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Notes on the Equivalence Theorem

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For example in the case of ϕ^3 theory we would choose to subtract $p^2\frac{C}{\epsilon}$. In fact usually when people say they are working in the $\overline{MS}$ they are referring to this exactly. Although there is nothing wrong with using the $\overline{MS}$ for the mass and the "on-shell" scheme for the wave function renormaliztion. But we digress. Suppose we work in the $\overline{MS}$ scheme and choose $O = \phi_R$, what is $\hat{Z}$? Then

$$\sqrt{\hat{Z}} = \hat{Z}^{-1/2} \lim_{p^2 \to m^2} \frac{(p^2 - m^2)}{i} \int d^4x e^{-ip\cdot x} \langle 0 \mid \phi(0)\phi(x) \mid 0 \rangle \tag{127}$$

So that

$$\hat{Z} = r \tag{128}$$

where r is the residue of the pole of the propagator in the $\overline{MS}$ scheme. Now when we use the LSZ formula the truncation of the external lines leaves over a finite residue which wasn't there in the on-shell scheme so the final result will have an overall factor of $\hat{Z}^{n/2}$ which will compensate for the fact that we didn't normalize the states properly. So if we define the Z factor in the $\overline{MS}$ (on shell) scheme as $Z_{\overline{MS}}$ (Z_{os}) then the final result for the S matrix element is given by

$$\langle k_1 ... k_n \mid S - 1 \mid l_1 ... l_m \rangle = \left(\frac{Z_{os}}{Z_{\overline{MS}}} \right)^{(n+m)/2} \overline{G}^*(k1, ...k_n; l_1......l_n) \tag{129}$$

where the $*$ denotes amputated, and I have used the fact that in the $\overline{MS}$ scheme the residue of the pole is given by $\frac{Z_{os}}{Z_{\overline{MS}}}$.

HW 3.4 Calculate, at lowest order in λ, the amplitude for two to two scattering in ϕ^3 theory in two ways. First in the usual way using ϕ as the interpolating field, then using ϕ^2 as the interpolating field and show that you get the same answer.

Finally, we can imagine the case where we use some other field to interpolate for the states and remained in the $\overline{MS}$ scheme for the fields in the action. Then there would be an overall factor of

$$\hat{Z}^{-(n+m)/2} \left(\frac{Z_{os}}{Z_{\overline{MS}}} \right)^{(n+m)} . \tag{130}$$

V. THE EQUIVALENCE THEOREM

We have gone a long way of getting rid of this notion of field particle duality. So far we have shown that we can use any field to interpolate for the external states, but we can go a step further in showing that the choice of fields in the action is also just a matter of convenience. Much as the choice of coordinate system is based upon convenience. We will now prove that there exist equivalence classes of actions, where each action within a given class yield the same S matrix[10]. Given an action in terms of a field ϕ we will show that all actions obtainable from ϕ via a transformation of the form

$$\phi = \psi + f(\psi) \tag{131}$$

will yield the same action. We restrict ourselves to this form of transformation so that we can make sense out of the action of the new theory. That is, we could imagine choosing a transformation where the right hand side does not start off with a monomial, but then the action for the new field would not have a familiar form. In this sense the transformations we are considering are like canonical transformations in classical mechanics, which are defined as those transformations which leave the phase space measure unchanged.

Now to the proof. We can write the generating functional of renormalized Greens functions as

$$Z[J] = \int d\pi d\phi exp\left(i\int d^4x(\pi\dot{\phi} - H(\pi,\phi) + \phi J\right) \tag{132}$$

We will see in a moment why I chose to write Z in this form [11].

Now consider the theory which arises from making the change of variables

$$\phi = \psi + f(\psi) \tag{133}$$

in the Hamiltonian. The generating functional for this theory is given by

$$Z_1[J] = \int d\pi_\psi d\psi exp\left(i\int d^4x(\pi_\psi\dot{\psi} - H_1(\pi_\psi,\psi) + \psi J\right) \tag{134}$$

We would like to show that the S matrix elements of this theory are the same as those of the theory for ψ. To compare the two we use the fact that

$$Z[J] = \int d\pi_\psi d\psi exp\left(i\int d^4x(\pi_\psi\dot{\psi} - H_1(\pi,\phi) + (\psi + F(\psi))J\right) \tag{135}$$

where I used the fact that the phase space measure is invariant as well as

$$\pi\dot{\phi} = \pi_\psi\dot{\psi}. \tag{136}$$

[10] In general they will not yield the same greens functions.

[11] Recall from QMIV that when we derive the path integral we always end up in this phase space form. The usual form in terms of just an integral over ϕ results from integrating over π.

Let's prove that the Jacobian is one.

$$\begin{aligned}
\frac{\partial\phi}{\partial\psi} &= 1 + f'(\psi), \\
\frac{\partial\pi_\phi}{\partial\psi} &= \frac{-\pi_\psi}{(1+f'(\psi))^2} f''(\psi) \\
\frac{\partial\phi}{\partial\pi_\psi} &= 0 \\
\frac{\partial\pi_\phi}{\partial\pi_\psi} &= \frac{1}{(1+f'(\psi))}.
\end{aligned} \tag{137}$$

where I used

$$\pi_\psi = \frac{\partial L}{\partial\dot{\psi}} = \frac{\partial L}{\partial\dot{\phi}}\frac{\partial\dot{\phi}}{\partial\dot{\psi}} = \pi_\phi(1+f'(\psi)). \tag{138}$$

Thus the Jacobian is one, and the transformation is indeed canonical since the Poisson bracket

$$\{\pi_\psi, \psi\} = 1 \tag{139}$$

is preserved. What we've actually shown here is that $\{\psi, \pi_\psi\} = 1$, but you can show that the jacobians are just inverses of each other. Suppose we had a transformation which was not just a polynomial in the field? Then there would be a jacobian (perhaps) and you could get ghosts (see artzs paper on arXive).

We can see that the Greens functions of the two theories will differ due to the factor of $F(\psi)$. We will now show that while this last statement is true, F will have no effects on the S matrix elements. First I will show that the two theories have the same pole structures in their two point functions and therefore the same masses. Consider

$$\begin{aligned}
\langle 0 \mid T(\phi(x)\phi(y)) \mid 0\rangle &= \langle 0 \mid T(\psi(x)\psi(y)) \mid 0\rangle + \langle 0 \mid T(F(\psi(x))\psi(y)) \mid 0\rangle \\
&+ \langle 0 \mid T(\psi(x)F(\psi(y))) \mid 0\rangle + \langle 0 \mid T(F(\psi(x))F(\psi(y))) \mid 0\rangle.
\end{aligned} \tag{140}$$

We can see that single poles are at $p^2 = m^2$ (assume we're working in the on-shell prescription) just as in the theory H_1. However, now the residue of the poles will be related by a factor of $(1+\Gamma)^2$, where Γ is the value of the blob at $p^2 = m^2$. In this sense the proof is almost trivial. The only non-trivial piece of information one needs is that Γ has no poles. We will prove in perturbation theory that Γ will only have cuts when we discuss the analytic structure of Feynman diagrams. In this sense the prove of the equivalence theorem presented here relies on perturbation theory. I have a non-perturbative proof but it wont fit in the margins.

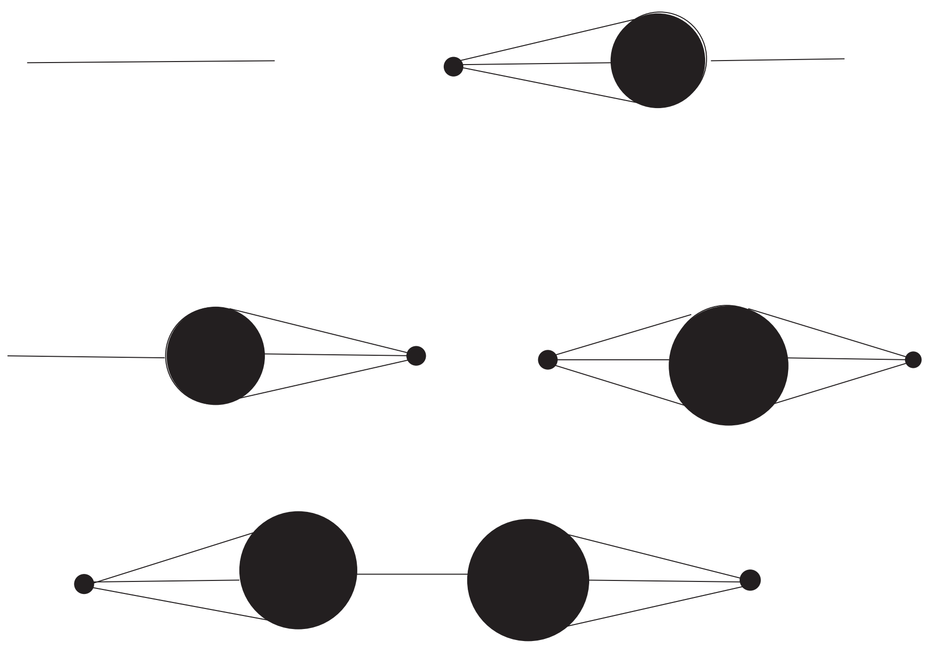

FIG. 2: The two point function as calculated with the original $Z[J]$. The propagators are the exact two point function, while the dark circle is the exact vertex function.

On might guess that for an n point function we would expect

$$G_{\phi}^{(n)} = (1+\Gamma)^n G_{\psi}^{(n)} \tag{141}$$

and indeed this is the case as can be seen from studying figure (V). The point is that to have the proper singularity structure, the Γ vertex must connect to the rest of the diagram through a single on-shell line. This forces Γ to be a constant which just shifts the normalization of the Greens function.

VI. THE ALGEBRA OF OBSERVABLES

We have seen that there is a lot of freedom in describing any given theory. We may choose to interpolate with any field which has non-zero overlap with the one particle state. So there really is no particle/field duality. We are even free to make canonical transformations leading to equivalence classes of actions. We would like to capture the structure of a given theory in as general a way as possible, clearly this would not correspond to choosing some fields and writing down an action.