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A tour to phases in condensed matter physics with entanglement entropy

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A tour to phases in condensed matter physics with entanglement entropy

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1. introduction
2. CFT in d=2 and in d>2
3. holography
4. non CFTs
5. discussion

Ref:
"Holographic Entanglement Entropy: An Overview",
T. Nishioka, SR, T. Takayanagi,
arXiv:0905.0932,
J. Phys. A42 504008 (2009)

phases and phase transitions in condensed matter systems

(partial and biased list)

classical phases

Ginzburg-Laudau theory
Nambu-Goldstone modes
finite T transition

quantum phases

gapless phases

- Fermi liquid
- Tomonaga-Luttinger liquid
- non-Fermi liquid (?)

gapped phases

-insulators
-topological insulators
-topological superconductors
-topological phases

quantum critical points (at T =0)

-relativistic conformal quantum critical point
- Lifshitz crtical point

quantum many-body physics beyond Landau-Ginzburg paradigm

- is it possible to have an (exhausive) classification of quantum phases in many-body systmes ?

- what is a good "order parameter" to distinguish all these phases ?

entanglement and entropy of entanglement

(i) bipartition the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

(ii) take partial trace $\rho_{\rm tot} = |\Psi\rangle\langle\Psi|$

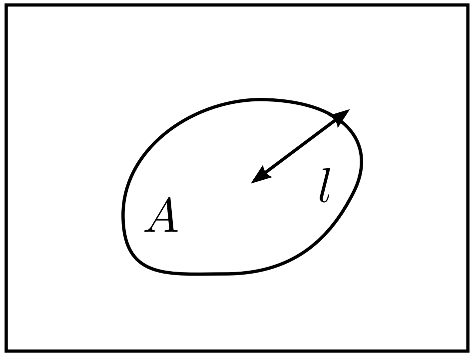
$$\rho_A = {\rm tr}_B |\Psi\rangle\langle\Psi| = \sum_j p_j |\psi_j\rangle_A \langle\psi_j|_A \quad \left(\sum_j p_j = 1\right)$$

(iii) entanglement entropy

$$S_A = -{\rm tr}_A \left[\rho_A \ln \rho_A\right] = -\sum_j p_j \ln p_j$$

application to many-body systms and field theories:

A, B : submanifold of the total system



--> (mainly) interested in scaling of EE

a few more quantities

(iv) entanglement spectrum

$$\rho_A = \mathrm{tr}_B |\Psi\rangle\langle\Psi| = \sum_j p_j |\psi_j\rangle_A \langle\psi_j|_A \quad \left(\sum_j p_j = 1\right)$$

$$p_i = \frac{1}{\sqrt{Z}} e^{-\xi_i/2}$$

(v) Renyi entropy

$$R_A^{(n)} = \frac{-1}{n-1} \ln \mathrm{tr}\left(\rho_A^n\right)$$

(vi) mutual information

$$I_{A:B} = S_A + S_B - S_{A\cup B}$$

entanglement entropy: some key properties

- when $\rho_{\rm tot} = \text{pure}$ $\quad B = A^{\rm complement}$ $\quad \Rightarrow \quad S_A = S_B$

- when $\rho_{\rm tot} = \text{mixed}$ $\quad B = A^{\rm complement}$ $\quad \Rightarrow \quad S_A \neq S_B$

- when $\rho_{\rm tot} = e^{-\beta H}$ $\quad A = \text{total system}$

  $\Rightarrow \quad S_A = \text{thermal entropy}$

- subadditivity

$$S_{AB} \leq S_A + S_B$$

- strong subadditivity Lieb-Ruskai (73)

$$S_B + S_{ABC} \leq S_{AB} + S_{BC}$$

motivation for entanglement entropy

EE can be a good "order parameter" for quantum systems (?)

- defined purely in terms of wavefunctions

  (i.e., always possible to define;
  EE measures a response to external gravity)

- best method to numerically compute central charge in (1+1) D CFT

$$S_A = \frac{c}{3}\log(l/a) + O(1)$$

- use computational complexity to classify quantum states ?

- EE spectrum: new tool to classify symmetry protected
  gapped phases

not sure how to measure it

rather difficult to compute !

entanglement entropy and quantum gravity

scaling of entanglement entropy

Area law (gapped systems, CFT in (d+1)D with d>1, etc.)

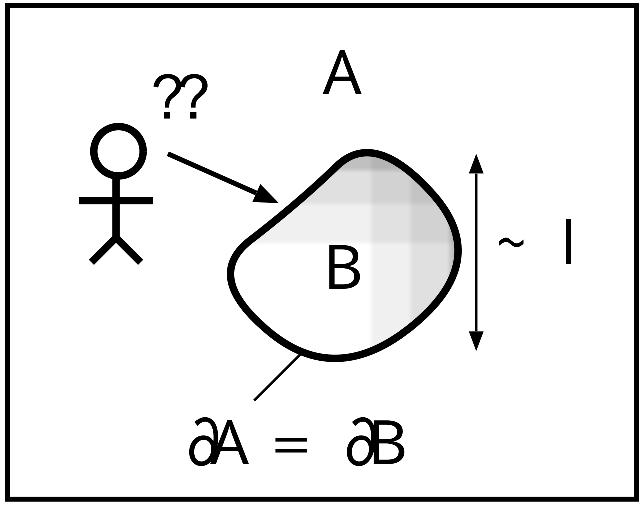
$$S_A = \text{const.} \left(\frac{l}{a}\right)^{d-1} + \cdots$$

[Bombelli, Koul, Lee and Sorkin (86)]
[Srednicki (93)]
[Callan, Wilczek (94)]
[Susskind, Uglum (94)]

$$\Updownarrow$$

Blackhole entropy (Beckenstein-Hawking)

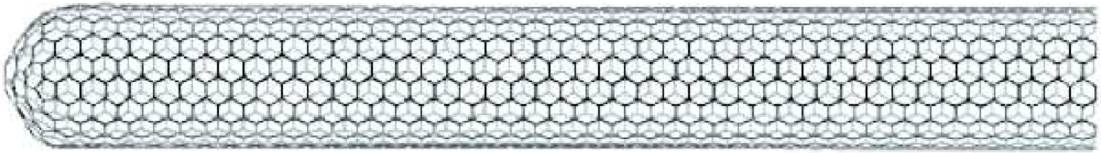
$$S_{BH} = \frac{\text{Area of horizon}}{4G_N}$$

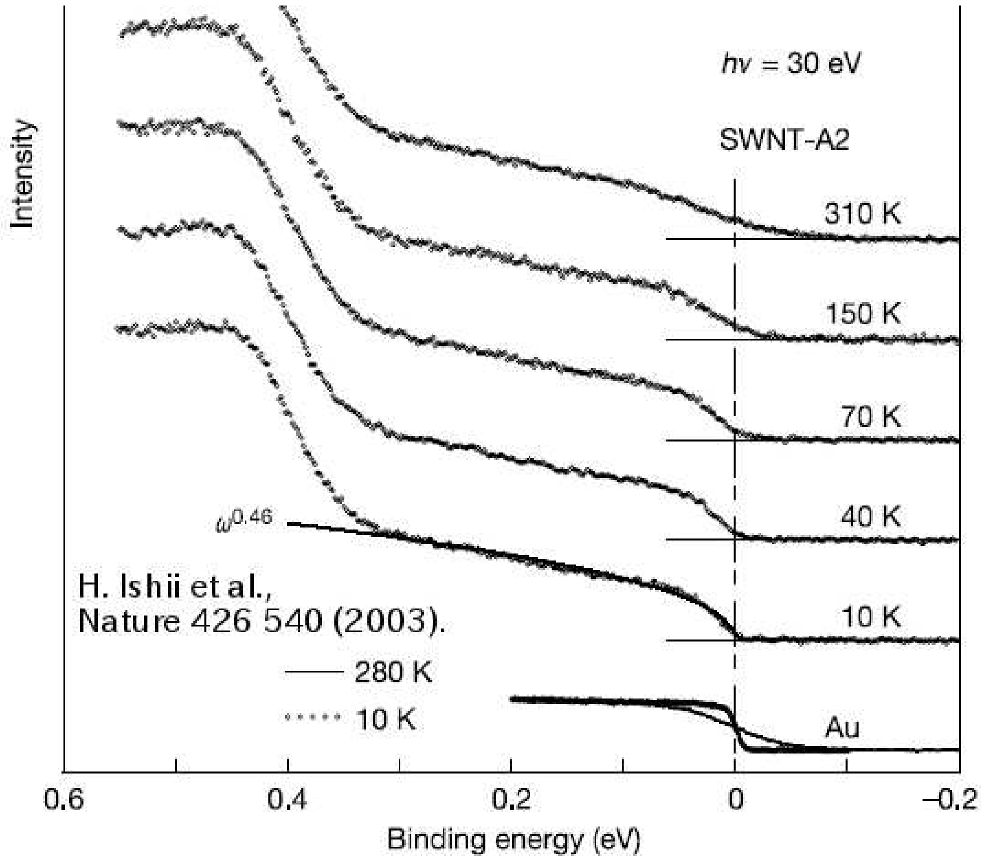


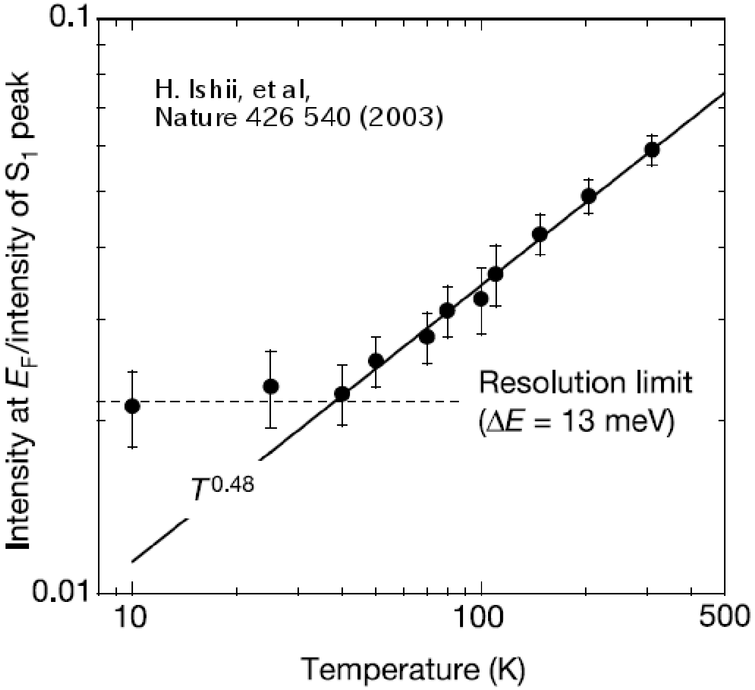
2. CFT in d=2 and in d>2

(1+1)D CFT in condensed matter -- quantum wire

carbon nanotube





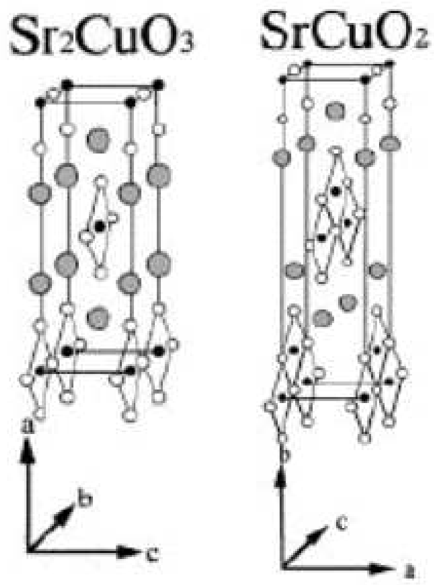


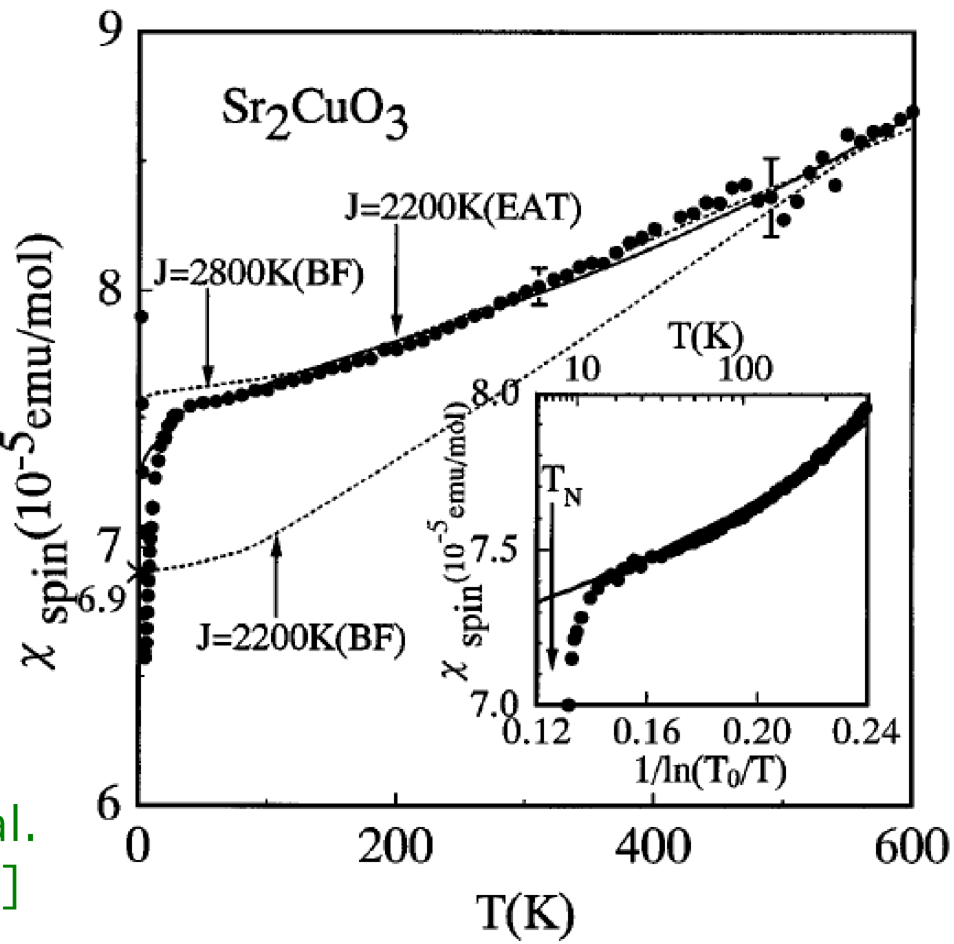
CFT description:
compactified boson

(1+1)D CFT in condensed matter -- quantum spin chain

S = 1/2 quantum Heisenberg model

$$H = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1}$$





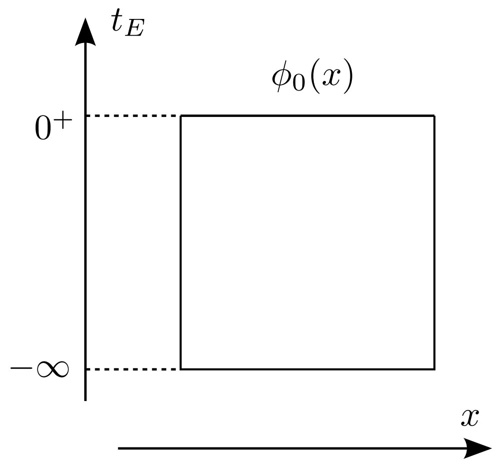
CFT description:
SU(2) level 1 WZW theory

[Motoyama et al. PRL 76 pp3212]

entanglement entropy in QFTs

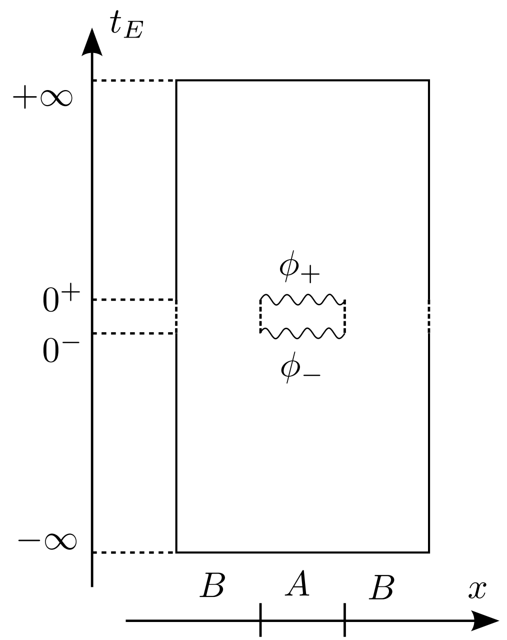
ground state wavefunctional:

$$\Psi[\phi_0(x)] = \mathcal{N}^{-1} \int_{-\infty}^{t=0} \mathcal{D}\phi e^{-S} \prod_{x \in A} \delta(\phi(0,x) - \phi_0(x))$$



reduced density matrix:

$$[\rho_A]_{\phi_+,\phi_-} = Z^{-1} \int_{-\infty}^{+\infty} \mathcal{D}\phi e^{-S}$$
$$\times \prod_{x \in A} \delta(\phi(+0,x) - \phi_+(x))\delta(\phi(-0,x) - \phi_-(x))$$

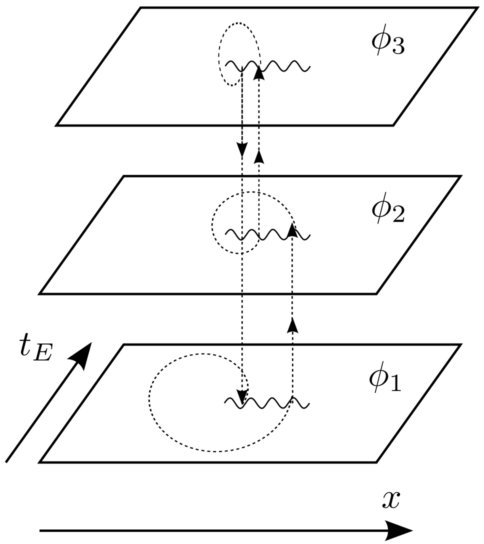


replica trick

replica trick

$$S_A = -\frac{\partial}{\partial n}\,\mathrm{tr}_A\rho_A^n\Big|_{n=1} = -\frac{\partial}{\partial n}\,\ln\mathrm{tr}_A\rho_A^n\Big|_{n=1}$$

$$\mathrm{tr}_A\rho_A^n = (Z_1)^{-n}\int_{\mathcal{R}_n}\mathcal{D}\phi e^{-S} = \frac{Z_n}{(Z_1)^n}$$



QFT on a singular curved space

conical defect with excess angle

EE in (1+1)D CFT -- Weyl rescaling

Weyl rescaling: $g_{\mu\nu} = \delta_{\mu\nu} e^{2\rho} \qquad l \sim e^{2\rho}$

$$
\begin{aligned}
l \frac{d}{dl} \ln \mathrm{tr}_A \rho_A^n &= 2 \int d^{d+1}x \, g_{\mu\nu}(x) \frac{\delta}{\delta g_{\mu\nu}(x)} \left(\ln Z_n - n \ln Z_1 \right) \\
&= -\frac{1}{2\pi} \left\langle \int d^{d+1}x \sqrt{g} T^\mu_\mu \right\rangle_{M_n} + \frac{n}{2\pi} \left\langle \int d^{d+1}x \sqrt{g} T^\mu_\mu \right\rangle_{T_1}
\end{aligned}
$$

$$
l \frac{d}{dl} S_A = -l \frac{d}{dl} \frac{\partial}{\partial n} \ln \mathrm{tr}_A \rho_A^n = -\frac{1}{2\pi} \frac{\partial}{\partial n} \left\langle \int d^{d+1}x \sqrt{g} T^\mu_\mu \right\rangle_{M_n} \qquad (n \to 1)
$$

2D CFT $\quad \left\langle T^\mu_\mu \right\rangle = -\frac{c}{12} R \quad \longrightarrow \quad \boxed{S_A = \frac{c}{3} \ln \frac{l}{a}}$

$$
R \sim 4\pi (1-n) \sum_i \delta^{(2)}(x - x_i)
$$

[Holzhey, Larsen, Wilczek (94)]
[Korepin (04)], [Clabrese, Cardy (04)]

(calculation is valid for small deficit angle)

EE in (1+1)D CFT -- multi-component picture and twsit field

introduce a multi-component field with BC:

$$\vec{\Phi}(x) = \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_n(x) \end{pmatrix} \qquad \vec{\Phi}(x, 0^-) = \sigma \vec{\Phi}(x, 0^+) \quad \text{for } x \in A$$

$$\sigma = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ \pm 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

BC can be implemented by a twist operator:

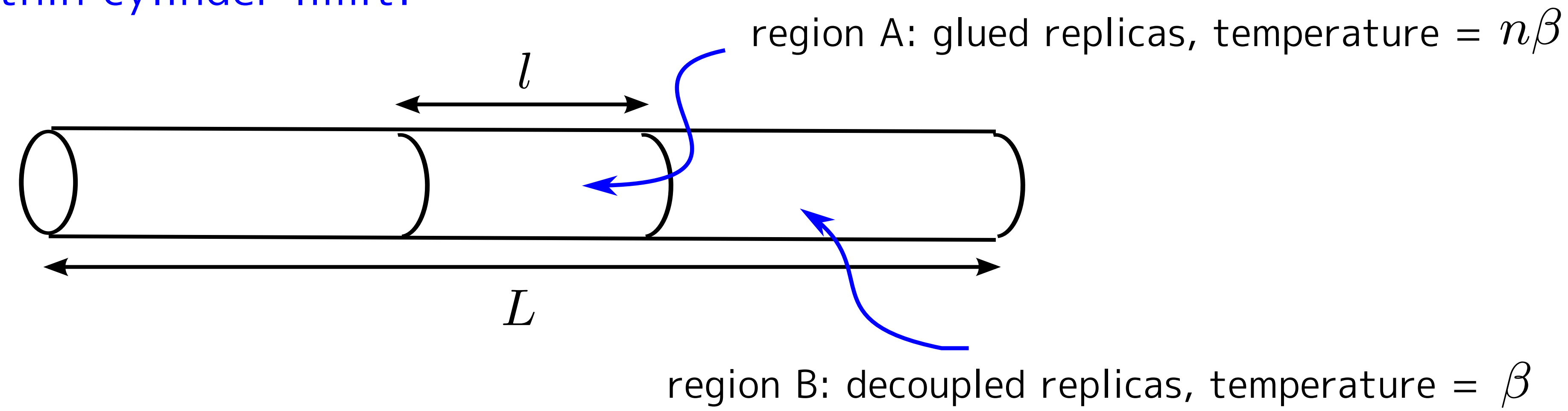
$$\mathcal{T}(z), \quad \tilde{\mathcal{T}}(z)$$

$$\mathrm{tr}\, \rho_A^n = \langle 1 \rangle_{\mathcal{R}_n} \qquad \mathrm{tr}\, \rho_A^n = \langle \mathcal{T}(u,0) \tilde{\mathcal{T}}(v,0) \rangle_{\mathbb{R}^d}$$

scaling dimension of the twist operator ?

EE in (1+1)D CFT -- multi-component picture and twsit field

thin cylinder limit:



$$Z = \frac{e^{\frac{\pi c}{6 n v \beta} l} \left[e^{\frac{\pi c}{6 v \beta}(L-l)} \right]^n}{\left[e^{\frac{\pi c}{6 v \beta}(L-l)} \right]^n} = e^{-\frac{2\pi l}{v\beta}\frac{c}{12}\left(n - \frac{1}{n}\right)}$$

c.f.

$$F_{\mathrm{CFT}}(\beta, \delta l) = \frac{\pi c \delta l}{6 v \beta^2}$$

can read off the scaling dimension by comparing with:

$$\langle \mathcal{O}(l) \mathcal{O}(0) \rangle_{\mathrm{cyl}} \sim e^{-2\pi \Delta_{\mathcal{O}} l / v\beta}$$

$$\Delta_n = \bar{\Delta}_n = \frac{c}{24}\left(n - \frac{1}{n}\right)$$

scaling dim. of twist operator

EE in (1+1)D CFT -- multi-component picture and twsit field

study correlator with stress tensor:

$$\frac{\langle T_{\rm tot}(z)\mathcal{T}(u)\tilde{\mathcal{T}}(v)\rangle_{\mathbb{C}}}{\langle \mathcal{T}(u)\tilde{\mathcal{T}}(v)\rangle_{\mathbb{C}}} = n\langle T(z)\rangle_{\mathcal{R}_n}$$

$$w = \left(\frac{z-u}{z-v}\right)^{1/n} \qquad T(z) = \left(\frac{dw}{dz}\right)^2 T(w) + \frac{c}{12}\{w,z\}$$

$$\frac{\langle T_{\rm tot}(z)\mathcal{T}(u)\tilde{\mathcal{T}}(v)\rangle_{\mathbb{C}}}{\langle \mathcal{T}(u)\tilde{\mathcal{T}}(v)\rangle_{\mathbb{C}}} = \frac{c}{24}\left(n-\frac{1}{n}\right)\frac{(u-v)^2}{(z-u)^2(z-v)^2}$$

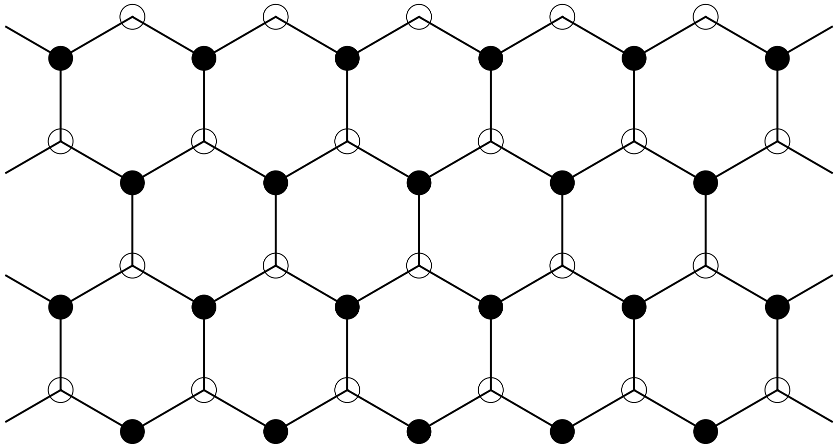
compare with conformal Ward identity:

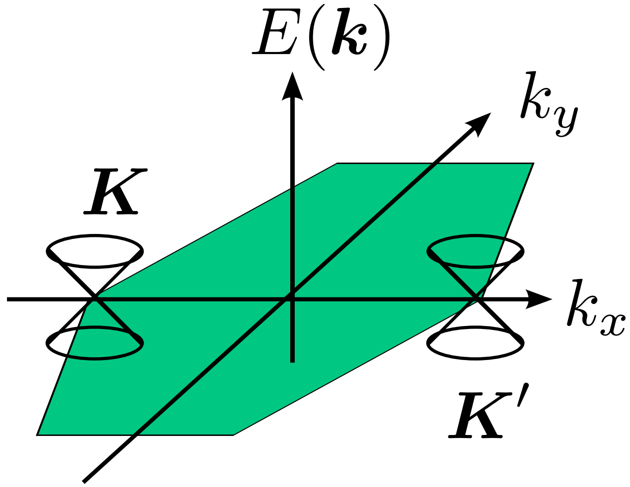
$$\langle T_{\rm tot}(z)\mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\rangle_{\mathbb{C}} = \sum_j \left(\frac{\Delta_n}{(z-x_j)^2} + \frac{1}{z-x_j}\frac{\partial}{\partial x_j} + \text{reg.}\right)\langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\rangle_{\mathbb{C}}.$$

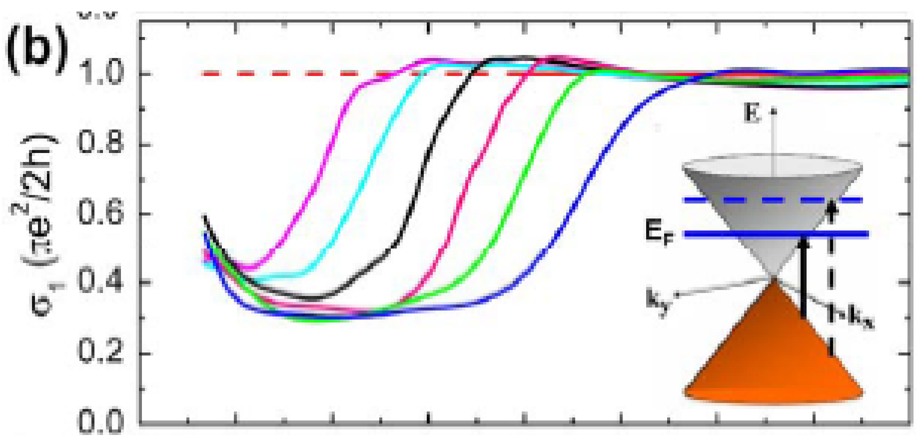
$$\Delta_n = \bar{\Delta}_n = \frac{c}{24}\left(n-\frac{1}{n}\right)$$

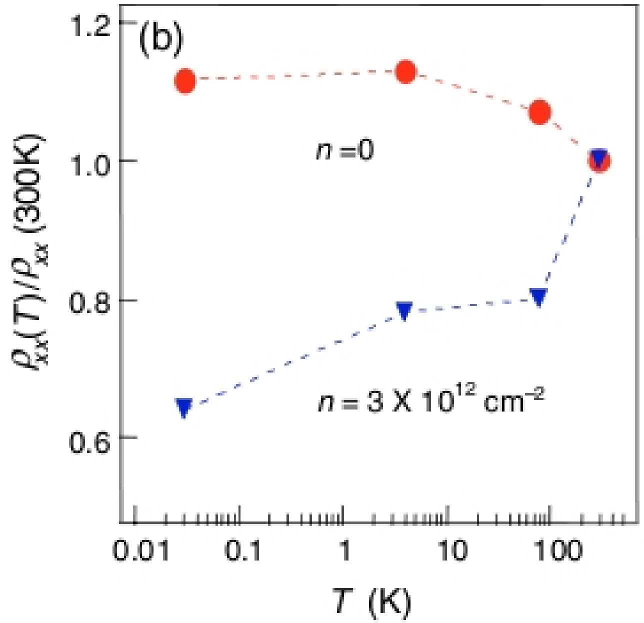
2. CFT in d=2 and in d>2

graphene -- (2+1)D CFT









Weyl semimetal -- (3+1)D CFT

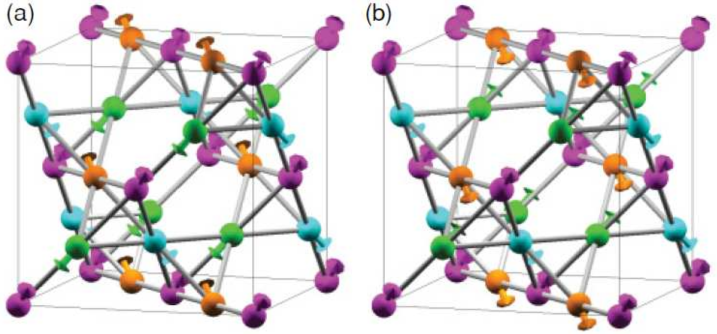
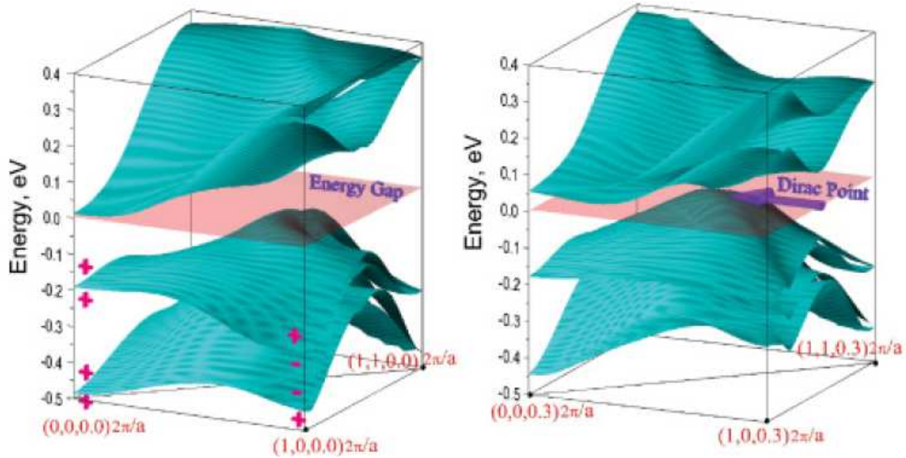


FIG. 2. (Color online) The pyrochlore crystal structure showing the Ir corner sharing tetrahedral network and two of the possible magnetic configurations. (a) The configuration that is predicted to occur for iridates, with all-in/all-out magnetic order. (b) An alternative, the 2-in/2-out configuration.



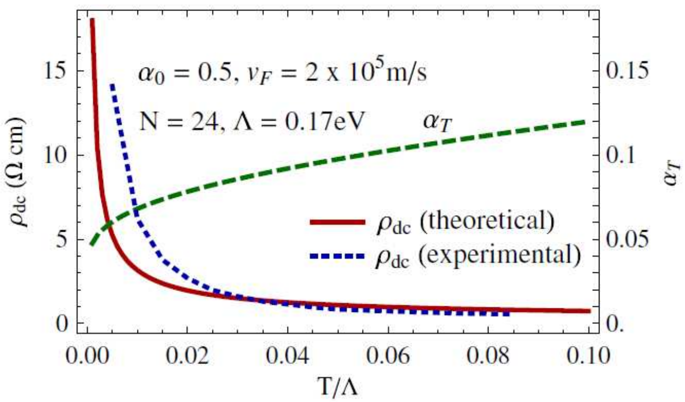
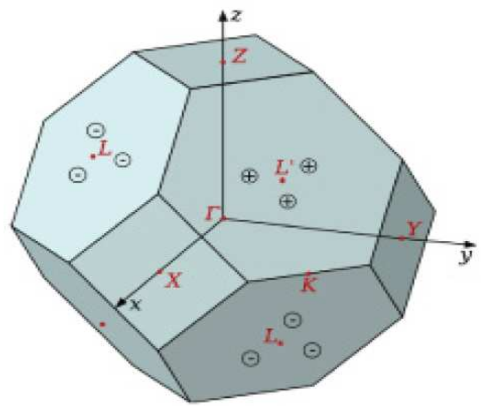


FIG. 2 (color online). $\rho_{dc} = \sigma_{dc}^{-1}$ and α_T (defined in the text) for the inset parameter values compared to experimental data from [12].



24 flavors of Weyl fermions

Weyl rescaling: $g_{\mu\nu} = \delta_{\mu\nu} e^{2\rho} \qquad l \sim e^{2\rho}$

$$\begin{aligned} l \frac{d}{dl} \ln \mathrm{tr}_A \rho_A^n &= 2 \int d^{d+1}x \, g_{\mu\nu}(x) \frac{\delta}{\delta g_{\mu\nu}(x)} \left(\ln Z_n - n \ln Z_1 \right) \\ &= -\frac{1}{2\pi} \left\langle \int d^{d+1}x \sqrt{g} T^\mu_\mu \right\rangle_{M_n} + \frac{n}{2\pi} \left\langle \int d^{d+1}x \sqrt{g} T^\mu_\mu \right\rangle_{T_1} \end{aligned}$$

$$l \frac{d}{dl} S_A = -l \frac{d}{dl} \frac{\partial}{\partial n} \ln \mathrm{tr}_A \rho_A^n = -\frac{1}{2\pi} \frac{\partial}{\partial n} \left\langle \int d^{d+1}x \sqrt{g} T^\mu_\mu \right\rangle_{M_n} \qquad (n \to 1)$$

entanglement entropy in CFTs

2D CFT $\left\langle T^{\mu}_{\mu} \right\rangle = -\frac{c}{12} R$ → $S_A = \frac{c}{3} \ln \frac{l}{a_0}$

[Holzhey, Larsen, Wilczek (94)] [Korepin (04)] [Calabrese, Cardy (04)]

4D CFT $\left\langle T^{\mu}_{\mu} \right\rangle = -\frac{c}{8\pi} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} + \frac{a}{8\pi} \tilde{R}_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma}$

→ $S_A = \gamma_1 \frac{l^2}{a_0^2} + \gamma_2 \ln \frac{l}{a_0} + \cdots$

e.g. free scalar field in 4d with sphere entangling surf. $\chi(\partial A) = 2$

$$a = \frac{1}{360}, \quad \gamma_2 = -\frac{1}{90}$$

[SR, Takayanagi (06)]

recently confirmed numerically

[Lohmayer-Neuberger-Schwimmer-Theisen (09)]
[Schwimmer-Theisen (08)]
[Casini-Huerta (09)]

odd dimentional CFTs: no central charges !?

3. holography

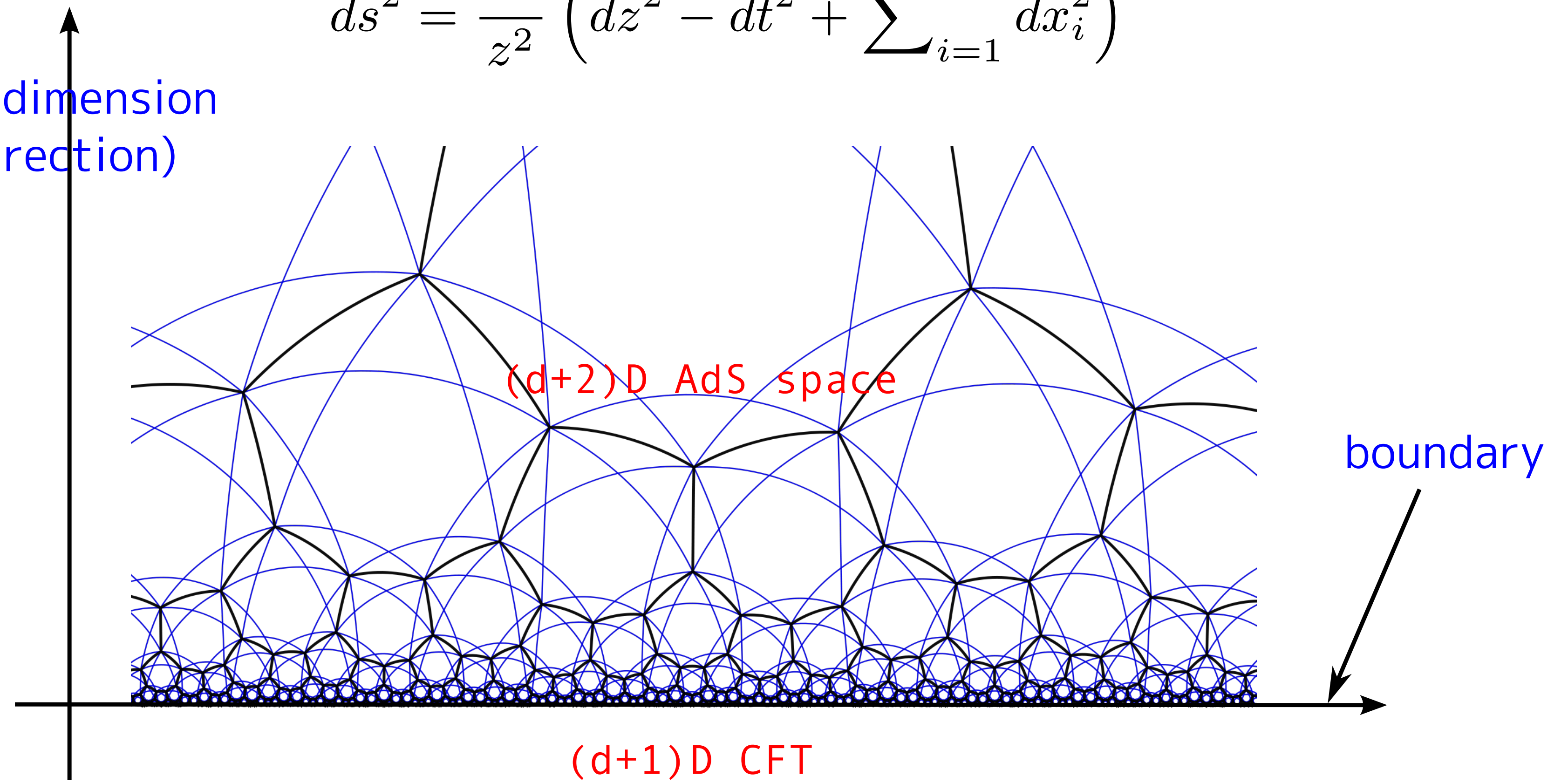
holography

t' Hooft (93'), Susskind (94') (holographic principle)

Maldacena conjecture (97') (AdS/CFT correspondence)

$$I_{\mathrm{EH}} = \frac{1}{16\pi G_N} \int d^{d+2}x \sqrt{-g}\, [R - \Lambda] \qquad \Lambda = -\frac{(d+1)d}{R^2}$$

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 - dt^2 + \sum_{i=1}^{d} dx_i^2\right)$$



holography and AdS/CFT

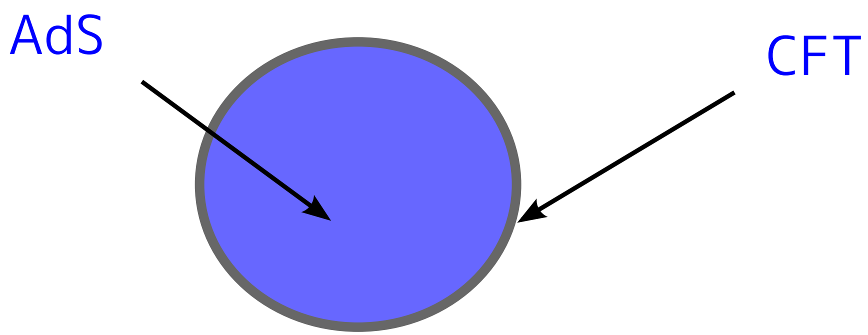
t' Hooft (93'), Susskind (94') (holographic principle)

Bekenstein-Hawking black hole entropy

$$S_{BH} = \frac{\text{Area of horizon}}{4G_N}$$

Maldacena conjecture (97') (AdS/CFT correspondence)

(quantum) gravity on d+2 dimensional AdS space
= d+1 dimensional CFT



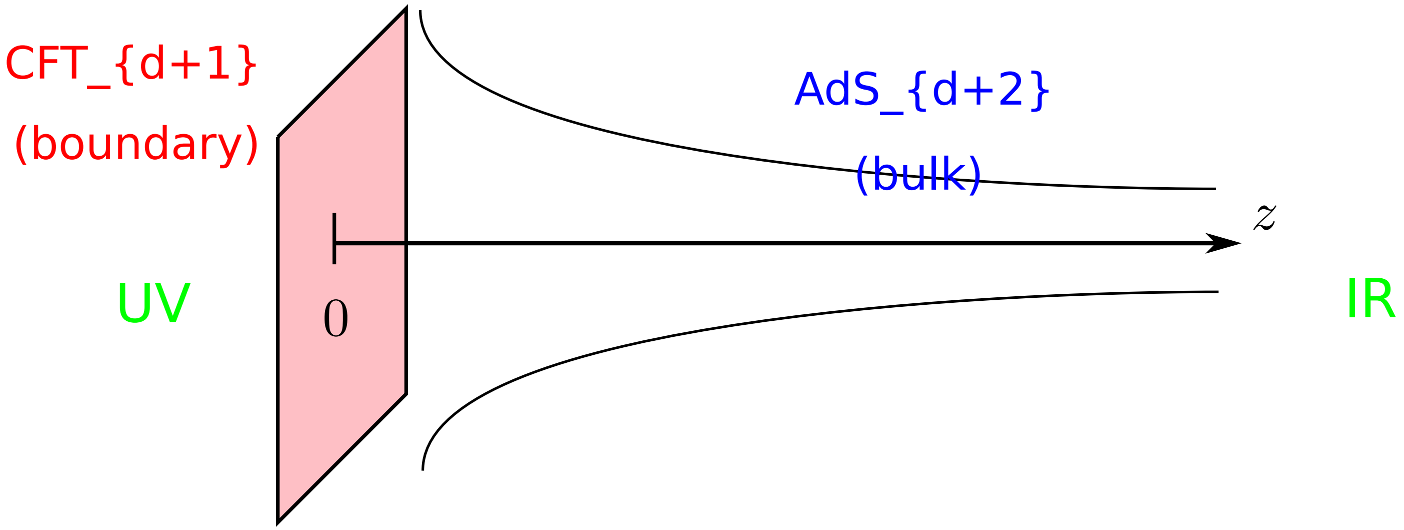
AdS/CFT

string theory in d+2 dimensional AdS space
= d+1 dimensional CFT

- correlation function   GKPW relation (98)

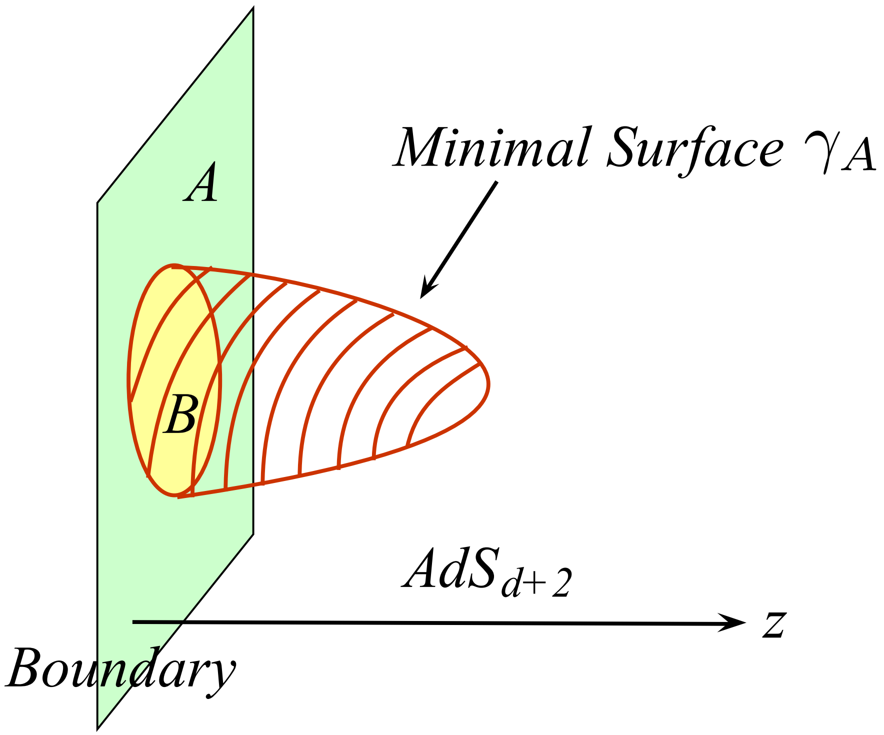
$$\left\langle e^{\int d^{d+1}x\, \varphi(x)\mathcal{O}(x)} \right\rangle_{\rm CFT} = Z_{\rm String}\big|_{\phi(x,z)|_{z=0}=\varphi(x)} \simeq e^{-I_{\rm SUGRA}[\phi]_{\phi(x,0)=\varphi(x)}}$$

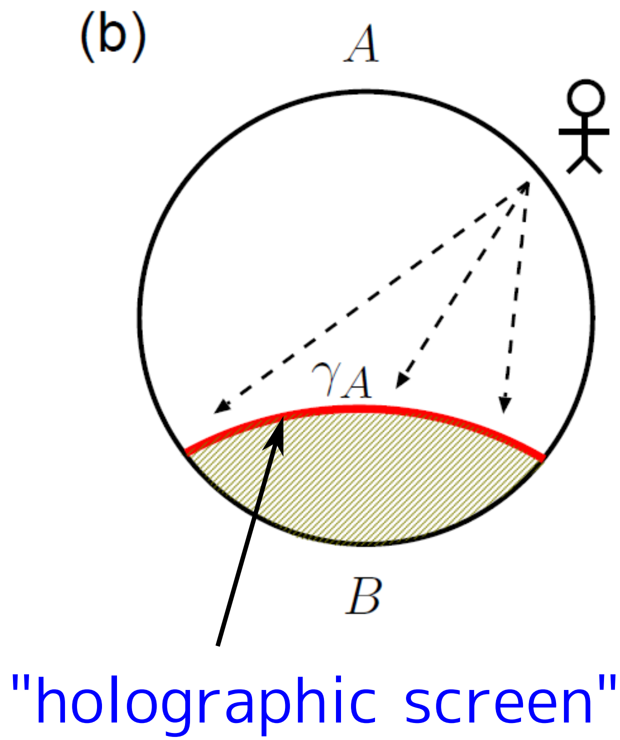
- geometrical realization of RG $\vec{x} \to \lambda\vec{x}, \quad z \to \lambda z$



holographic formula for entanglement entropy

$$S_A = \frac{\text{Area of minimal surface } \gamma_A}{4G_N}$$





heuristic argument

AdS/CFT dictionary:

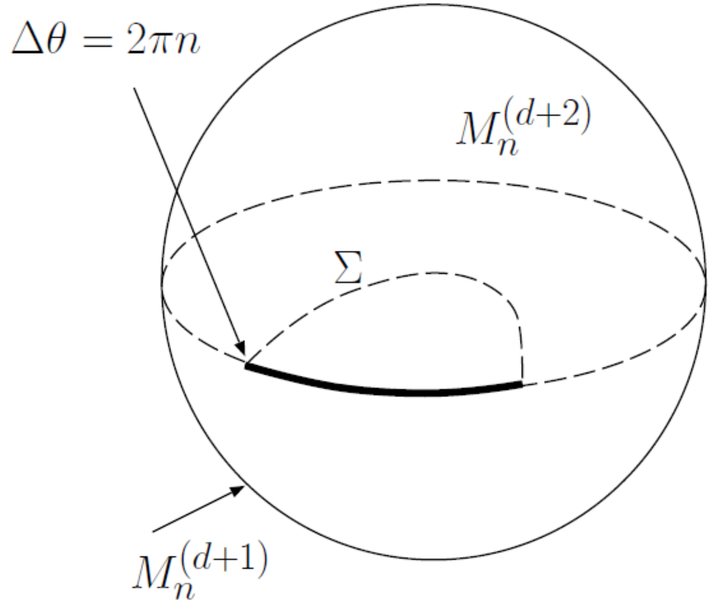
$$Z_{\mathrm{CFT},M_n^{(d+1)}} = e^{-S_{\mathrm{SUGRA}}}\Big|_{g_{\mu\nu}(x,r)|_{r=r_{\mathrm{UV}}}=g_{\mu\nu}(x)|_{M_n^{(d+1)}}}$$

If the conical defects propagate into the bulk as

$$R \sim 4\pi(1-n)\delta^2(x)$$

$$\int_{M_n^{d+2}} d^{d+2}x\,\sqrt{g}R$$
$$= \int_{\Sigma} d^{d+2}x\,\sqrt{g}R + \int_{M_n^{d+2}/\Sigma} d^{d+2}x\,\sqrt{g}R$$
$$= 4\pi(1-n)A[\Sigma] + \int_{M_n^{d+2}/\Sigma} d^{d+2}x\,\sqrt{g}R$$

conical singularity



EE in AdS3/CFT2

$$ds^2 = R^2 \frac{-dt^2 + dx^2 + dz^2}{z^2}$$

$$c = \frac{3R}{2G_N}$$

Brown-Henneaux

area (length) functional:

$$\int_{(x,z)=(-l/2,0)}^{(x,z)=(l/2,0)} ds = R \int \frac{\sqrt{dx^2 + dz^2}}{z}$$

$$= R \int dx \frac{\sqrt{1 + \left(\frac{dz}{dx}\right)^2}}{z}$$

UV cutoff:

$$z_{\rm UV} = l\epsilon/2 \sim a$$

"energy" conservation:

$$\frac{d}{dx}\left[L - z' \frac{\partial L}{\partial z'}\right] = 0$$

$$\Rightarrow L - z' \frac{\partial L}{\partial z'} = \text{const.}$$

$$\text{Length} = 2R \log \frac{l}{a}$$

$$S = \frac{\text{Length}}{4G_N} = \frac{c}{3} \log \frac{l}{a}$$

integration constant:

$$\frac{l}{2} = \int dx = \int_0^{z_*} dz \frac{dx}{dz}$$

EE in AdS3/CFT2 -- finite system of length L

global coordinate:

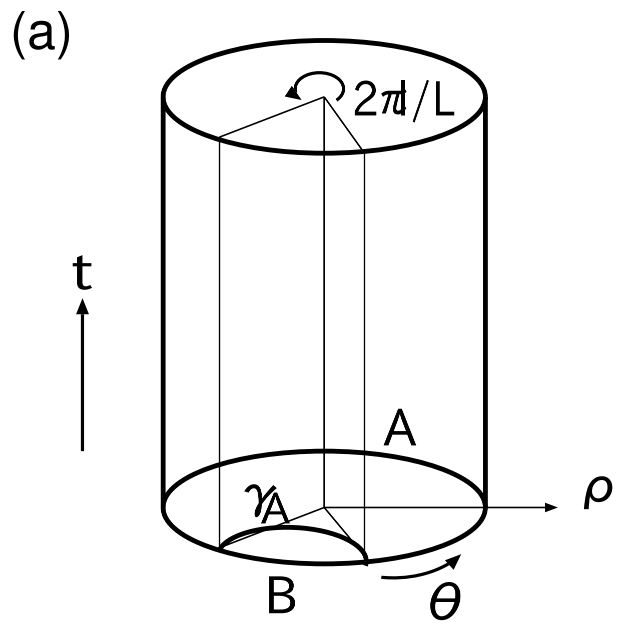
$$ds^2 = R^2\left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\theta^2\right)$$

$$\text{Length} = R\int\sqrt{(d\rho)^2 + \sinh^2\rho(d\theta)^2}$$

$$L - \rho'\frac{\partial L}{\partial \rho'} = \text{const.}$$

$$\frac{1}{2}\times\frac{2\pi l}{L} = \int_{\pi l/L}^{2\pi l/L} d\theta$$

$$= \int_{\rho^*}^{\infty} d\rho\frac{d\theta}{d\rho}$$



$$\sim \frac{c}{3}\ln\left[e^{\rho_{\mathrm{UV}}}\sin\frac{\pi l}{L}\right]$$

finite temperature -- similar calculations

holographic derivation of entanglement entropy

$$c = \frac{3R}{2G_N^{(3)}}$$

- minimal surface = geodesic

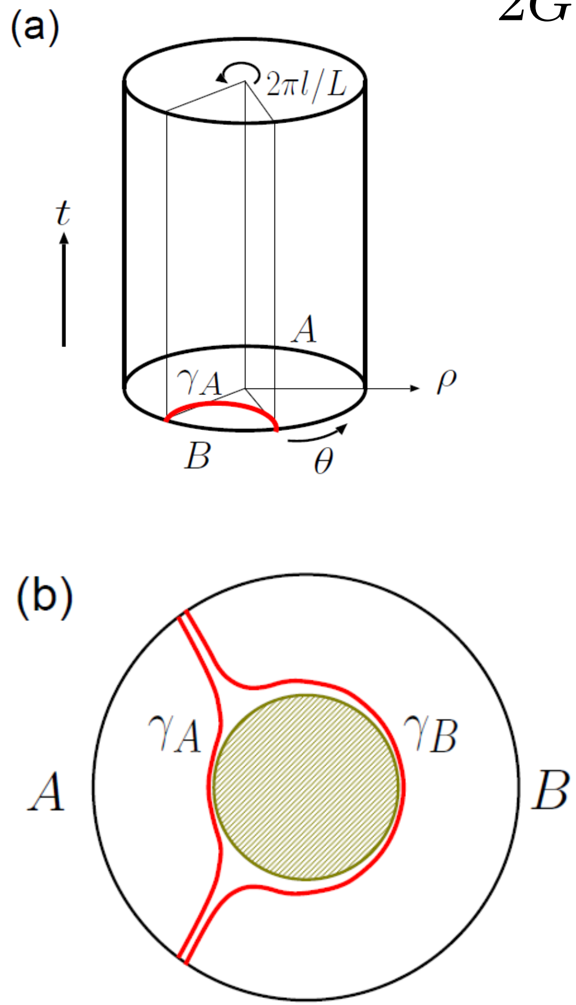
$$S_A = \frac{c}{3}\log(l/a) + O(1)$$

- finite system

$$S_A = \frac{c}{3}\log\left(\frac{L}{\pi a}\sin\frac{\pi l}{L}\right) + O(1)$$

- finite temperature

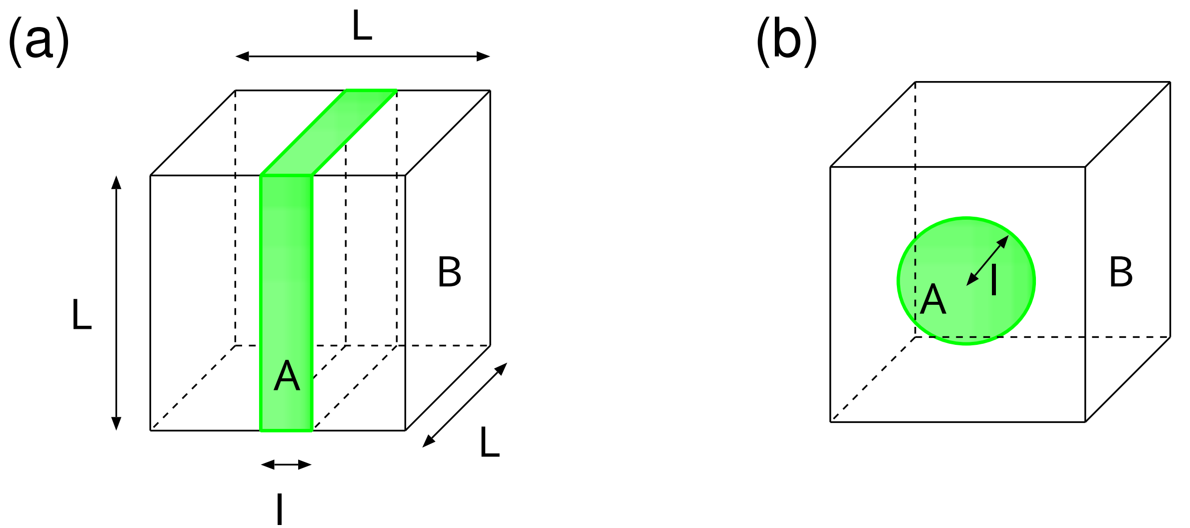
$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\frac{\pi l}{\beta}\right) + O(1)$$

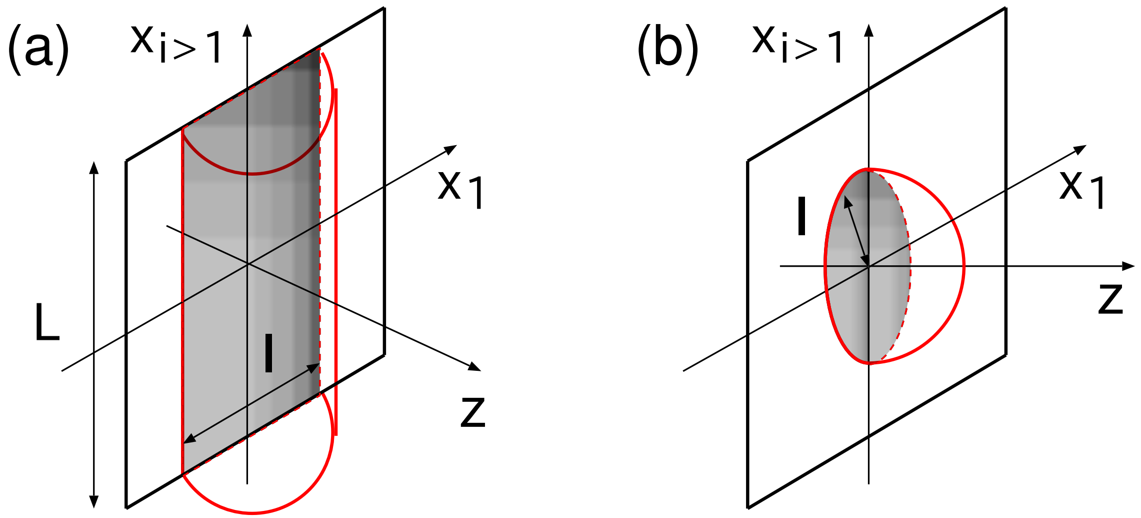


EE in AdS_{d+2}/CFT_{d+1}

straight belt of length L and width l

disc (ball) of radius l





EE in AdS_{d+2}/CFT_{d+1}

- straight belt of length L and width l

$$\text{Area} = R^d L^{d-1} \int_{-l/2}^{l/2} dx \frac{\sqrt{1+(\frac{dz}{dx})^2}}{z^d}$$

$$S_{A_S} = \frac{1}{4G_N^{(d+2)}} \left[\frac{2R^d}{d-1} \left(\frac{L}{a}\right)^{d-1} - \frac{2^d \pi^{d/2} R^d}{d-1} \left(\frac{\Gamma(\frac{d+1}{2d})}{\Gamma(\frac{1}{2d})} \right)^d \left(\frac{L}{l}\right)^{d-1} \right]$$

area law universal

- disc (ball) of radius l

$$\text{Area}_{A_D} = R^d \cdot \text{Vol}(\text{S}^{d-1}) \cdot \int_0^l dr r^{d-1} \frac{\sqrt{1+(\frac{dz}{dr})^2}}{z^d}$$

EE in AdS_{d+2}/CFT_{d+1}

d+1 = even:

$$S_A = p_1 \left(\frac{l}{a}\right)^{d-1} + p_3 \left(\frac{l}{a}\right)^{d-3} + \cdots + p_{d-2} \left(\frac{l}{a}\right)^{2} + q \log l/a + O(1)$$

d+1 = odd:

$$S_A = p_1 \left(\frac{l}{a}\right)^{d-1} + p_3 \left(\frac{l}{a}\right)^{d-3} + \cdots + p_{d-1} \left(\frac{l}{a}\right)^{1} + p_d + O(a/l)$$

q and p_d : universal and conformal invariant [Graham+Witten (99)]

q: related to central charge in even dim. CFT

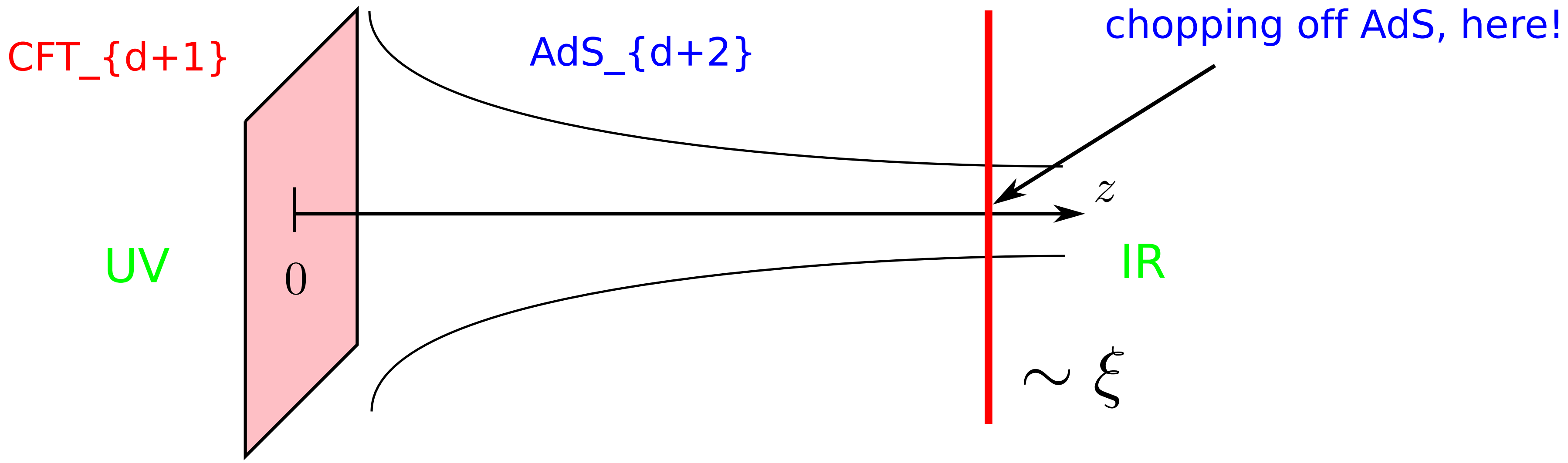
p_d: universal although no central charge in odd dim. CFT

an alternative of central charge ?

4. non CFTs

1. confinement
2. topological phases
3. fermi surface

massive deformation

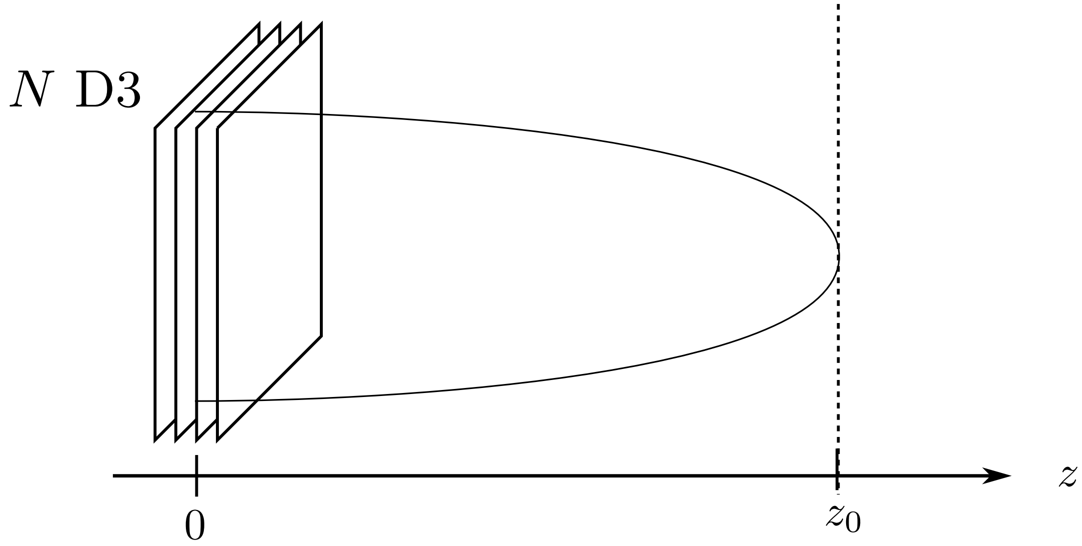


for straight line entangling surface

$$\ell \leq \xi \qquad \text{Area} \sim R^d L^{d-1} \int_{a_0}^{z_*} dz \frac{\sqrt{(dx/dz)^2+1}}{z^d}$$

$$\ell \geq \xi$$

pure Yang-Mills in (2+1)D ⟷ AdS soliton



$$ds^2 = \frac{R^2 dr^2}{r^2 f(r)} = \frac{r^2}{R^2}\left(-dt^2 + f(r) d\chi^2 + dx_1^2 + dx_2^2\right)$$

$$f(r) = 1 - (r_0/r)^4$$

QFT dual:

(i) start from N=4 SYM in d+1 = 4 dim
(ii) compactify one direction, get rid of fermions (susy) by APBC
(iii) scalars get massive by radiative correction

Witten (98)

minimal surface in AdS soliton

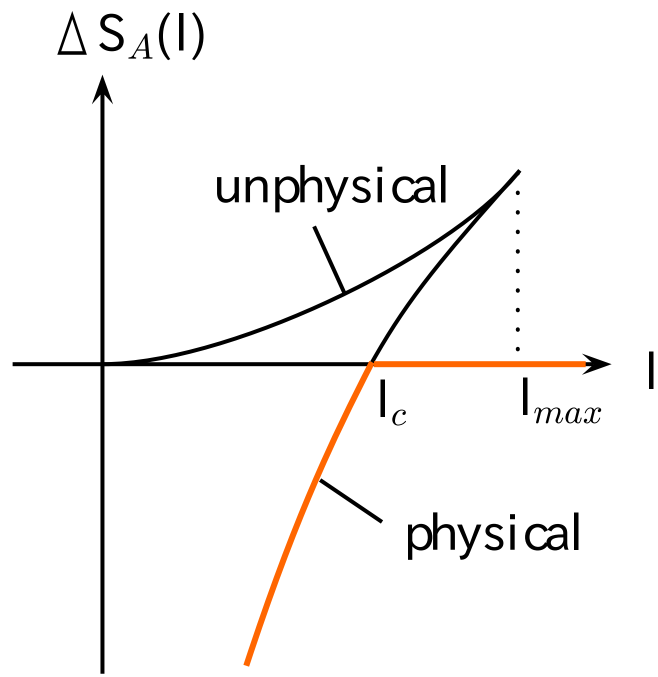
[Nishioka-Takayanagi (06)]

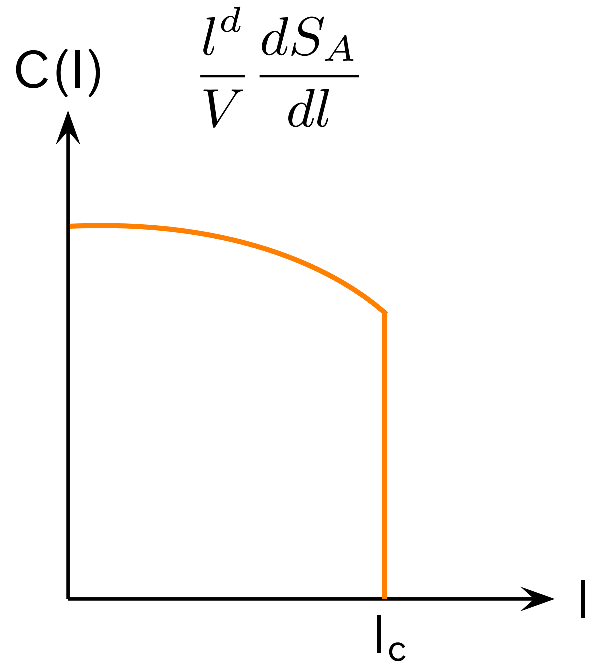
connected ansatz

$$\text{Area} = VL \int_{-l/2}^{l/2} dx_1 \frac{r}{R} \sqrt{\frac{r^4 f}{R^4} + \left(\frac{dr}{dx}\right)^2}$$

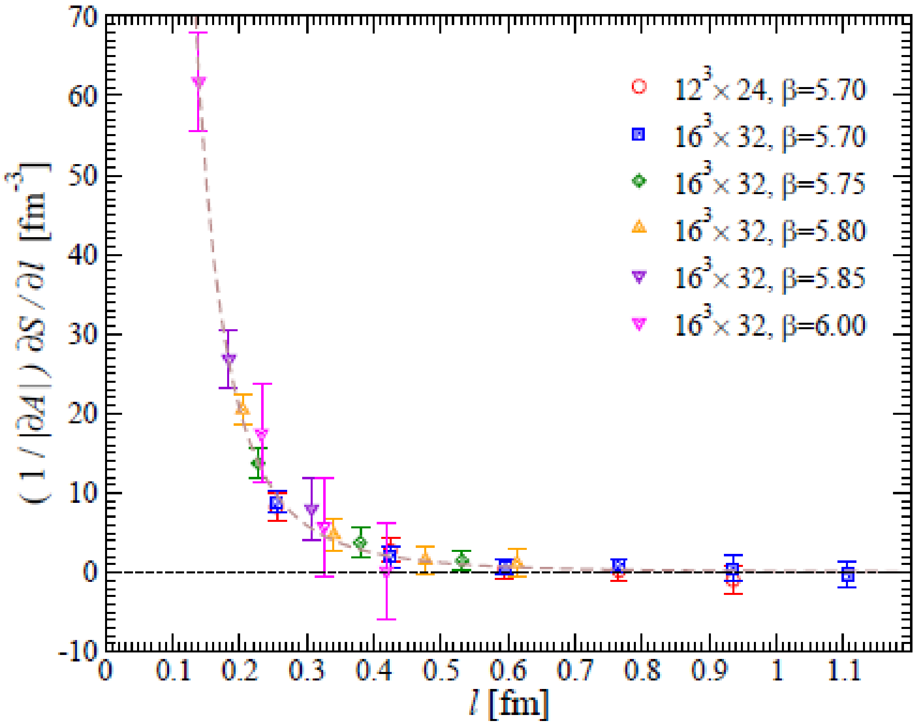
disconnected ansatz

$$\text{Area} = VL \int dr \frac{r}{R}$$

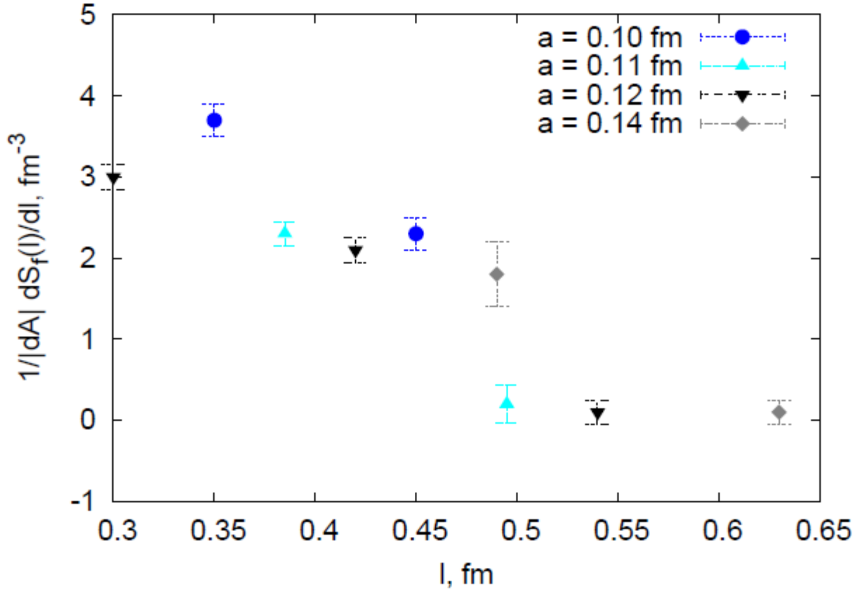




EE in pure 4D lattice gauge theory



EE in 4D SU(3) pure LGT



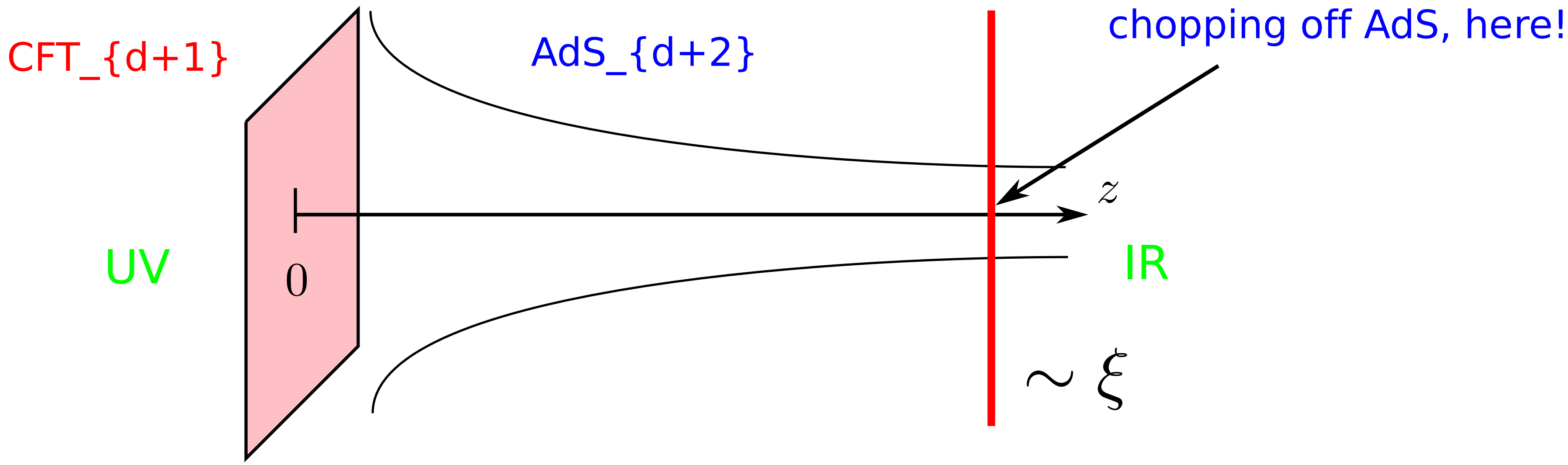
EE in 4D SU(2) pure LGT

Buividovich, Polikarpov (NPB802, pp458, 2008)

Nakagawa-Nakamura-Motoki-Zakharov (09)

holographic calculations: Nishioka, Takayanagi (2006,2007),
Klebanov, Kutasov, Murugan (2007)

massive deformation



for circular loop boundary

$$\mathrm{Area} = R^d \mathrm{vol}(S^{d-1}) \int_{a_0}^{l} dz r^{d-1} \frac{\sqrt{z^2 + r^2}}{z^d} \qquad \ell \leq \xi$$

$$\mathrm{Area} = R^d \mathrm{vol}(S^{d-1}) \int_{a_0}^{\xi} dz r^{d-1} \frac{\sqrt{z^2 + r^2}}{z^d} \qquad \ell \geq \xi$$

$$S_A = p_1 \left(\frac{l}{a_0}\right)^{d-1} + p_3 \left(\frac{l}{a_0}\right)^{d-3} + \cdots$$
$$-p_1 \left(\frac{l}{\xi}\right)^{d-1} - p_3 \left(\frac{l}{\xi}\right)^{d-3} + \cdots$$

-when d+1 = even = n, log(xi) appears:

$$\frac{-1}{d-2n-1} \left(\frac{\xi}{l}\right)^{-d+2n+1} - \frac{-1}{d-2n-1} \left(\frac{l}{a_0}\right)^{d-2n-1}$$
$$= \log \frac{\xi}{l} - \log \frac{a_0}{l} = \log \frac{\xi}{a_0}$$

- scaling of EE is almost unchanged by massive deformation,
  for all most all dimensions, except for (1+1)d.

$$S_A^{d=1} = \frac{c}{6} \log \xi / a_0$$

- as before, there is some sort of even-odd effect in dimensions
  log (xi) appears for d+1;
  however, no constant term for d+1=odd.

4. non CFTs

1. confinement
2. topological phases
3. fermi surface

topological phases

- fully gapped

- "quantum liquid" phase

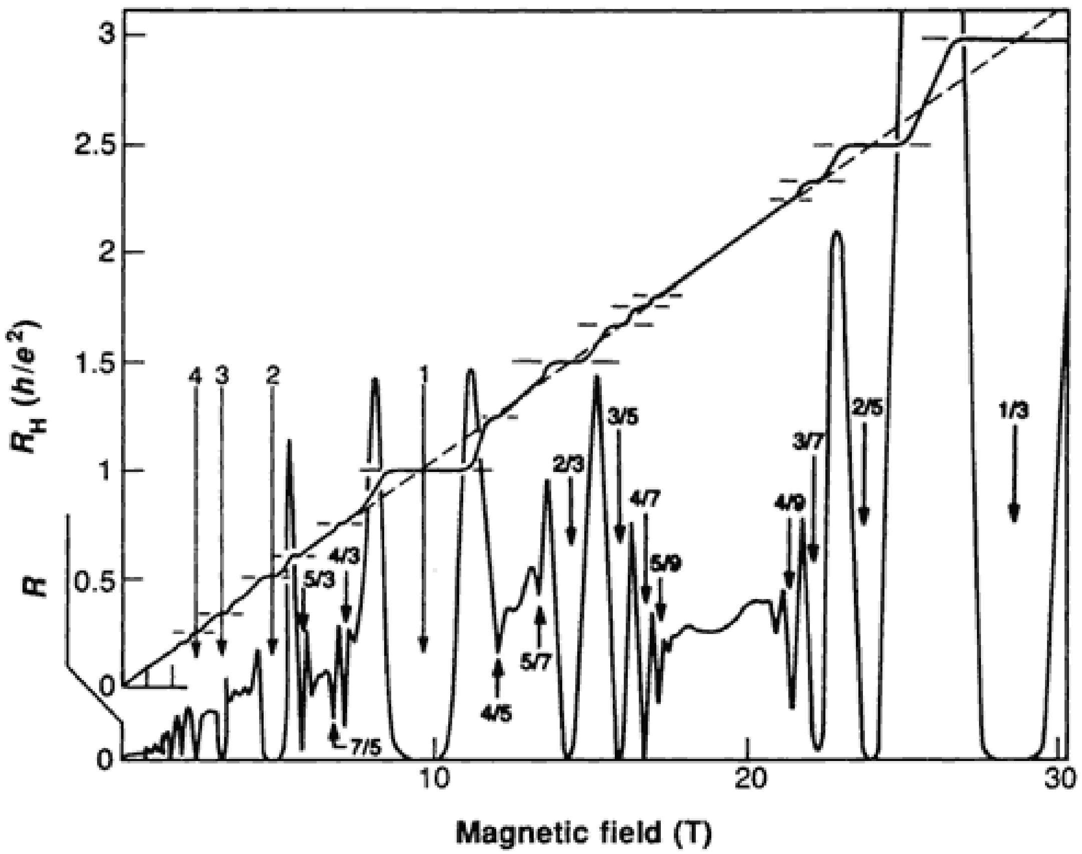
  not characterized by local order parameter
  no classical analogue -- fully quantum mechanical state

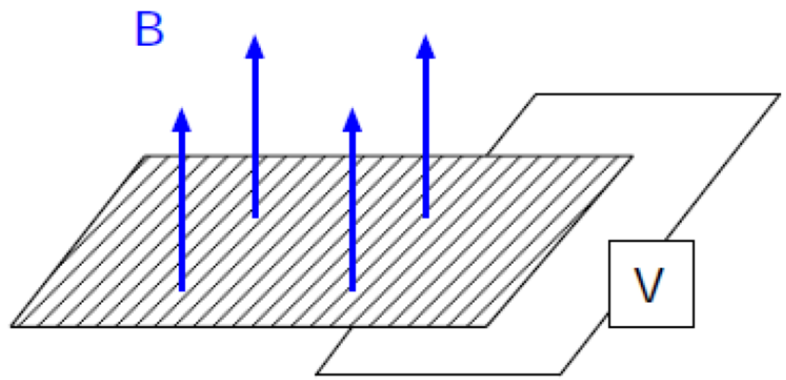
- ground state degeneracy depending on topology of spatial manifold

- fractionally charged excitations

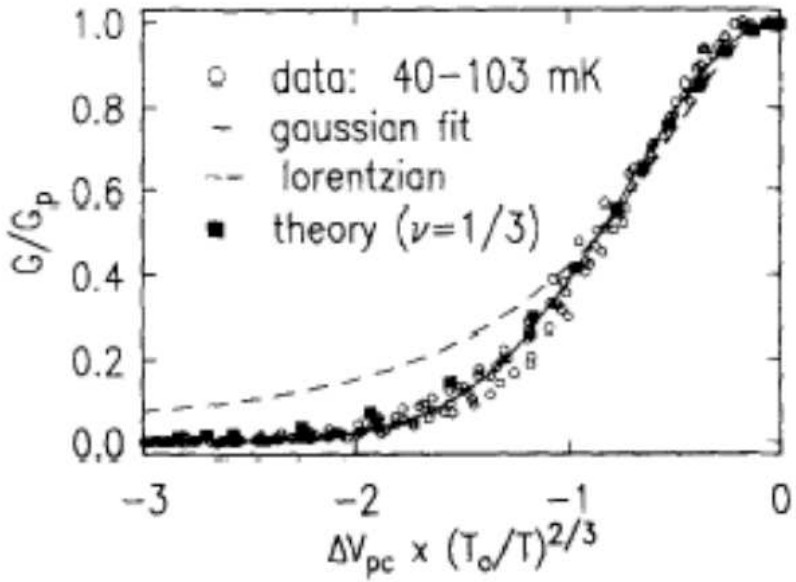
- non-trivial statistics (Abelian/non-Abelian fractional statistics) "anyon"

Fractional quantum Hall effect (FQHE)





QFT description:
(roughly) Chern-Simons type
topological quantum field theory



topological entanglement entropy

Kitaev & Preskill Levin & Wen (2006)

$$S_A = \gamma \frac{l}{a} - \log(D)$$

"total quantum dimension"

$$D = \sqrt{\sum_a d_a^2}$$

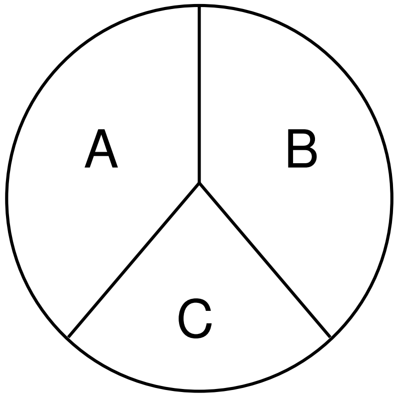
quantum dimension (d_a)

quasi-particle type (a)

$\log D = \log \sqrt{q}$ FQHE at nu = 1/q (Chern-Simons theory)

$\log D = \log 2$ Z2 lattice gauge theory

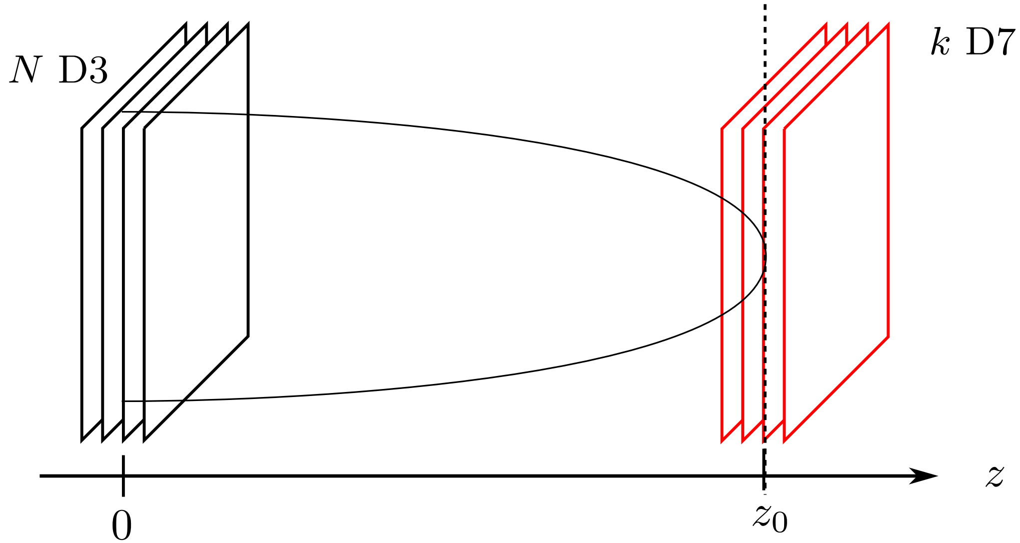
$$\begin{aligned} S_{\rm top} &= S_A + S_B + S_C \\ &\quad - S_{AB} - S_{BC} - S_{CA} \\ &\quad + S_{ABC} \\ &= -\log(D) \end{aligned}$$



Kitaev-Preskill

holographic dual of Chern-Simons

Fujita-Li-SR-Takayanagi (2009)



holographic dual of topological phase

$$S_{\mathrm{D3}} = \frac{k}{4\pi} \int \mathrm{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

$$S_{top} \sim \frac{k^2}{2} \log N$$

Need backreacted geometry

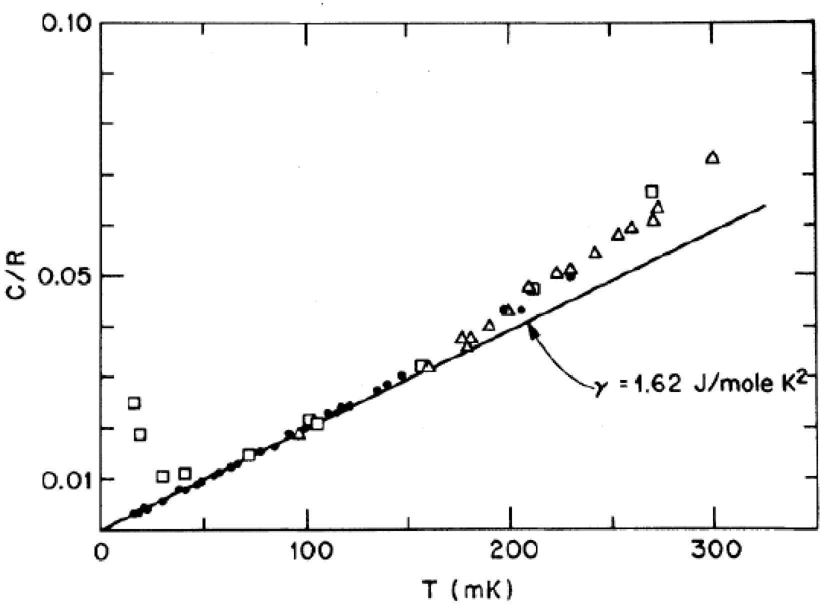
4. non CFTs

1. confinement
2. topological phases
3. fermi surface

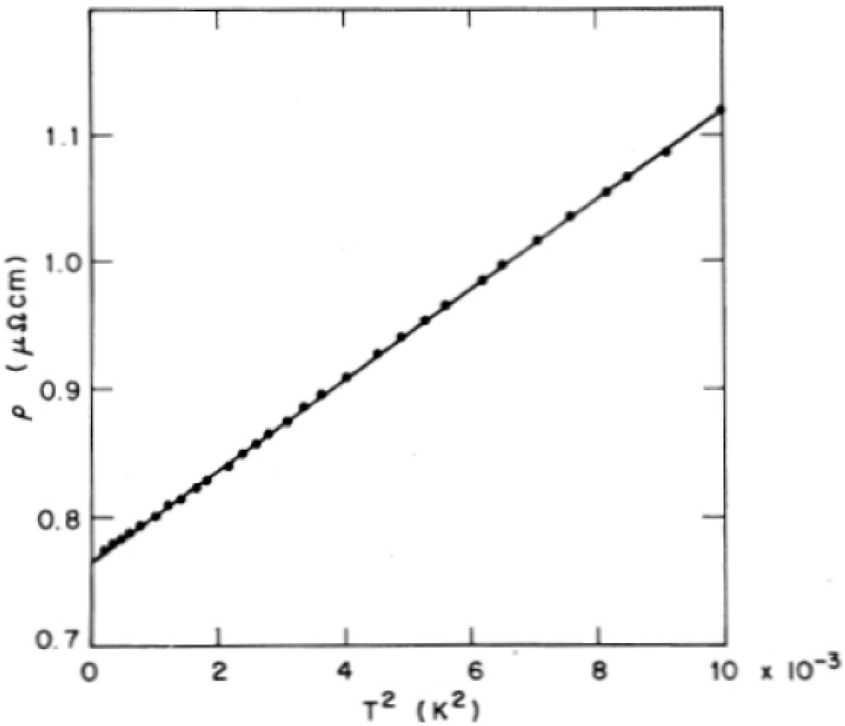
fermi liquid behavior CeAl3

specific heat

$$C = \gamma_0 T$$

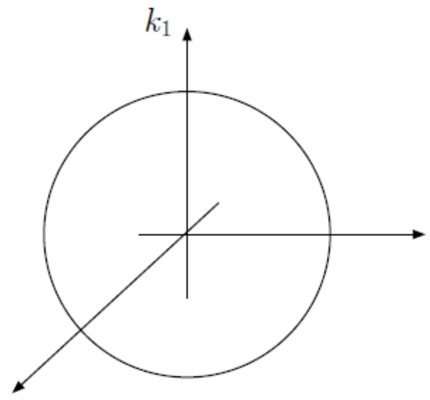


resistivity $\rho = \rho_0 + AT^2$



EE for free fermions with fermi surface

- for entangling surface parallel to x_2, ..., x_d directions
  (staight belt of length L)

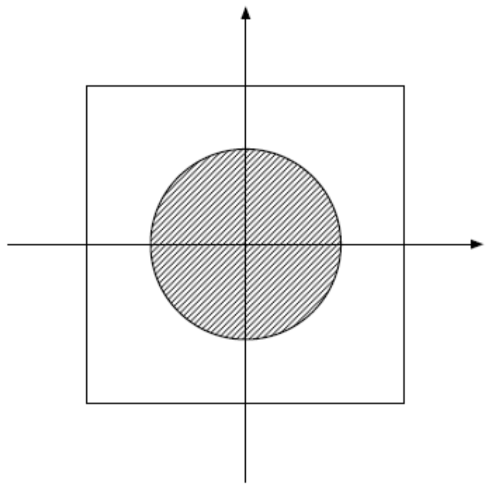


k_1, ..., k_d are a good quantum number
for each (k_2, ... k_d), there is a 1d system

$$S_A = \sum_{k_y,\cdots,k_d}^{\xi\leq l} \frac{c}{6}\log\frac{\xi}{a} + \sum_{k_y,\cdots,k_d}^{\xi\geq l} \frac{c}{3}\log\frac{l}{a}$$

$$S_A^{(\mathrm{FS})} = pL\log\frac{l}{a_0} - q\frac{L}{a_0} + \cdots$$

$$S_A^{(\mathrm{CFT})} = \gamma\frac{L}{a_0} - \alpha\frac{L}{\ell}$$



- for compact entangling surface

Gioev & Klich, Wolf (2006)

$$S_A = Cl^{d-1}\log(l/a_0)$$

$$C \propto \int_{\partial A}\int_{\mathrm{FS}} |\mathbf{n}_r\cdot\mathbf{n}_k| dS_r dS_k$$

holographic model for fermi surface

Ogawa-Takayanagi-Ugajin (2011)

start from an ansatz:

$$ds^2 = \frac{R^2}{z^2}\left(-f(z)dt^2 + g(z)dz^2 + dx^2 + dy^2\right)$$

for straight belt entangling surface: $A = \{(x,y)| -l/2 \leq x \leq l/2, 0 \leq y \leq L\}$

$$\text{Area} = 2R^2 L \int_{\epsilon}^{z_*} \frac{dz}{z^2}\sqrt{g(z) + (x'(z))^2}$$

if we choose:

$$g(z) \simeq \left(\frac{z}{z_F}\right)^{2n}, \quad z \gg z_F$$
$$\simeq 1, \quad z \ll z_F$$

we reproduce the log (n=1):

$$S_A = \frac{R^2 L}{2G_N^{(4)}\epsilon} + k_1 \frac{R^2}{G_N^{(4)}}\frac{L}{z_F}\ln\frac{l}{z_F} + \mathcal{O}(l^0)$$

null energy condition:

$$T_{\mu\nu}N^\mu N^\nu \geq 0$$

$$C \propto S \propto T^{\frac{1}{\hat{z}}} \quad \hat{z} \geq \frac{3}{2}$$

non-fermi liquid!

5. discussion

status of holographic entanglement entropy

Although the holographic formula has passed several non-trivial checks,
it has not been proven yet.

- reproduces area law

- reproduces EE of an interval in 1+1 d CFT

- reproduces a log part of EE in even dim CFT

- agrees with BH formula for high temperature

- (in)equalities, strong subadditivity,

- two intervals (mutual information)

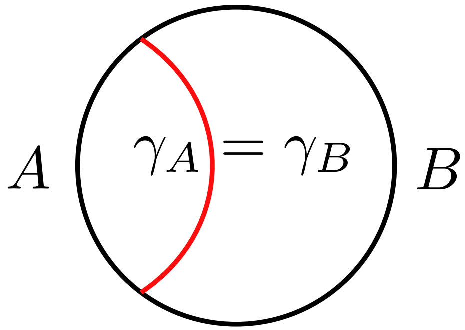
- proof when A = ball (dA = sphere)

Other possible checks ?

holographic entanglement entropy: some key properties

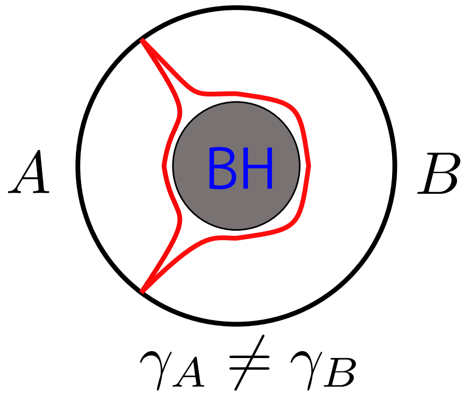
- when $\rho_{\rm tot} = \text{pure} \quad B = A^{\rm complement}$

$$S_A = S_B$$



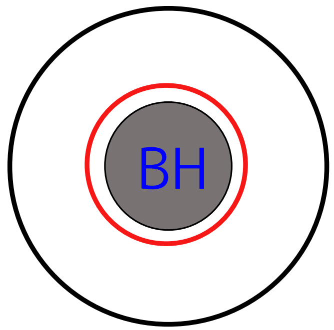
- when $\rho_{\rm tot} = \text{mixed} \quad B = A^{\rm complement}$

$$S_A \neq S_B$$



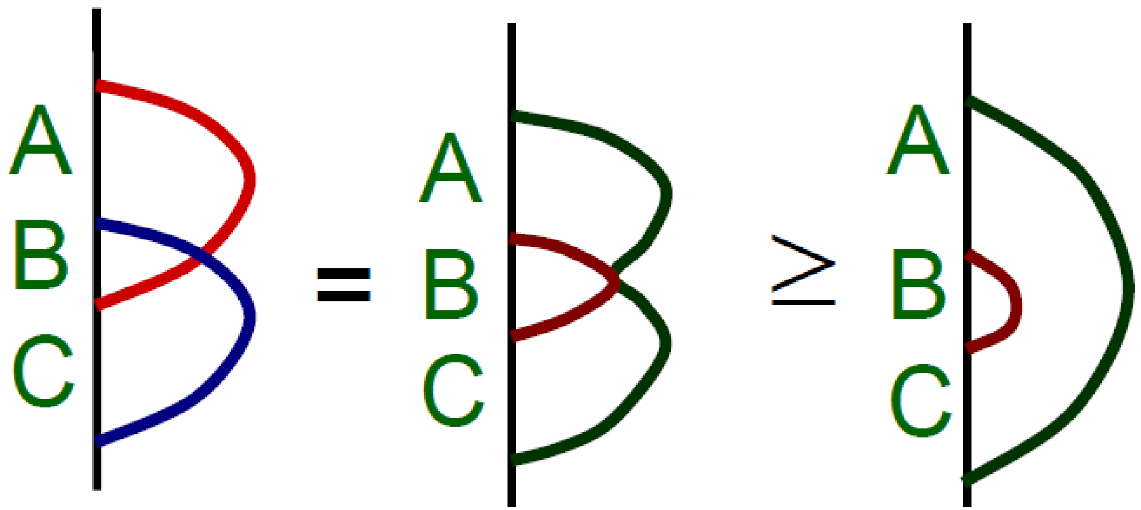
- when $\rho_{\rm tot} = e^{-\beta H} \quad A = \text{total system}$

$$S_A = \text{thermal entropy}$$



- strong subadditivity

$$S_{AB} + S_{BC} \geq S_B + S_{ABC}$$



Headrick-Takayanagi (07)

tensor-network representation of quantum states

$$|\Psi\rangle = \sum_{s_1,s_2,s_3,s_4\cdots} C^{s_1,s_2,s_3,s_4\cdots}|s_1,s_2,s_3,s_4\ldots\rangle$$

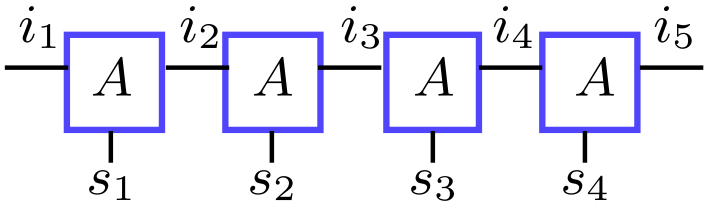
product state:

$$|\Psi\rangle = \sum_{\{s_a\}} A^{s_1}A^{s_2}A^{s_3}A^{s_4}\cdots|s_1,s_2,s_3,s_4\ldots\rangle = \prod_i A^{s_i}|s_i\rangle$$

EE = 0

matrix product state :

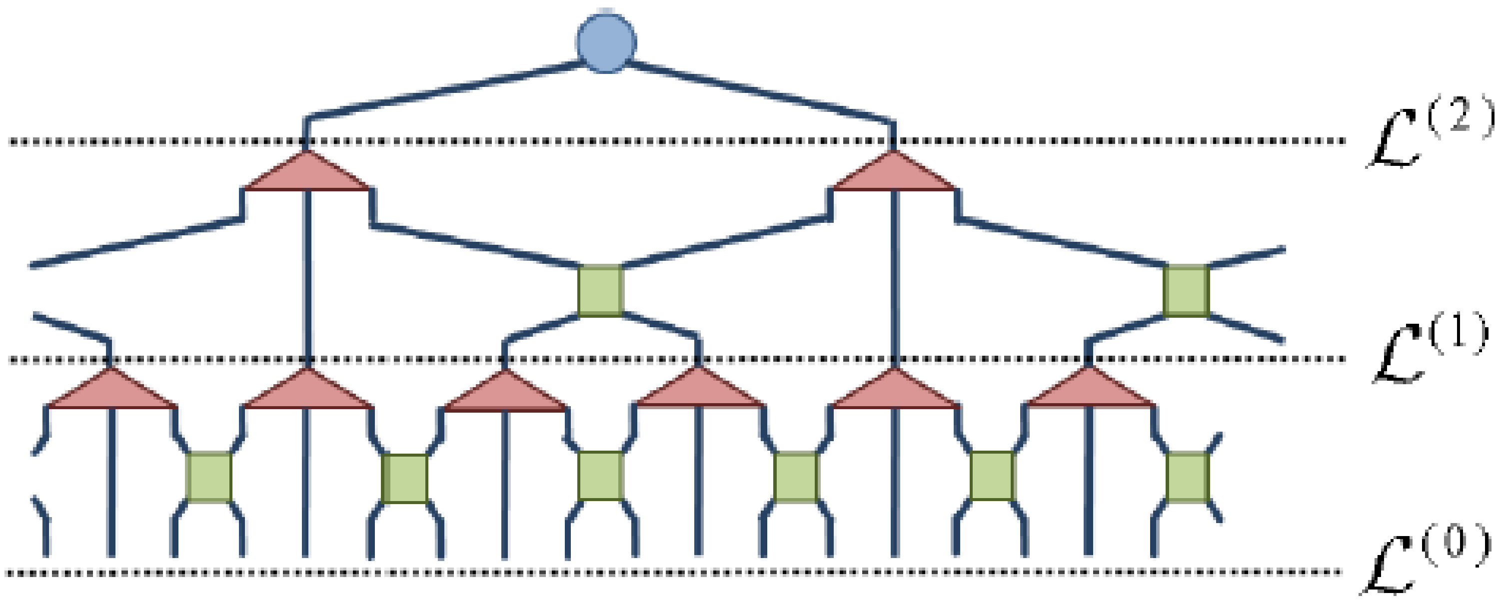
$$|\Psi\rangle = \sum_{\{s_a\}}\sum_{\{i_n=1,\cdots,\chi\}} A^{s_1}_{i_1,i_2}A^{s_2}_{i_2,i_3}A^{s_3}_{i_3,i_4}A^{s_4}_{i_4,i_5}\cdots|s_1,s_2,s_3,s_4\ldots\rangle$$



EE: Area law

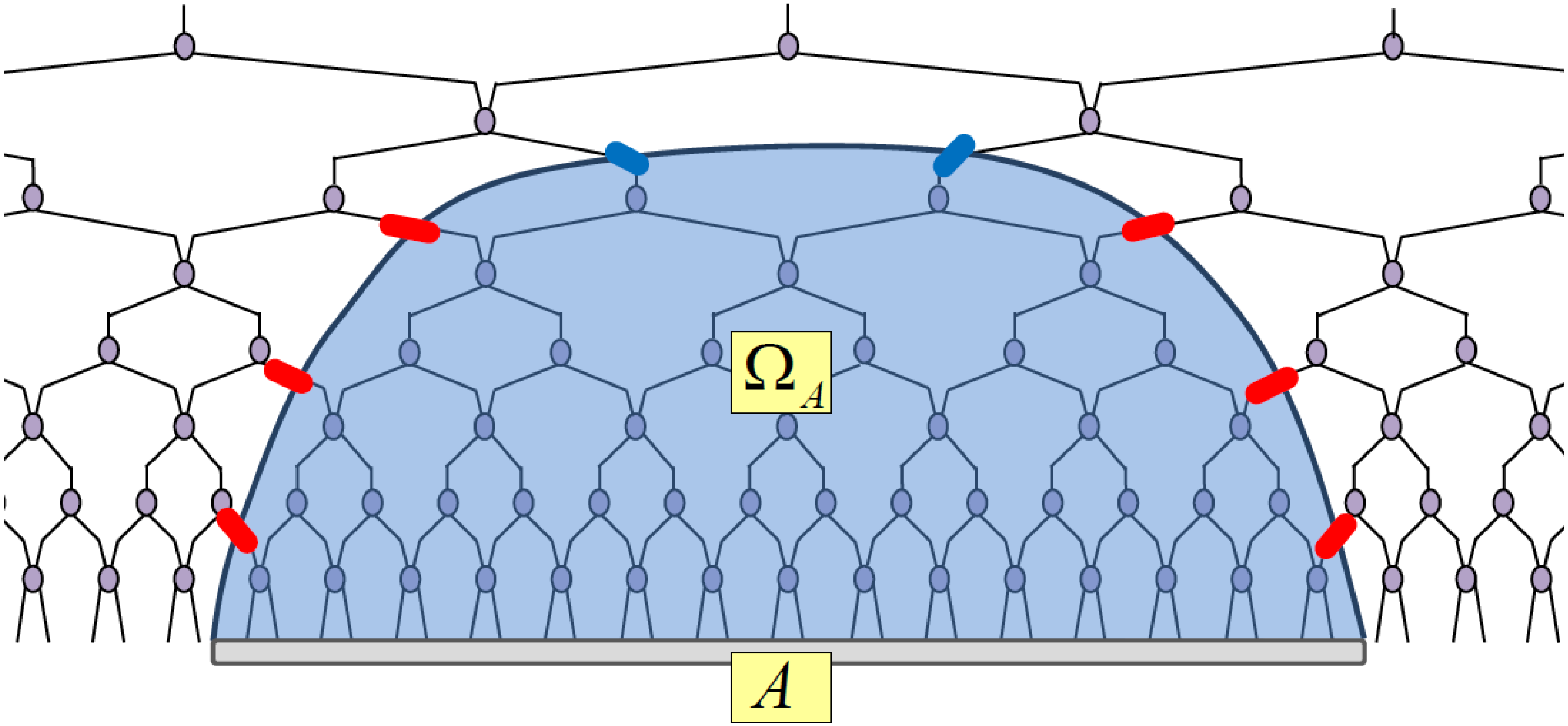
multiscale entanglement renomalization ansatz (MERA)

[Vidal (07-08)]



MERA and holographich entanglement entropy

[Swingle (09)]

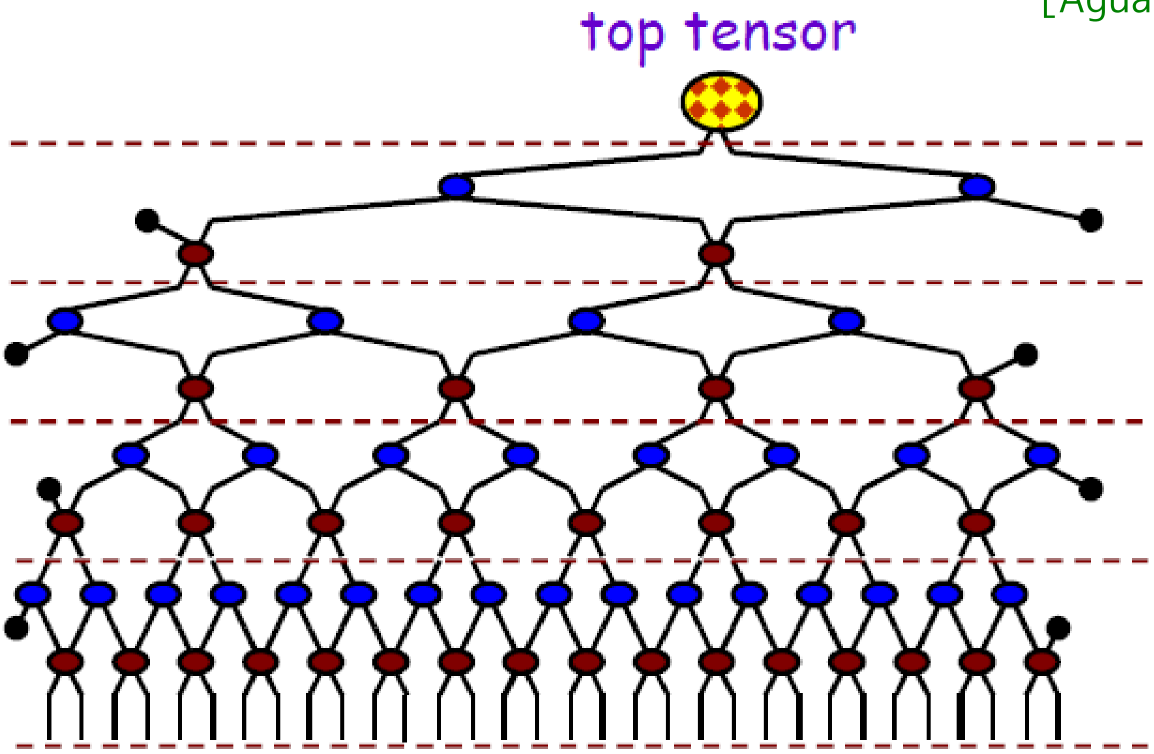


EE: $S_A \sim \log(l/a)$

MERA for topological phase

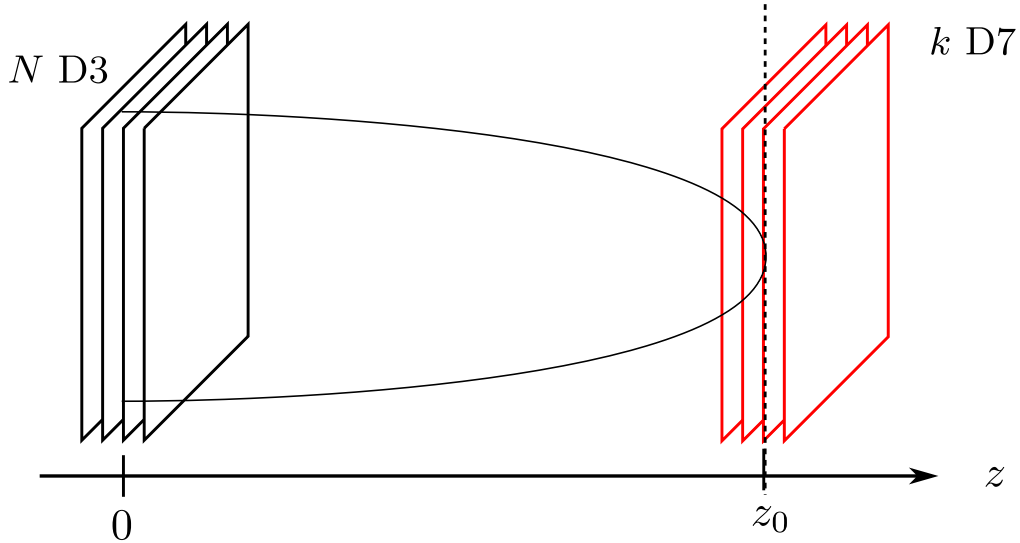
Topological infomation is strored in "top tensor"

[Koenig, Reichardt, Vidal (08)]
[Aguado-Vidal (09)]



holographic dual of Chern-Simons

Fujita-Li-SR-Takayanagi (2009)



holographic dual of topological phase

$$S_{\mathrm{D3}} = \frac{k}{4\pi}\int \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$

$$S_{top} \sim \frac{k^2}{2}\log N$$

Need backreacted geometry

D7-brane = "Top tensor" ?

summary

- entanglement entropy as an "order parameter"

  classical phases = group theory
  quantum phases = geometry ?

- holographic formula for EE

  -- not yet proved, but passed various tests,
     and has already been useful

  -- predictions and subsequend confirmations;
     CFTs, confinment, etc.

- how can geometry emerge from a quantum many-body state ?