The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific and
Cultural Organization

IAEA
International Atomic Energy Agency

2331-4

Spring School on Superstring Theory and RelatedTopics

19 - 27 March 2012

Extremal Black Hole Entropy

A. Sen
Harish-Chandra Research Institute

Extremal Black Hole Entropy

Ashoke Sen

Harish-Chandra Research Institute, Allahabad, India

ICTP, March 2012

Introduction

A black hole is a classical solution in general theory of relativity with special properties.

It is surrounded by an event horizon which acts as a one way membrane.

Nothing can escape from inside the event horizon to the outside.

Thus in classical general theory of relativity a black hole behaves as a perfect black body at zero temperature and is an infinite sink of entropy.

It has been known since the work of Bekenstein, Hawking and others that in quantum theory a black hole behaves as a thermodynamic system with finite temperature, entropy etc.

$$S_{BH} = \frac{A}{4\,G_N}$$

Bekenstein, Hawking

A: Area of the event horizon

G_N: Newton's gravitational constant

Our units: $\hbar = c = k_B = 1$

For ordinary objects the entropy of a system has a microscopic interpretation.

We fix the macroscopic parameters (e.g. total electric charge, energy etc.) and count the number of quantum states – known as microstates – each of which has the same charge, energy etc.

d_{micro}: number of such microstates

Define microscopic (statistical) entropy:

$$S_{micro} = \ln d_{micro}$$

Question: Does the entropy of a black hole have a similar statistical interpretation?

The best tests involve a class of supersymmetric extremal black holes in string theory, also known as BPS states.

Strategy:

1. Identify a supersymmetric black hole carrying a certain set of electric charges $\{Q_i\}$ and magnetic charges $\{P_i\}$ and calculate its entropy $S_{BH}(Q, P)$ using the Bekenstein-Hawking formula.

2. Identify the supersymmetric quantum states in string theory carrying the same set of charges and calculate the number $d_{micro}(Q, P)$ of these states.

3. Compare $S_{micro} \equiv \ln d_{micro}(Q, P)$ with $S_{BH}(Q, P)$.

For these one indeed finds a match:

$$\mathbf{A}/\mathbf{4G_N} = \ln \mathbf{d_{micro}}$$

Strominger, Vafa, ...

However this agreement also opens up new questions.

1. Both $A/4G_N$ and d_{micro} are computed in the large charge approximation.

On the black hole side this is needed to keep the curvature at the horizon small so that we can use classical Bekenstein-Hawking formula.

On the microscopic side the large charge approximation is needed so that we can use some asymptotic formula for estimating $\ln d_{micro}$.

Does the agreement between the microscopic and the macroscopic results hold beyond the large charge limit?

– need tools for more accurate computation of entropy on both sides.

2. On the microscopic side we can compute the entropy in different ensembles, *e.g.* grand canonical, canonical, microcanonical etc.

They all agree in the large charge limit, but differ from each other for finite charges.

Which of these entropies should we compare with the black hole entropy?

3. The computation of the entropy on the black hole side is valid when gravity is sufficiently strong so that the horizon radius is much larger than the compton wavelength.

The microscopic computation is valid in the opposite limit.

How can we compare the two?

Suggested remedy: Use supersymmetric index $\sim \mathrm{Tr}(-1)^F$

Protected from quantum corrections and is easier to compute on the microscopic side.

Is it reasonable to compare this with black hole entropy which counts $\mathrm{Tr}(1)$?

4. Do black holes carry more information than just the total number of states?

Example 1: Can we tell if most of the black holes are bosonic or fermionic, i.e. is $\mathrm{Tr}(-1)^F$ positive or negative?

Example 2: Suppose the theory has a discrete $\mathbb{Z}_N$ symmetry generated by g.

Can the black holes tell us the answer for $\left\{\mathrm{Tr}(-1)^F g\right\}$?

$\Leftrightarrow$ distribution of $\mathbb{Z}_N$ quantum numbers among the microstates.

Hope

Once we understand these issues we may gain more insight into the physics of non-extremal black holes.

Plan

1. First describe the analysis and results from the black hole side.

2. Then discuss the microscopic results and compare with the results from the black hole side.

Reviews:

A.S., arXiv:0708.1270, Mandal, A.S. arXiv:1008.3801

Gomes: arXiv:1111.2025

Precision analysis of black hole entropy

Goal:

1. Develop tools for computing the degeneracy of extremal black holes beyond the large charge limit.

2. Extend this to computation of index.

3. Repeat the analysis for g-twisted index.

Computation of black hole degeneracy.

To leading order it is given by $\exp[\mathbf{S_{BH}}(\mathbf{Q})]$**.**

Our goal will be to study corrections to this formula.

In string theory the Bekenstein-Hawking formula receives two types of corrections:

1. **Higher derivative (α') corrections in classical string theory.**

2. **Quantum (g_s) corrections.**

Note: Since the metric and the dilaton at the horizon are fixed by the charges, both the higher derivative corrections and string loop corrections are controlled by appropriate combination of the charges.

α' and g_s expansion $\Rightarrow$ an expansion in inverse power of charges.

Thus it is essential to understand these corrections if we want to go beyond the large charge limit.

How can we calculate the higher derivative and quantum corrections to the entropy?

Of these the higher derivative corrections are captured by Wald's modification of the Bekenstein-Hawking formula.

We need to develop an algorithm for computing quantum corrections.

Strategy: Use euclidean path integral formulation and make use of the presence of AdS_2 in the near horizon geometry.

– will include both types of corrections.

Origin of $\mathrm{AdS_2}$: Reissner-Nordstrom solution in $D = 4$

$$
\begin{aligned}
ds^2 &= -(1-\rho_+/\rho)(1-\rho_-/\rho)d\tau^2 \\
&\quad + \frac{d\rho^2}{(1-\rho_+/\rho)(1-\rho_-/\rho)} \\
&\quad + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)
\end{aligned}
$$

Define

$$
2\lambda = \rho_+ - \rho_-, \quad t = \frac{\lambda\tau}{\rho_+^2}, \quad r = \frac{2\rho - \rho_+ - \rho_-}{2\lambda}
$$

and take $\lambda \to 0$ limit keeping r, t fixed.

$$
ds^2 = \rho_+^2\left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right] + \rho_+^2(d\theta^2 + \sin^2\theta d\phi^2)
$$

$\mathrm{AdS_2}$ $\times$ S^2

This feature holds for all known extremal black hole solutions.

Postulate: Any extremal black hole has an AdS_2 factor / $SO(2,1)$ isometry in the near horizon geometry.

– partially proved

Kunduri, Lucietti, Reall; Figueras, Kunduri, Lucietti, Rangamani

The full near horizon geometry takes the form
$AdS_2 \times K$

K: some compact space that includes the S^2 factor.

In the presence of AdS_2 factor, Wald's formula takes a simple form.

First of all we can write down the most general field configuration consistent with the isometries of the near horizon geometry in terms of few constants.

e.g. if there are scalar fields ϕ_s, gauge fields $A_\mu^{(k)}$ and metric $g_{\mu\nu}$ then we have

$$ds^2 = a^2 \left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1} \right], \quad \phi_s = u_s, \quad F_{rt}^{(k)} = e_k$$

a, u_s, e_k : constants

Suppose $\mathcal{L}$ is the two dimensional classical effective Lagrangian density after dimensional reduction on the compact space K.

Construct the 'entropy function':

$$\mathbf{E} = 2\pi(\mathbf{q_k e_k} - \mathbf{a}^2\mathcal{L})|_{\mathbf{horizon}}$$

q_k: electric charge associated with the gauge field $A_\mu^{(k)}$.

Then the near horizon parameters are determined by solving the algebraic equations:

$$\frac{\partial \mathbf{E}}{\partial \mathbf{e_i}} = \frac{\partial \mathbf{E}}{\partial \mathbf{a}} = \frac{\partial \mathbf{E}}{\partial \mathbf{u_s}} = 0$$

The Wald entropy is given by

$$\mathbf{E}|_{\mathbf{extremum}}$$

Example: extremal Reissner-Nordstrom black hole

$$\mathbf{S} = \int \mathbf{d^4x}\sqrt{\det \mathbf{g}}\left[\frac{\mathbf{1}}{\mathbf{16\pi G_N}}\mathbf{R} - \frac{\mathbf{1}}{\mathbf{4}}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right]$$

General form of the near horizon geometry:

$$\mathbf{ds^2 = a^2\left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right] + b^2(d\theta^2 + \sin^2\theta d\phi^2)}$$

$$\mathbf{F_{rt} = e}, \qquad \mathbf{F}_{\theta\phi} = \frac{\mathbf{p}}{\mathbf{4\pi}}\sin\theta$$

Substituting this into the action and integrating over θ, ϕ we get the two dimensional Lagrangian density:

$$\mathbf{a^2\mathcal{L} = \frac{1}{2G_N}(a^2-b^2) + 2\pi b^2 a^{-2} e^2 - 2\pi a^2 b^{-2}(p/4\pi)^2}$$

$$\mathbf{E = 2\pi\left[q\,e - \frac{1}{2G_N}(a^2-b^2) - 2\pi b^2 a^{-2} e^2 + 2\pi a^2 b^{-2}(p/4\pi)^2\right]}$$

$$\mathbf{E} = 2\pi \left[\mathbf{q}\,\mathbf{e} - \frac{1}{2\mathbf{G_N}}(\mathbf{a}^2 - \mathbf{b}^2) - 2\pi \mathbf{b}^2\mathbf{a}^{-2}\mathbf{e}^2 + 2\pi \mathbf{a}^2\mathbf{b}^{-2}(\mathbf{p}/4\pi)^2 \right]$$

$$\partial\mathbf{E}/\partial\mathbf{e} = \partial\mathbf{E}/\partial\mathbf{a} = \partial\mathbf{E}/\partial\mathbf{b} = \mathbf{0}$$

$$\Rightarrow \mathbf{e} = \mathbf{q}/4\pi, \quad \mathbf{a}^2 = \mathbf{b}^2 = \mathbf{G_N}(\mathbf{q}^2 + \mathbf{p}^2)/4\pi$$

Thus

$$\mathbf{Entropy} = \mathbf{E} = (\mathbf{q}^2 + \mathbf{p}^2)/4$$

What about quantum corrections?

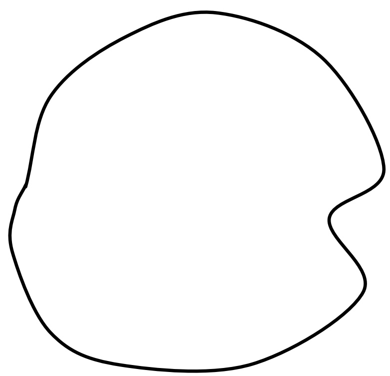
Presence of the AdS_2 factor allows us to apply the rules of AdS/CFT correspondence.

1. Consider the euclidean AdS_2 metric:

$$
\begin{aligned}
ds^2 &= a^2\left((r^2-1)d\theta^2 + \frac{dr^2}{r^2-1}\right), \quad 1 \le r < \infty,\ \theta \equiv \theta + 2\pi \\
&= a^2(\sinh^2\eta\, d\theta^2 + d\eta^2), \quad r \equiv \cosh\eta, \quad 0 \le \eta < \infty
\end{aligned}
$$

Regularize the infinite volume of AdS_2 by putting a cut-off $r \le r_0 f(\theta)$ for some smooth periodic function $f(\theta)$.

This makes the AdS_2 boundary have a finite length L.



2. Define the partition function:

$$Z_{AdS_2 \times K} = \int D\varphi \exp[-\text{Action}]$$

φ: set of all string fields

Boundary condition: Asymptotically the field configuration should approach the near horizon geometry of the black hole.

By AdS_2/CFT_1 correspondence:

$$Z_{AdS_2 \times K} = Z_{CFT_1}$$

CFT_1: dual (0+1) dimensional CFT obtained by taking the infrared limit of the quantum mechanical system underlying the black hole microstates.

3. Note on boundary condition:

Near the boundary of AdS_2, the θ independent solution to the Maxwell's equation has the form:

$$A_r = 0, \quad A_\theta = C_1 + C_2 r$$

C_1 (chemical potential) represents normalizable mode

C_2 (electric charge) represents non-normalizable mode

$\rightarrow$ the path integral must be carried out keeping C_2 (charge) fixed and integrating over C_1 (chemical potential).

Two consequences:

(a) The AdS_2 path integral computes the CFT_1 partition function in the microcanonical ensemble where all charges are fixed.

(b) This also forces us to to include a Gibbons-Hawking type boundary term in the path integral

$$\exp[-i q_k \oint_{\partial(AdS_2)} d\theta \, A_\theta^{(k)}]$$

$A_\mu^{(k)}$: gauge fields on AdS_2.

q_k: associated electric charge

4.

$$Z_{AdS_2\times K} = Z_{CFT_1} = Tr(e^{-LH}) = d_{hor}\, e^{-L\, E_0}$$

H: Hamiltonian of dual CFT_1 at the boundary of AdS_2.

(d_{hor}, E_0): (degeneracy, energy) of the states of CFT_1.

5. Thus we can define d_{hor} by expressing $Z_{AdS_2\times K}$ as

$$Z_{AdS_2\times K} = e^{CL} \times d_{hor} \quad \text{as } L \to \infty$$

C: A constant

d_{hor}: 'finite part' of $Z_{AdS_2\times K}$.

We identify $(\ln d_{hor})$ as the quantum corrected black hole entropy S_{macro}

Classical limit

$$Z_{AdS_2\times K} = \exp[-\text{Classical Action} - iq_k \oint d\theta\, A_\theta^{(k)}]$$

$$= \exp\left[-\int_1^{r_0} dr \int_0^{2\pi} d\theta [\sqrt{\det g}\,\mathcal{L}_E + iq_k\, F_{r\theta}^{(k)}]\right]$$

$\mathcal{L}_E$: Euclidean Lagrangian density integrated over K.

Now in the near horizon geometry:

$$\sqrt{\det g} = a^2, \quad \mathcal{L}_E = \text{constant}, \quad F_{r\theta}^{(k)} = -ie_k$$

Thus

$$Z_{AdS_2\times K} = \exp\left[-(a^2\mathcal{L}_E + q_k e_k)\int_1^{r_0} dr \int_0^{2\pi} d\theta\right]$$

$$\int_1^{r_0} dr \int_0^{2\pi} d\theta = 2\pi(r_0 - 1)$$

Length of the boundary of AdS_2 is

$$L = \int_0^{2\pi} \sqrt{g_{\theta\theta}}\, d\theta = 2\pi a \sqrt{r_0^2 - 1} = 2\pi r_0 a + \mathcal{O}(1/r_0)$$

Thus

$$\int_1^{r_0} dr \int_0^{2\pi} d\theta = L/a - 2\pi + \mathcal{O}(L^{-1})$$

$$\begin{aligned} Z_{AdS_2 \times K} &= \exp\left[-(a^2 \mathcal{L}_E + q_k e_k) \int_1^{r_0} dr \int_0^{2\pi} d\theta\right] \\ &= \exp\left[-(a^2 \mathcal{L}_E + q_k e_k)(L/a - 2\pi)\right] \end{aligned}$$

$$\Rightarrow d_{hor} = \exp[2\pi(a^2 \mathcal{L}_E + q_k e_k)] = \exp[S_{wald}]$$

Algorithm for computing quantum black hole entropy

1. Regularize infinte volume of AdS_2 by putting a cut-off $r \leq r_0$ so that the boundary has a finite length L.

2. Calculate

$$Z_{AdS_2\times K} = \int D\varphi\, \exp[-\text{Action} - i q_k \oint_{\partial(AdS_2)} d\theta\, A_\theta^{(k)}]$$

3. Define d_{hor} through:

$$Z_{AdS_2\times K} = e^{CL}\, d_{hor} \quad \text{as } L \to \infty$$

4. Identify $\ln d_{hor}$ as the quantum corrected entropy.

We can compute quantum corrections to d_{hor} by computing quantum corrections to $Z_{AdS_2 \times K}$.

Example: Logarithmic corrections to the black hole entropy

– corrections of order $\ln \Lambda$ if all charges scale as Λ

– arise from one loop contribution to $Z_{AdS_2 \times K}$ from massless fields.

General procedure

Let $\mathcal{K}$ be the kinetic operator for fluctuating massless fields around the black hole background.

Contribution to $Z_{\mathrm{AdS_2 \times K}}$ from the non-zero modes:

$$(\mathrm{sdet}'\, \mathcal{K})^{-1/2}$$

sdet′: remove contribution from zero modes.

This can be computed using the heat kernel of $\mathcal{K}$.

Zero modes of $\mathcal{K}$ are associated with pure gauge deformations with gauge transformation parameters which do not vanish at infinity.

The contribution from integration over the zero modes is obtained by changing variables from field deformations to gauge transformation parameters and keeping track of the Jacobian.

Details can be found in

Banerjee, Gupta, A.S., arXiv:1005.3044;
Banerjee, Gupta, Mandal, A.S., arXiv:1106.0080;
A.S. arXiv:1108.3842, arXiv:1109.3706
Bhattacharyya, Panda, A.S., to appear

Some details of the computation

Let $\{\psi_r\}$ denote the set of fluctuating massless fields around the near horizon background.

Let the eigenfunctions of the kinetic operator $\mathcal{K}$ be:

$$\psi_r = f_r^{(n)}(x), \qquad x \in AdS_2 \times S^2$$

with eigenvalue κ_n.

$$\mathcal{K} f^{(n)} = \kappa_n f^{(n)}$$

$$\int d^4x \sqrt{g} \sum_r f_r^{(n)}(x) f_r^{(m)}(x) = \delta_{mn}$$

$$\mathcal{K}\mathbf{f}^{(n)} = \kappa_n \mathbf{f}^{(n)}, \quad \int d^4x \sqrt{g} \sum_r f_r^{(n)}(x) f_r^{(m)}(x) = \delta_{mn}$$

Heat kernel sans zero modes:

$$K'(x, x', s) \equiv {\sum_{n,r}}' e^{-\kappa_n s} f_r^{(n)}(x) f_r^{(n)}(x')$$

One loop correction to ln Z from non-zero modes:

$$\Delta \ln Z = -\frac{1}{2} \ln \det{}' \mathcal{K} = -\frac{1}{2} {\sum_n}' \ln \kappa_n = \frac{1}{2} \int_\epsilon^\infty \frac{ds}{s} {\sum_n}' e^{-\kappa_n s}$$

ϵ: a string scale UV cut-off.

$$\Delta \ln Z = \frac{1}{2} \int_\epsilon^\infty \frac{ds}{s} {\sum_n}' e^{-\kappa_n s} = \frac{1}{2} \int_\epsilon^\infty \frac{ds}{s} \int d^4x \sqrt{\det g}\, K'(x, x; s)$$

$$\Delta \ln Z = \frac{1}{2}\int_{\epsilon}^{\infty}\frac{ds}{s}\int d^4x\,\sqrt{\det g}\;K'(x,x;s)$$

Homogeneity of $AdS_2 \times S^2$

$\Rightarrow K'(x,x;s)$ is independent of x.

$$\int d^4x\sqrt{\det g} = 4\pi a^2 \times 2\pi a^2 (r_0 - 1) \simeq 8\pi^2 a^4 (\frac{L}{2\pi a} - 1)$$

Drop the part proportional to L.

One loop correcton to entropy from non-zero modes:

$$-4\pi^2 a^4 \int_{\epsilon}^{\infty}\frac{ds}{s}K'(x,x;s)$$

$$K'(x,x',s) = \sum_{n,r}{}' e^{-\kappa_n s} f_r^{(n)}(x) f_r^{(n)}(x')$$

Since the eigenvalues κ_n are proportional to a^{-2}, and $f_r^{(n)}(x) \propto a^{-2}$, $a^4\, K'(x,x;s)$ is a function of $\bar{s} = s/a^2$.

One loop correcton to entropy from non-zero modes:

$$-4\pi^2 a^4 \int_\epsilon^\infty \frac{ds}{s} K'(x,x;s) = -4\pi^2 a^4 \int_{\epsilon/a^2}^\infty \frac{d\bar{s}}{\bar{s}} K'(x,x;s)$$

The logarithmic correction $\propto \ln a$ comes from the $\mathcal{O}(\bar{s}^0)$ term in the small $\bar{s}$ expansion of $K'(x,x;s)$.

If C_0 denotes the s-independent term in the small s expansion of $a^4 K'(x,x;s)$ then

$$\Delta \ln Z = -4\pi^2\, C_0 \ln a^2$$

Note: $\kappa_n = 0$ modes must be removed.

Zero mode contribution:

The path integral over the fields is defined with the standard general coordinate invariant measure, *e.g.* for gauge fields:

$$\int [\mathbf{DA}_\mu]\mathbf{exp}\left[-\int \mathbf{d^4x}\sqrt{\det \mathbf{g}}\,\mathbf{g}^{\mu\nu}\mathbf{A}_\mu\mathbf{A}_\nu\right] = \mathbf{1}$$

SInce $\sqrt{\det \mathbf{g}}\,\mathbf{g}^{\mu\nu} \sim \mathbf{a}^2$ this shows that $[\mathbf{aA}_\mu]$ has a independent measure.

Zero modes of $\mathbf{A}_\mu$ are of the form $\partial_\mu\Lambda$ with Λ not vanishing at ∞.

Changing variables from $\mathbf{aA}_\mu$ to $\Lambda \Rightarrow$ 'a' per zero mode

Net contribution to $\mathbf{Z}_{\mathbf{AdS_2\times K}}$ from gauge field zero modes is $\mathbf{a}^{\mathbf{N_z}}$ where $\mathbf{N_z}$ is the number of zero modes.

Computation of N_z:

Let

$$A_\mu(x) = h_\mu^{(k)}(x), \qquad k = 1, 2, \cdots$$

be the zero mode wave functions

Camporesi, Higuchi

$$N_z = \sum_k 1 = \int d^4x \sqrt{\det g}\, g^{\mu\nu} \sum_k h_\mu^{(k)}(x) h_\nu^{(k)}(x)$$

$c_z \equiv a^4 g^{\mu\nu} \sum_k h_\mu^{(k)}(x) h_\nu^{(k)}(x)$ is independent of x and a after summing over k.

$$\begin{aligned} N_z &= c_z\, a^{-4} \int d^4x \sqrt{\det g} = 8\pi^2 c_z (r_0 - 1) \\ &= 8\pi^2 c_z \left(\frac{L}{2\pi a} - 1 + \mathcal{O}(L^{-1}) \right) \end{aligned}$$

$$N_z = 8\pi^2 c_z \left(\frac{L}{2\pi a} - 1 + \mathcal{O}(L^{-1}) \right)$$

$\Rightarrow$ gauge field zero mode contribution to $Z_{AdS_2 \times K}$:

$$a^{N_z} = \exp\left[8\pi^2 c_z \ln a \left(\frac{L}{2\pi a} - 1 + \mathcal{O}(L^{-1}) \right) \right]$$

Comparing with $Z_{AdS_2 \times K} = d_{hor}\, e^{-E_0 L}$ we get the logarithmic contribution to $\ln d_{hor}$ from the zero modes:

$$-8\pi^2 c_z \ln a$$

Contributions from other zero modes can be found similarly.

Final results:

S. Banerjee, Gupta, Mandal, A.S.; Ferrara, Marrani; A.S.

The theory	scaling of charges	logarithmic contribution
$\mathcal{N} = 4$ with n_V matter	$Q_i \sim \Lambda, \quad A_H \sim \Lambda^2$	0
$\mathcal{N} = 8$	$Q_i \sim \Lambda, A_H \sim \Lambda^2$	$-8 \ln \Lambda$
$\mathcal{N} = 2$ with n_V vector and n_H hyper	$Q_i \sim \Lambda, \quad A_H \sim \Lambda^2$	$\frac{1}{6}(23 + n_H - n_V) \ln \Lambda$
$\mathcal{N} = 6$	$Q_i \sim \Lambda, \quad A_H \sim \Lambda^2$	$-4 \ln \Lambda$
$\mathcal{N} = 5$	$Q_i \sim \Lambda, \quad A_H \sim \Lambda^2$	$-2 \ln \Lambda$
$\mathcal{N} = 3$ with n_V matter	$Q_i \sim \Lambda, \quad A_H \sim \Lambda^2$	$2 \ln \Lambda$
BMPV in type IIB on $T^5/\mathbb{Z}_N$ or $K3 \times S^1/\mathbb{Z}_N$ with n_V vectors	$Q_1, Q_5, n \sim \Lambda$ $J \sim \Lambda^{3/2}, \quad A_H \sim \Lambda^{3/2}$	$-\frac{1}{4}(n_V - 3) \ln \Lambda$
BMPV in type IIB on $T^5/\mathbb{Z}_N$ or $K3 \times S^1/\mathbb{Z}_N$ with n_V vectors	$Q_1, Q_5, n \sim \Lambda$ $J = 0, \quad A_H \sim \Lambda^{3/2}$	$-\frac{1}{4}(n_V + 3) \ln \Lambda$

Blue entries: Tested against microscopic results

One loop correction due to massive string loops

Integrating out massive string modes gives a local one loop correction to the effective action.

The contribution of this term to $\ln d_{hor}$ is identical to the correction to the Wald entropy due to this local correction to the effective action.

Caution: Often only some special one loop correction to the effective Lagrangian is known and we can make further progress by assuming that only these terms contribute to the entropy at this order.

Index and twisted index

We shall now discuss computation of the index and twisted index from the black hole side.

For definiteness we shall mostly consider black holes in asymptotically (3+1) dimensional Minkowski space but the analysis can be easily generalized to other dimensions.

Index

The microscopic analysis is always done in a region of the moduli space where gravity is weak and hence the states do not form a black hole.

In order to be able to compare it with the results from the black hole side we must focus on quantities which do not change as we change the coupling from small to large value.

– needs appropriate supersymmetric index.

The appropriate index in D=4 is the helicity trace index.

Bachas, Kiritsis

Suppose we have a BPS state that breaks 4n supersymmetries.

$\rightarrow$ there will be 4n fermion zero modes (goldstino) on the world-line of the state.

Consider a pair of fermion zero modes ψ_0, $\psi_0^\dagger$ satisfying

$$\{\psi_0, \psi_0^\dagger\} = 1$$

If $|0\rangle$ is the state annihilated by ψ_0 then

$$|0\rangle, \qquad \psi_0^\dagger|0\rangle$$

give a degenerate pair of states with $J_3 = \pm 1/4$ and hence

$$(-1)^F = (-1)^{2J_3} = (-1)^{\pm 1/2} = \pm i$$

Thus

$$\mathrm{Tr}(-1)^F = 0, \qquad \mathrm{Tr}(-1)^F(2J_3) = i$$

Lesson: Quantization of the fermion zero modes produces Bose-Fermi degenerate states and make $\mathrm{Tr}(-1)^F$ vanish.

Remedy: Define

$$B_{2n} = \frac{(-1)^n}{(2n)!}\mathrm{Tr}(-1)^F(2J_3)^{2n} = \frac{(-1)^n}{(2n)!}\mathrm{Tr}(-1)^{2J_3}(2J_3)^{2n}$$

Since there are 2n pairs of zero modes,

$$\begin{aligned}
B_{2n} &= \frac{(-1)^n}{(2n)!}\mathrm{Tr}_{\mathrm{rest}}\mathrm{Tr}_{\mathrm{zero}}(-1)^{2J_3^{(1)}+\cdots 2J_3^{(2n)}+2J_3^{\mathrm{rest}}} \\
&\quad \times \left(2J_3^{(1)} + \cdots 2J_3^{(2n)} + 2J_3^{\mathrm{rest}}\right)^{2n} \\
&= (-1)^n\mathrm{Tr}_{\mathrm{rest}}\mathrm{Tr}_{\mathrm{zero}}(-1)^{2J_3^{(1)}+\cdots 2J_3^{(2n)}+2J_3^{\mathrm{rest}}} \times 2J_3^{(1)} \times \cdots 2J_3^{(2n)} \\
&= \mathrm{Tr}_{\mathrm{rest}}(-1)^{2J_3^{\mathrm{rest}}}
\end{aligned}$$

$$B_{2n} = \mathrm{Tr}_{rest}(-1)^{2J_3^{rest}}$$

Thus B_{2n} effectively counts $\mathrm{Tr}_{rest}(-1)^F$, with the trace taken over modes other than the 4n fermion zero modes associated with broken supersymmetry.

Note: B_{2n} does not receive any contribution from non-BPS states which break more than 4n supersymmetries and hence have more than 4n fermion zero modes.

Due to this property B_{2n} is protected from quantum corrections.

Question: Can we compute B_{2n} from the black hole side?

A.S.; Dabholkar, Gomis, Murthy, A.S.

For this it is crucial to understand the origin of the fermion zero modes on the black hole side.

The fermion zero modes associated with the broken supersymmetry generators have support outside the event horizon.

As a result d_{hor} does not count the contribution from these fermion zero modes.

Proof: Take a black hole solution and deform it by infinitesimal local supersymmetry transformation with parameter $\epsilon(\mathbf{x})$ such that

$$\epsilon(\mathbf{x}) \to \epsilon_0 \quad \text{as } \mathbf{x} \to \infty$$

$$\epsilon(\mathbf{x}) = 0 \quad \text{for } |\mathbf{x}| < R_0 \text{ for some } R_0$$

1. Deformations have support outside the sphere of radius R_0.

2. This is not a pure gauge deformation if ϵ_0 is not the asymptotic value of a Killing spinor.

Now let us compute B_{2n} for the black hole.

$$B_{2n} = \frac{(-1)^n}{2n!} \mathrm{Tr}(-1)^{2h}(2h)^{2n} = \frac{(-1)^n}{2n!} \mathrm{Tr}(-1)^{h_{hor}+h_{zero}}(2h_{hor} + 2h_{zero})^{2n},$$

$$h \equiv J_3$$

For black hole with four unbroken supersymmetries:

SUSY $+ SL(2,R)$ isometry of $AdS_2 \rightarrow SU(1,1;2)$ supergroup

– symmetry group of the near horizon geometry.

$$SU(1,1;2) \supset SU(2)$$

$\rightarrow$ horizon must be spherically symmetric.

Furthermore since the black hole is in the microcanonical ensemble,

spherical symmetry $\rightarrow$ zero angular momentum

$\rightarrow h_{hor} = 0.$

$$B_{2n} = \frac{(-1)^n}{2n!} \mathrm{Tr}(-1)^{2h}(2h)^{2n} = \frac{(-1)^n}{2n!} \mathrm{Tr}(-1)^{h_{hor}+h_{zero}}(2h_{hor} + 2h_{zero})^{2n}$$

$$h_{hor} = 0$$

Thus

$$B_{2n} = \frac{(-1)^n}{2n!} \mathrm{Tr}(-1)^{h_{zero}}(2h_{zero})^{2n}$$

$$\| \qquad \mathrm{Tr} \Rightarrow \mathrm{Tr}_{hor}\mathrm{Tr}_{zero}$$

$$\mathrm{Tr}_{hor} 1 \times \frac{(-1)^n}{2n!} \mathrm{Tr}_{zero}(-1)^{h_{zero}}(2h_{zero})^{2n} = d_{hor}$$

$$B_{2n} = d_{hor}$$

– explains why we can compare the microscopic index with the macroscopic entropy.

A.S.; Dabholkar, Gomes, Murthy, A.S.

In addition this predicts positive sign of B_{2n}.

This can in principle be tested on the microscopic formulæ for the index, if known.

Twisted index

Suppose the theory under consideration has some discrete $\mathbb{Z}_N$ symmetry generated by g that commutes with the 4n unbroken supersymmetries of the black hole.

Then

$$B^g_{2n} = \frac{(-1)^n}{(2n)!} \mathrm{Tr}\left[(-1)^{2h} (2h)^{2n} g\right]$$

is protected.

Can we compute this from the black hole side?

After separating out the contribution from the hair degrees of freedom, and using $h_{hor} = 0$, we see that the relevant quantity associated with the horizon is

$$\mathrm{Tr}_{hor}(g)$$

By following the logic of AdS/CFT correspondence we find that we need to again compute the partition function on AdS_2, but this time with a g twisted boundary condition on the fields under $\theta \to \theta + 2\pi$.

Other than this the asymptotic boundary condition must be identical to that of the original near horizon geometry since the charges have not changed

The ‘finite part’ of this partition function gives us $Tr_{hor}(g)$.

Recall AdS_2 metric:

$$\mathbf{ds^2 = a^2 \left[(r^2 - 1) d\theta^2 + \frac{dr^2}{r^2 - 1} \right] = v \left[\sinh^2 \eta \, d\theta^2 + d\eta^2 \right]}$$

The circle at infinity, parametrized by θ, is contractible at the origin $r = 1$.

Thus a g twist under $\theta \to \theta + 2\pi$ is not admissible.

$\to$ the $AdS_2 \times S^2$ geometry is not a valid saddle point of the path integral.

Question: Are there other saddle points which could contribute to the path integral?

Constraints:

1. It must have the same asymptotic geometry as the $AdS_2 \times S^2$ geometry.

2. It must have a g twist under $\theta \to \theta + 2\pi$.

3. It must preserve sufficient amount of supersymmetries so that integration over the fermion zero modes do not make the integral vanish.

Beasley, Gaiotto, Guica, Huang, Strominger, Yin; N. Banerjee, S. Banerjee, Gupta, Mandal, A.S.

There are indeed such saddle points in the path integral, constructed as follows.

1. Take the original near horizon geometry of the black hole.

2. Take a $\mathbb{Z}_N$ orbifold of this background with $\mathbb{Z}_N$ generated by simultaneous action of

a) $\theta \to \theta + 2\pi/N$

a) $\phi \to \phi + 2\pi/N$ (needed for preserving SUSY)

c) g.

N. Banerjee, Jatkar, A.S.; A.S.; Pioline, Murthy

To see that this satisfies the required boundary condition we make a rescaling:

$$\theta \to \theta / \mathbf{N}, \qquad \mathbf{r} \to \mathbf{N}\,\mathbf{r}$$

The metric takes the form:

$$\mathbf{a}^2 \left((\mathbf{r}^2 - \mathbf{N}^{-2}) \mathbf{d}\theta^2 + \frac{\mathbf{dr}^2}{\mathbf{r}^2 - \mathbf{N}^{-2}} \right)$$

Orbifold action: $\theta \to \theta + 2\pi$, $\phi \to \phi + 2\pi / \mathbf{N}$, g

The g transformation provides us with the correct boundary condition.

The ϕ shift can be regarded as a Wilson line, and hence is an allowed fluctuation in AdS_2.

The classical action associated with this saddle point, after removing the divergent part proportional to the length of the boundary, is S_{wald}/N.

Thus the leading contribution to the twisted partition function from this saddle point is

$$Z_g^{finite} = \exp\left[S_{wald}/N\right]$$

This gives a prediction for the twisted index B_{2n}^g in the microscopic theory.

Some exact microscopic results in D=4

Exact microscopic results are known for

1. Type II on T^6,

2. Heterotic on T^6 or equivalently type II on $K3 \times T^2$,

3. Some special orbifolds of the above theories with 16 unbroken supersymmetries

– known as CHL models

Chaudhuri, Hockney, Lykken

Relevant index

Type II on T^6 has 32 supersymmetries.

1/8 BPS black holes break 28 of the supersymmetries.

Thus the relevant index is B_{14}.

Heterotic on T^6 (or type II on $K3 \times T^2$) has 16 supersymmetries.

1/4 BPS black holes break 12 supersymmetries.

Thus the relevant index is B_6.

Type II on T^6

This theory has 12 NSNS sector gauge fields and 16 RR sector gauge fields.

Consider a dyon carrying NSNS sector charges.

– characterized by 12 dimensional electric and magnetic charge vectors Q and P.

Q and P transform as vectors under the T-duality group $SO(6, 6; \mathbb{Z})$

$Q^2, P^2, Q \cdot P$: T-duality invariant inner products.

$$Q^2 = 2\sum_{i=1}^{6} n_i w_i, \quad P^2 = 2\sum_{i=1}^{6} N_i W_i, \quad Q\cdot P = \sum_{i=1}^{6}(n_i N_i + w_i W_i)$$

n_i, w_i: (momentum, winding) along i-th circle

N_i, W_i: (KK monopole, H-monopole) charge along i-th circle

Define $\Delta = Q^2P^2 - (Q\cdot P)^2$

– invariant also under S-duality group

Restrict to states satisfying $\gcd\{Q_iP_j - Q_jP_i\} = 1$

Dabholkar, Gaiotto, Nampuri

Then

$$\mathbf{B_{14}} = (-1)^{\mathbf{Q\cdot P+1}} \sum_{\mathbf{s|Q^2/2,P^2/2,Q\cdot P}} \mathbf{s}\,\widehat{\mathbf{c}}(\Delta/\mathbf{s^2})$$

where $\widehat{\mathbf{c}}(\mathbf{u})$ is defined through

$$-\vartheta_1(\mathbf{z}|\tau)^2\,\eta(\tau)^{-6} \equiv \sum_{\mathbf{k,l}} \widehat{\mathbf{c}}(4\mathbf{k}-\mathbf{l}^2)\,\mathbf{e}^{2\pi \mathbf{i}(\mathbf{k}\tau+\mathbf{l}\mathbf{z})}$$

Shih, Strominger, Yin

ϑ_1: Jacobi theta function η: Dedekind eta function

$$\widehat{\mathbf{c}}(-1)=1,\ \widehat{\mathbf{c}}(0)=-2,\ \widehat{\mathbf{c}}(3)=8,\ \widehat{\mathbf{c}}(4)=-12,\ \widehat{\mathbf{c}}(7)=39$$

$$\widehat{\mathbf{c}}(8)=-56,\ \widehat{\mathbf{c}}(11)=152,\ \widehat{\mathbf{c}}(12)=-208,\cdots$$

B_{14} is positive and for large charges we have

$$\log[\mathbf{B_{14}}] = \pi\sqrt{\Delta} - 2\ln\Delta + \cdots$$

Although we have stated the results for black holes carrying only NSNS sector charges, it also covers many other black holes carrying purely RR charges or both NSNS and RR charges, since U-duality symmetry relates many of these black holes.

$$\mathbf{B_{14}} > \mathbf{0}, \quad \mathbf{log}[\mathbf{B_{14}}] = \pi\sqrt{\Delta} - \mathbf{2\,ln}\,\Delta + \cdots$$

Bekenstein-Hawking entropy S_{BH} of a black hole carrying the same charges is given by

$$\pi\sqrt{\Delta}$$

1. Agreement between black hole entropy and $\ln B_{14}$ at leading order as predicted by general arguments.

2. The sign of B_{14} is positive as predicted from the black hole side.

3. The subleading $-2\ln\Delta$ correction is precisely in agreement with the logarithmic correction computed from the black hole side.

Heterotic string theory on T^6

This theory has 28 U(1) gauge fields.

Thus a generic charged state is characterized by 28 dimensional electric charge vector Q and magnetic charge vector P.

The theory has T-duality symmetry $O(6, 22; \mathbb{Z})$ under which Q and P transform as vectors.

This allows us to define T-duality invariant bilinears in the charges:

$$Q^2, \qquad P^2, \qquad Q \cdot P$$

More general class of $\mathcal{N}=4$ supersymmetric string theories can be constructed by taking orbifolds of heterotic string theory on T^6.

– CHL models

Chaudhuri, Hockney, Lykken

These theories have $(r+6)$ U(1) gauge fields for different values of r.

Thus Q and P are (r+6) dimensional vectors.

We can again construct $O(r,6)$ invariant bilinears

$$Q^2, \qquad P^2, \qquad Q \cdot P$$

In each of these theories, the index $\mathbf{B_6(Q,P)}$ has been computed for a wide class of charge vectors $\mathbf{(Q,P)}$.

In each case the result is expressed as Fourier expansion coefficients of some well known functions $\mathbf{Z}(\rho,\sigma,\mathbf{v})$, called Siegel modular forms:

$$\mathbf{B_6} = (-1)^{\mathbf{Q.P}+1} \int \mathbf{d}\rho \int \mathbf{d}\sigma \int \mathbf{dv}\, \mathbf{e}^{-\pi i(\rho \mathbf{Q^2}+\sigma \mathbf{P^2}+2\mathbf{v}\mathbf{Q}\cdot\mathbf{P})} \mathbf{Z}(\rho,\sigma,\mathbf{v})$$

$\mathbf{Z}(\rho,\sigma,\mathbf{v})$: explicitly known in each of the examples, and transform as modular forms of certain weights under subgroups of $\mathrm{Sp}(2,\mathbb{Z})$.

Dijkgraaf, Verlinde, Verlinde; Shih, Strominger, Yin; David, Jatkar, A.S.; Dabholkar, Gaiotto, Nampuri;

S. Banerjee, Srivastava, A.S.; Dabholkar, Gomes, Murthy; Govindarajan, Gopala Krishna; $\cdots$

Some microscopic results for B_6 in heterotic on T^6 (Fourier coefficients of a Siegel modular form)

$(Q^2, P^2) \backslash Q.P$	-2	2	3	4	5	6	7
(2,2)	-209304	648	327	0	0	0	0
(2,4)	-2023536	50064	8376	-648	0	0	0
(2,6)	-15493728	1127472	130329	-15600	972	0	0
(4,4)	-16620544	3859456	561576	12800	3272	0	0
(4,6)	-53249700	110910300	18458000	1127472	85176	-6404	0
(6,6)	2857656828	4173501828	920577636	110910300	8533821	153900	26622
(2,10)	-510032208	185738352	16844421	-2023536	315255	-31104	1620

Red entries: Negative B_6

Blue entries: $\Delta \equiv Q^2P^2 - (Q.P)^2 < 0$ for which there are no black hole solutions.

Presence of negative B_6 is an apparent violation of the prediction from the black hole side.

However we must take into account that the $B_6 > 0$ prediction was made for single centered black holes.

In general B_6 can receive contribution from both single and multi-centered black holes.

This complication was absent for type II on T^6 since in that case the multi-centered black holes do not contribute to B_{14} for $\Delta > 0$.

A.S.

Strategy: Calculate the contribution to the index from multi-centered black holes and subtract from the above result.

Result after subtraction

$(Q^2, P^2) \backslash Q.P$	-2	2	3	4	5	6	7
(2,2)	648	648	0	0	0	0	0
(2,4)	50064	50064	0	0	0	0	0
(2,6)	1127472	1127472	25353	0	0	0	0
(4,4)	3859456	3859456	561576	12800	0	0	0
(4,6)	110910300	110910300	18458000	1127472	0	0	0
(6,6)	4173501828	4173501828	920577636	110910300	8533821	153900	0
(2,10)	185738352	185738352	16844421	16491600	0	0	0

No more negative index or $\Delta < 0$ states.

Similar results hold for other $\mathcal{N} = 4$ supersymmetric CHL models.

The above results illustrate the power of black holes to explain features of black hole microstates beyond the leading Bekenstein-Hawking entropy.

Proving these positivity relations for all $(\mathbf{Q}^2, \mathbf{P}^2, \mathbf{Q}.\mathbf{P})$ remains a challenging problem for the mathematicians and reflects some non-trivial properties of the Siegel modular forms.

Partial progress by Bringmann and Murthy

It is also possible to find the systematic expansion of B_6 for large charges.

In each case we find $\underline{B_6 > 0}$ in large charge limit.

– consistent with the fact that the single centered black hole contrbution dominates in the large charge limit.

$$\ln \mathbf{B_6} \;=\; \pi\sqrt{\mathbf{Q^2P^2} - (\mathbf{Q.P})^2} + \mathbf{f}\left(\frac{\mathbf{Q.P}}{\mathbf{P^2}}, \frac{\sqrt{\mathbf{Q^2P^2} - (\mathbf{Q.P})^2}}{\mathbf{P^2}}\right)$$
$$+\mathcal{O}(\mathbf{charge}^{-2})$$

f: a known function.

Cardoso, de Wit, Kappeli, Mohaupt; David, Jatkar, A.S.

For example, for heterotic string theory compactified on a six dimensional torus,

$$\mathbf{f}(\tau_1, \tau_2) = \mathbf{12}\ln\tau_2 + \mathbf{24}\ln\eta(\tau_1 + \mathbf{i}\tau_2) + \mathbf{24}\ln\eta(-\tau_1 + \mathbf{i}\tau_2)$$

η: **Dedekind function**

$$\ln B_6 = \pi\sqrt{Q^2P^2-(Q.P)^2}+f\left(\frac{Q.P}{P^2},\frac{\sqrt{Q^2P^2-(Q.P)^2}}{P^2}\right)$$
$$+\mathcal{O}(\text{charge}^{-2})$$

Bekenstein-Hawking entropy S_{BH} of a black hole carrying the same charges is given by

$$\pi\sqrt{Q^2P^2-(Q.P)^2}$$

1. The absence of logarithmic corrections is precisely in agreement with the corresponding results from the black hole side.

2. The finite subleading corrections agree with the one loop contribution to d_{hor} due to massive string states.

There are closely related results in 4+1 non-compact dimensions e.g. in type II on T^5, type II on $K3 \times S^1$ and their orbifolds.

Maldacena, Moore, Strominger; Dijkgraaf, Moore, Verlinde, Verlinde; Jatkar, David, A.S.

Logarithmic corrections to the entropy in these theories agree with the prediction from the black hole side.

Twisted index

On special subspaces of the parameter space of the $\mathcal{N}=8$ and $\mathcal{N}=4$ supersymmetric string theories in (3+1) dimensions, the theory develops $\mathbb{Z}_N$ discrete symmetry generated by an element g which commutes with 16 supersymmetries.

Example: For heteroric on T^6 we can have N=2,3,4,5,6,7,8

On these special subspaces we can define the twisted index:

$$B_6^g = -\frac{1}{6!}\mathrm{Tr}\left[(-1)^{2h}(2h)^6 g\right]$$

Like B_6, this index is also protected.

In each case we can calculate the twisted index B_6^g, and find that the result is again given by Fourier integrals of modular forms of subgroups of $Sp(2, \mathbb{Z})$.

$$B_6^g = (-1)^{Q.P+1} \int d\rho \int d\sigma \int dv\, e^{-\pi i(\rho Q^2 + \sigma P^2 + 2v Q\cdot P)} Z_g(\rho, \sigma, v)$$

Z_g are known functions.

Furthermore for large charges we find

$$B_6^g = \exp[\pi \sqrt{Q^2 P^2 - (Q\cdot P)^2}/N + \cdots]$$

This is exactly in accordance with the prediction from the black hole side.

Conclusion

Quantum gravity in the near horizon geometry contains detailed information about not only the total number of microstates. but also finer details *e.g.* the $\mathbb{Z}_N$ quantum numbers carried by the microstates, the sign of the index etc..

Thus at least for extremal black holes there seems to be an exact duality between

Gravity description $\Leftrightarrow$ Microscopic description

General lesson

Euclidean quantum gravity can be trusted beyond the classical approximation.

Inspired by the success we can try to extend the Euclidean gravity techniques to non-supersymmetric black holes.

Example: An extremal Kerr black hole in D=4 has logarithmic correction:

$$\frac{16}{45} \ln A_H$$

Can Kerr/CFT correspondence explain this microscopically?