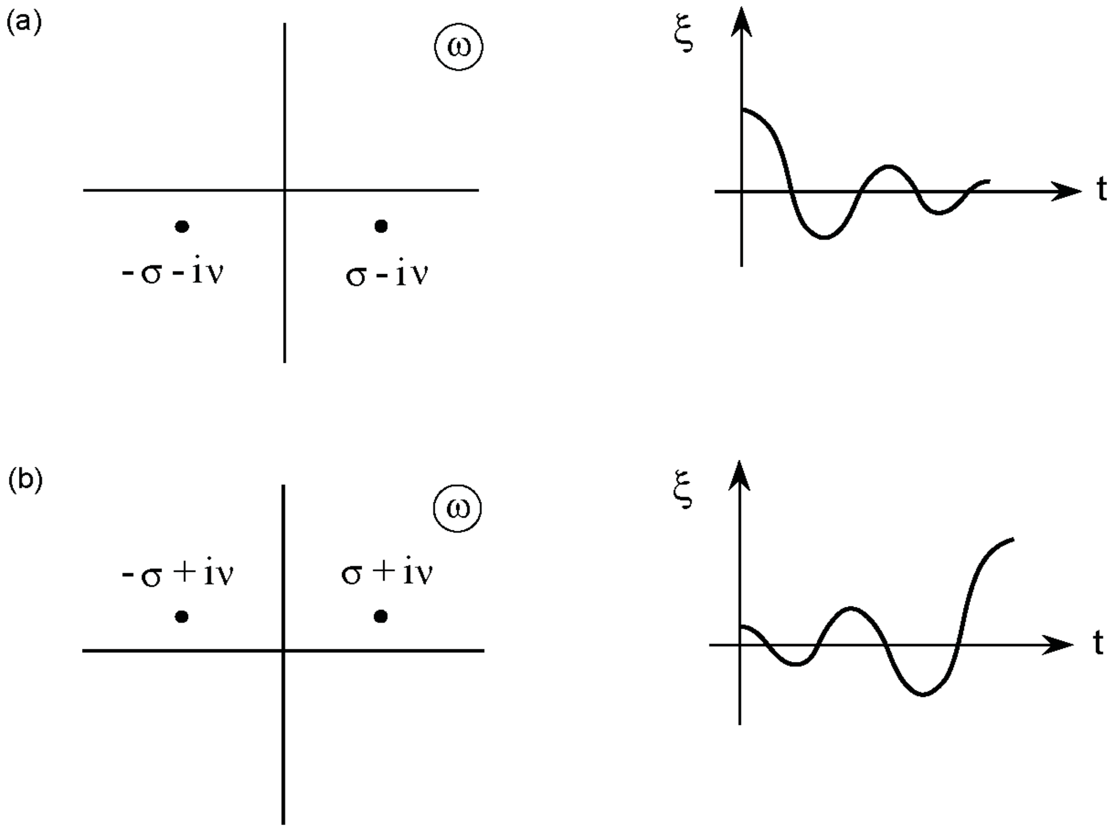
$\Rightarrow$ Crudely, $\mathbf{F}(\hat{\boldsymbol{\xi}}) \sim -\hat{\boldsymbol{\xi}}$ for $\omega^2 > 0$ and $\sim \hat{\boldsymbol{\xi}}$ for $\omega^2 < 0$ (cf. intuitive picture).

Instability

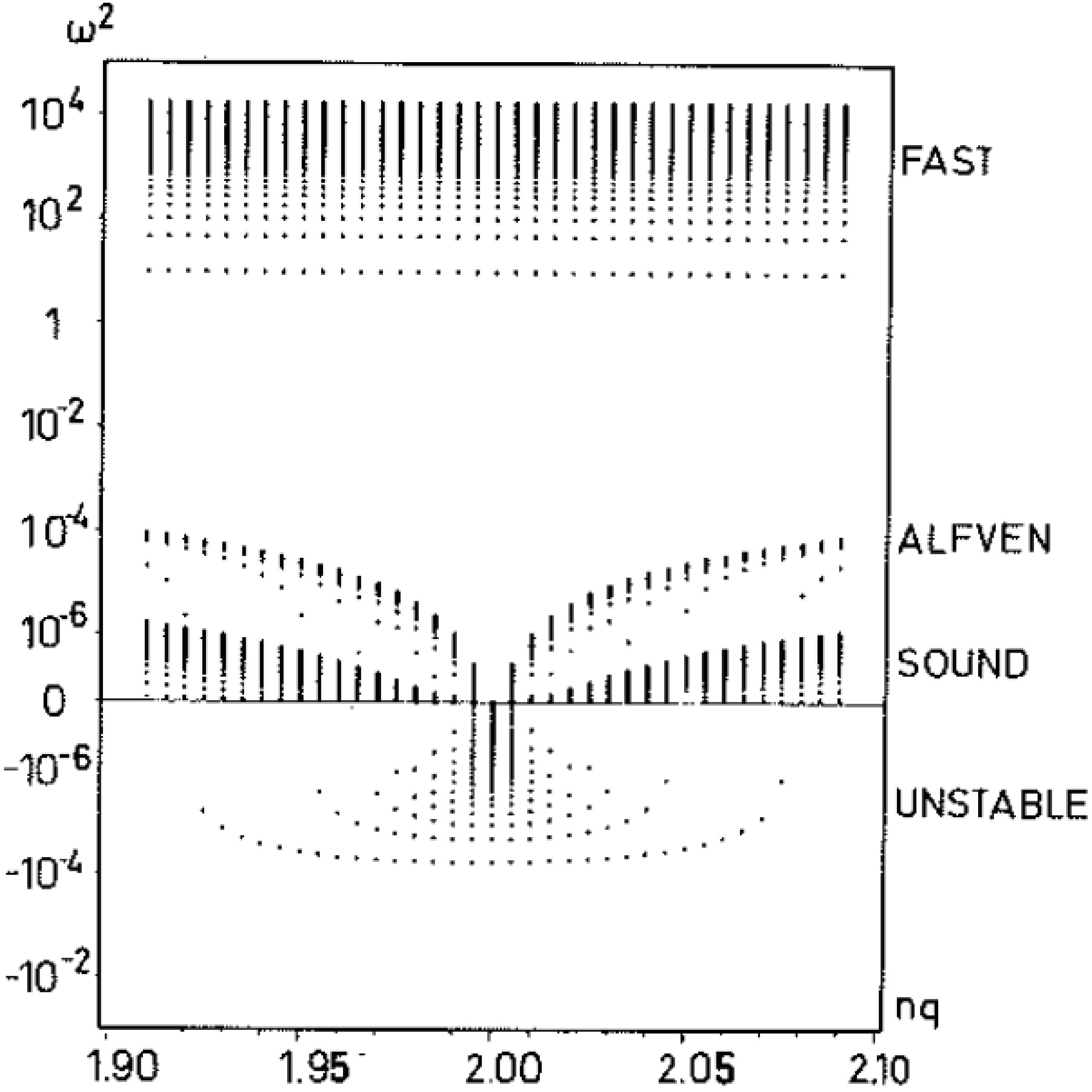
Dissipative MHD

- In resistive MHD, operators no longer self-adjoint, eigenvalues ω^2 complex.

  $\Rightarrow$ *Stable, damped waves* and *'overstable' modes ($\equiv$ instabilities):*



Example 1

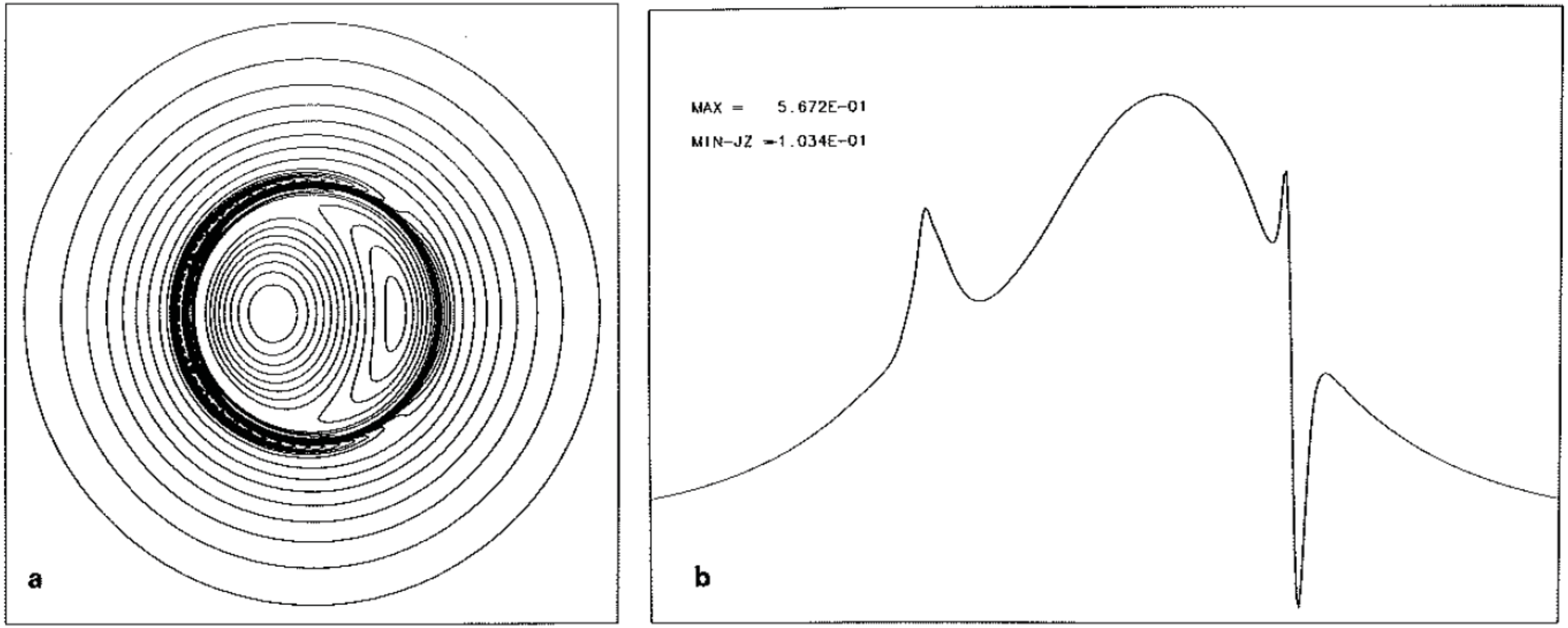


Complete ideal spectrum of a constant current cylindrical plasma column

for $n = 1$, $m = -2$, $k = 0.1$. Three branches occur: fast and slow magnetoacoustic and Alfvén waves. Negative values of ω^2 indicate exponentially growing instabilities.

[from Kerner 1989]

Example 2



Nonlinear evolution of an internal kink mode for finite resistivity $\eta = 10^{-6}$. (a) Contour plots of the magnetic flux with a pronounced $m = 1$ island; (b) longitudinal current J_z as a function of the radius in the plane $z = 0$ showing a current sheet at the rational surface $q = 1.0$.

[from Kerner 1989]

Resistivity

Basic equations

- We now present the **resistive normal mode analysis** of the plane slab. Starting point is the nonlinear resistive MHD equations:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \,, \tag{20}$$

$$\rho\,(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p + \rho \mathbf{g} + \mathbf{j} \times \mathbf{B} \,, \qquad \mathbf{j} = \nabla \times \mathbf{B} \,, \tag{21}$$

$$\frac{\partial p}{\partial t} = -\mathbf{v} \cdot \nabla p - \gamma p \nabla \cdot \mathbf{v} + (\gamma - 1)\, \eta |\, \mathbf{j}|^2 \,, \tag{22}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta\, \mathbf{j}) \,. \tag{23}$$

  Resistivity causes Ohmic dissipation term in the pressure equation and **resistive diffusion in the flux equation.** The latter completely changes the stability analysis.

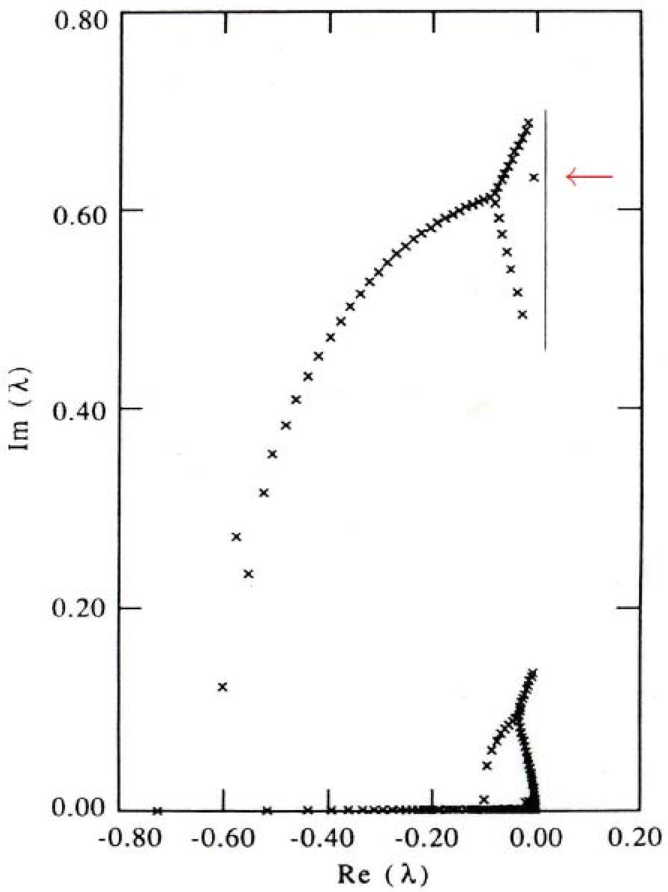
- We linearise the equations for perturbations about static equilibrium. Strictly, this assumption is not justified since resistivity causes magnetic field to decay. However, *the magnetic Reynolds number $R_m \equiv \mu_0 l_0 v_A/\eta$ is usually very large* so that this is a very slow process: $\tau \sim R_m \cdot \tau_A$, where τ_A is the characteristic Alfvén time for ideal MHD phenomena. The resistive modes grow on the much faster time scale $\sim (R_m)^\nu$, where $0 < \nu < 1$, so that the equilibrium may be considered static.

Resistivity

Resistive spectrum: surprise

- Resistivity changes order of system so that the singularities due to vanishing coefficient in front of highest derivative disappear. Hence, one should expect that *the ideal MHD continua split up in discrete modes.*
- This is what happens, but in a totally unexpected way: *multitude of discrete modes on triangular paths* appear in the complex $\lambda \equiv -\mathrm{i}\omega$ plane.
- *Collective effect of ideal MHD continua appears as the damped quasi-mode* inside triangle. This mode is robust: damping remains in the limit $\eta \longrightarrow 0$!

[Poedts & Kerner, PRL **66**, 2871 (1991)]



Conclusions

- MHD wave and instability theory is **mathematically sound**
- **Inhomogeneity** leads to continuous spectra & instabilities
- **Other effects**, such as
  - Dissipation
  - Background flow
  - Nonlinearity
  - Partial ionization, etc.

  complicate the MHD wave theory and its applications, e.g. to the solar corona
- Nevertheless, nice results are obtained, e.g. in coronal seismology & CME initiation (cf. later this week)

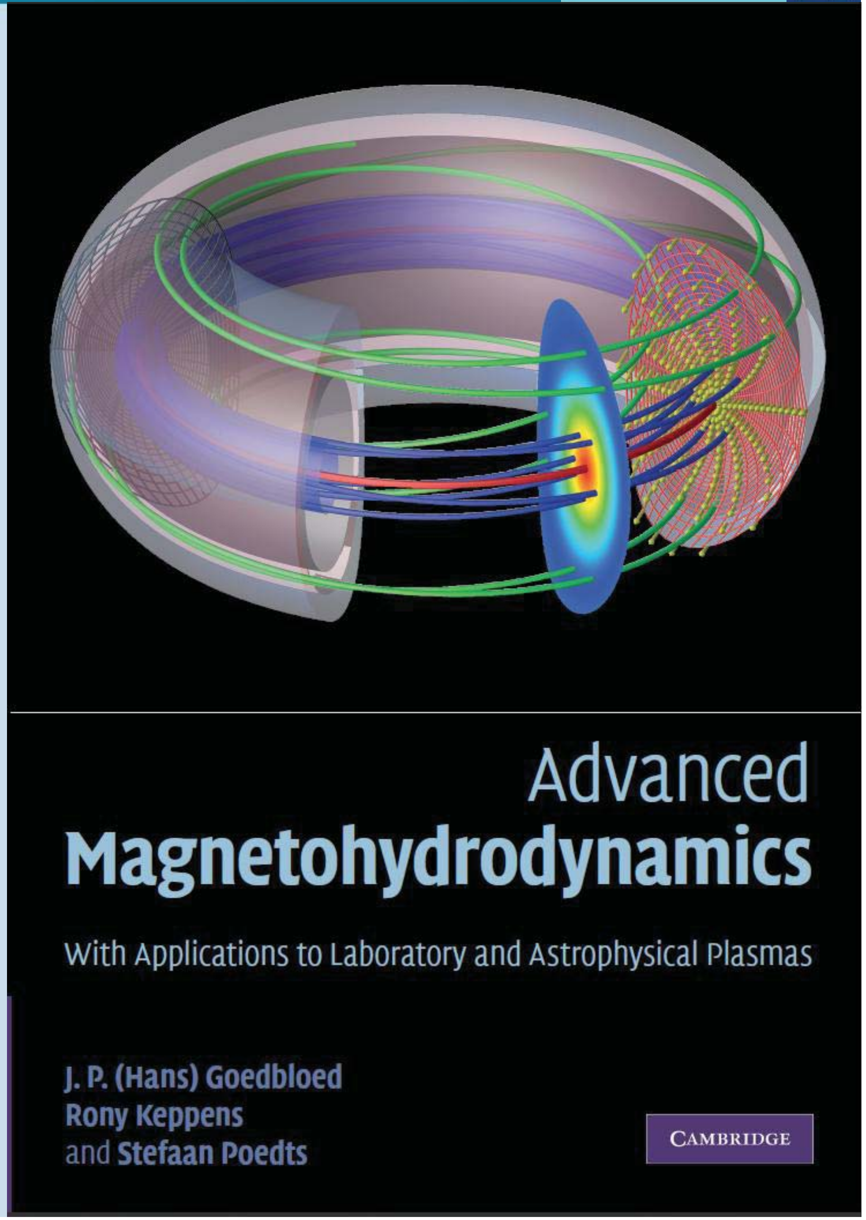
More in…

“Advanced Magnetohydrodynamics: with applications to laboratory and astrophysical plasmas",

J.P. Goedbloed, R. Keppens and S. Poedts

Cambridge University Press, 2010.

ISBN-13: 9780521879576 (hardback),
ISBN-13: 9780521705240 (paperback).



Conclusions

Thank you!