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Z. Komargodski
The Weizmann Institute of Science

1. Two-Dimensional Theories

1.1. Scale Invariant Theories

Lorentz invariant two-dimensional theories enjoy the two-dimensional Poincare group, which contains Lorentz transformations in $SO(1,1)$. Let us consider a conformally invariant theory, that is, a theory with $SO(2,2)$ symmetry. It has an energy-momentum tensor which is conserved and trace-less

$$\partial^\mu T_{\mu\nu} = 0 \ , \qquad T^\mu_\mu = 0 \ .$$

These equations should be regarded as operator relations, that is, they hold in all correlation functions at separated points. At coincident points there may be contact-terms (i.e. various delta functions). If these equations can be maintained at coincident points we say that the conformal group is free of anomalies. However, we will now prove that these equations cannot be maintained at coincident points.

Consider the two point function $\langle T_{\mu\nu}(q)T_{\rho\sigma}(-q)\rangle$ and decompose it in the most general way consistent with Lorentz invariance

$$\langle T_{\mu\nu}(q)T_{\rho\sigma}(-q)\rangle = (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho})f_1(q^2) + \eta_{\mu\nu}\eta_{\rho\sigma}f_2(q^2)$$

$$+(q_\mu q_\rho\eta_{\nu\sigma}+q_\mu q_\sigma\eta_{\nu\rho}+q_\nu q_\rho\eta_{\mu\sigma}+q_\nu q_\sigma\eta_{\mu\rho})f_3(q^2)+(q_\mu q_\nu\eta_{\rho\sigma}+q_\rho q_\sigma\eta_{\mu\nu})f_4(q^2)+q_\mu q_\nu q_\rho q_\sigma f_5(q^2) \ .$$

We will now impose the conservation equation. This gives relations among these functions. We get the following relations $f_1 + q^2 f_3 = 0$, $f_2 + q^2 f_4 = 0$, $2f_3 + f_4 + q^2 f_5 = 0$. We now plug them back and reparametrize the most general conserved two-point function as

$$\langle T_{\mu\nu}(q)T_{\rho\sigma}(-q)\rangle = -q^2(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - 2\eta_{\mu\nu}\eta_{\rho\sigma})f_3(q^2) + q^4\eta_{\mu\nu}\eta_{\rho\sigma}f_5(q^2)$$

$$+(q_\mu q_\rho\eta_{\nu\sigma}+q_\mu q_\sigma\eta_{\nu\rho}+q_\nu q_\rho\eta_{\mu\sigma}+q_\nu q_\sigma\eta_{\mu\rho}-2q_\mu q_\nu\eta_{\rho\sigma}-2q_\rho q_\sigma\eta_{\mu\nu})f_3(q^2)-q^2(q_\mu q_\nu\eta_{\rho\sigma}+q_\rho q_\sigma\eta_{\mu\nu})f_5(q^2)$$

$$+q_\mu q_\nu q_\rho q_\sigma f_5(q^2) \ .$$

Equivalently,

$$\langle T_{\mu\nu}(q)T_{\rho\sigma}(-q)\rangle = -q^2(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - 2\eta_{\mu\nu}\eta_{\rho\sigma})f_3(q^2)$$

$$+(q_\mu q_\rho\eta_{\nu\sigma} + q_\mu q_\sigma\eta_{\nu\rho} + q_\nu q_\rho\eta_{\mu\sigma} + q_\nu q_\sigma\eta_{\mu\rho} - 2q_\mu q_\nu\eta_{\rho\sigma} - 2q_\rho q_\sigma\eta_{\mu\nu})f_3(q^2)$$

$$+(q_\mu q_\nu - q^2\eta_{\mu\nu})(q_\rho q_\sigma - q^2\eta_{\rho\sigma})f_5(q^2) \ .$$

Let us see what would happen if we tried to impose the tracelessness of $T_{\mu\nu}$. If we imposed $\langle T^\mu_\mu(q) T_{\rho\sigma}(-q)\rangle = 0$ for all q we would get

$$f_5(q^2) = 0$$

(More precisely the equation is $q^2 f_5(q^2) = 0$. This will be important soon.) So if we insist on the tracelessness of T we must substitute $f_5 = 0$ but then we would remain only with the terms proportional to f_3. Hence we would get $\langle T_{11}(q) T_{11}(-q)\rangle = 0$, which violates unitarity.

Thus we cannot have the trace to be zero.

In a scale invariant theory we can use dimensional analysis to fix

$$f_5 = -c/q^2 \ .$$

Then

$$\langle T^\mu_\mu(q) T_{\rho\sigma}(-q)\rangle = c(q_\rho q_\sigma - q^2 \eta_{\rho\sigma})$$

This is a contact term. Hence, $T = 0$ holds as an operator equation, but it is violated at coincident points. Also note that

$$\langle T^\mu_\mu(q) T^\rho_\rho(-q)\rangle = -cq^2 \ .$$

This is a very important equation. It means that the two point function of the traces is a contact term. However, note that so far we only assumed scaling symmetry. Therefore scaling symmetry alone implies that at separated points $\langle T(x) T(0)\rangle = 0$. This means that $T = 0$ as an operator. Thus, in two dimensions scaling symmetry implies the full conformal group. We have thus re-derived Polchinski's theorem.

Now let us go back to $\langle T^\mu_\mu(q) T_{\rho\sigma}(-q)\rangle = c(q_\rho q_\sigma - q^2 \eta_{\rho\sigma})$. Couple the 2d theory to background metric. This is done to linear order via $\sim \int d^2x T^{\mu\nu} g_{\mu\nu}$. Hence in the presence of a background metric

$$\langle T^\mu_\mu(0)\rangle_g \sim \int d^2x \langle T^\mu_\mu(0) T^{\rho\sigma}(x)\rangle_0 g_{\rho\sigma}(x) \sim c\int d^2x \left(\partial^\rho\partial^\sigma \delta^2(x) - \eta_{\rho\sigma}\partial^2\delta^2(x)\right) g_{\rho\sigma}$$

$$\sim c(\partial^\rho\partial^\sigma - \eta^{\rho\sigma}\partial^2) g_{\rho\sigma} \sim cR \ .$$

The equation

$$T = -\frac{c}{24\pi} R$$

is the familiar trace anomaly in two dimensions, c is the central charge. It is also easy to show that in every conformal field theory $c > 0$ (in the appropriate convention). One again looks at $\langle T_{11}T_{11}\rangle$ and finds that this is proportional to c. Hence $c > 0$.

There is another useful interpretation of c. Consider a conformal field theory compactified on a two-sphere S^2 of radius a

$$ds^2 = \frac{4a^2}{(1+|x|^2)^2}\sum_{i=1}^{2}(dx_i)^2 \ , \qquad |x|^2 = \sum_{i=1}^{2}(x_i)^2 \ .$$

The Ricci scalar is $R = 2/a^2$. Because of the quantum anomaly, the partition function

$$Z_{S^2} = \int [d\Phi]e^{-\int_{S^2}\mathcal{L}(\Phi)}$$

depends on a. We find that

$$\frac{d}{d\log a}\log Z_{S^2} = -\int_{S^2}\sqrt{g} < T >= \frac{c}{24\pi}\int_{S^2}\sqrt{g}R = \frac{c}{24\pi}\frac{2}{a^2}Vol(S^2) = \frac{c}{3} \ .$$

Thus the logarithmic derivative of the partition function yields the c anomaly. The appearance of c in the path integral over the theory compactified on a two-sphere turns out to be natural in some respects that we will understand later.

We will now consider non-scale invariant theories.

1.2. Massive Theories

Conformal fixed points can be perturbed by relevant operators (or marginally relevant). This triggers a flow CFT_{UV} to CFT_{IR}. We will now study the correlation functions of the stress tensor in such a case. Hence, we no longer impose scale invariance, just diff invariance. To avoid having to discuss contact terms (which were very important above!) we switch to position space now and discuss only two-point functions of the stress tensor. It is further convenient to work with the complex coordinate $z = x^1 + ix^2$.

Then the conservation equations are $\partial_{\overline{z}}T_{zz} = -\partial_z T$, $\partial_z T_{\overline{zz}} = -\partial_{\overline{z}}T$. We can parameterize the most general two point functions consistent with Lorentz

$$\langle T_{zz}(z)T_{zz}(0)\rangle = \frac{F(z\overline{z}, M)}{z^4}$$

$$\langle T(z)T_{zz}(0)\rangle = \frac{G(z\overline{z}, M)}{z^3\overline{z}}$$

$$\langle T(z)T(0)\rangle = \frac{H(z\overline{z}, M)}{z^2\overline{z}^2}$$

However, from our analysis of the implications of diff-invariance in momentum space we already know that there are only two real independent functions. So there must be relations between F,G,H. A little algebra shows that the relation are $\dot{F} = -\dot{G} + 3G$, $\dot{H} - 2H = -\dot{G} + G$, where $\dot{X} \equiv |z^2|\frac{dX}{d|z|^2}$.

Using these relations one finds that the combination $C \equiv F - 2G - 3H$ satisfies the following differential equation

$$\dot{C} = -6H \ , \tag{1.1}$$

however, since H is positive definite, C decreases monotonically as we increase the distance. Let us now identify C at very short and very long distances. At very short and very long distances it tends to the appropriate values in the conformal field theory. However, in the conformal field theory G, H are contact terms. Hence, G, H are subleading at separated points compared to $F \sim c$. (It is easy to check that F is insensitive to the function f_3 in the conformal field theory and only knows about f_5.)

Hence, we have established a monotonic decreasing function that starts from c_{UV} and flows to c_{IR}. This means that the space of CFTs is foliated. No cycles. Also implies that degrees of freedom are lost along every flow. c is a measure of degrees of freedom. We can integrate the equation (1.1) to obtain

$$c_{uv} - c_{ir} \sim \int d\log|z^2|H \sim \int d^2z|z^2|\langle T(z)T(0)\rangle > 0 \ . \tag{1.2}$$

Since c can also be understood as the path integral over the two-sphere, the inequality can also be interpreted as a statement about the partition function of the massive theory on S^2.

2. Three-Dimensional Theories

The problem of identifying a quantity that would generalize Zamolodchikov's inequality $c_{uv} > c_{ir}$ has been open for several decades. It is still open, but since 2010 there is at least a plausible conjecture. We will review the conjecture and some of the evidence for it. The conjecture arose independently via studies in AdS/CFT and via studies in 3d field theories.

2.1. The Conjecture and Checks for Perturbative Flows

Any conformal field theory on R^3 can be canonically mapped to a theory on the curved space S^3. This is because S^3 is stereographically equivalent to flat space and (thus the metric on S^3 is conformal to R^3). In three dimensions there are no trace anomalies, and hence the partition function over S^3 has no logarithms of the radius.

Indeed, consider

$$Z_{S^3} = \int [d\Phi] e^{-\int_{S^3} \mathcal{L}(\Phi)} \ .$$

This is generally divergent and takes the form (for a three-sphere of radius a)

$$\log Z_{S^3} = c_1(\Lambda a)^3 + c_2(\Lambda a) + F \ . \tag{2.1}$$

Terms with inverse powers of Λ are dropped since they are not part of the continuum theory. Since this is a conform field theory Λ is the only scale (of course a fictitious scale!), and so nothing can contaminate the constant F. (Re-scalings of Λ only allow to dial $c_{1,2}$.) Hence, F is part of the continuum theory, independent of the radius of the sphere.

Imagine a three-dimensional flow from some CFT_{uv} to some CFT_{ir}. Then we can compute F_{uv} and F_{ir} via the procedure above. In fact, the F's can be complex numbers. The conjecture is

$$|F_{uv}| > |F_{ir}|$$

Let us consider the computation of F is simple examples. Take a free massless scalar $\mathcal{L} = \frac{1}{2}(\partial\Phi)^2$. To put it in a curved background while preserving conformal invariance we write (in d dimensions)

$$S = \frac{1}{2}\int d^3x\sqrt{g}\left((\nabla\Phi)^2 + \frac{d-2}{4(d-1)}R[g]\phi^2\right) \ .$$

This coupling to the Ricci scalar is necessary to preserve Weyl invariance (which is just the generalization of the conformal group to curved space)

$$g \to e^{2\sigma} g \ , \qquad \phi \to e^{-\frac{d-2}{2}\sigma}\phi \ .$$

The partition function in three dimensions is thus

$$-\log Z_{S^3} = \frac{1}{2}\log\det\left(-\nabla^2 + \frac{1}{8}R\right) \ ,$$

and the Ricci scalar is related to the radius in three dimensions via $R(=\frac{d(d-1)}{a^2})=\frac{6}{a^2}$. The spherical harmonics are all known. The eigenvalues are

$$\lambda_n=\frac{1}{a^2}\left(n+\frac{3}{2}\right)\left(n+\frac{1}{2}\right)$$

and their respective multiplicities

$$m_n=(n+1)^2$$

The free energy on the three sphere due to a single conformally coupled scalar is therefore

$$-\log Z_{S^3}=\frac{1}{2}\sum_{n=0}^{\infty}m_n\left(-2\log(\mu_0 a)+\log\left(n+\frac{3}{2}\right)+\log\left(n-\frac{1}{2}\right)\right)$$

This sum clearly diverges and needs to be regulated. One choses judiciously the zeta function. One finds that with this regulator $\sum_{n=0}m_n=\zeta(-2)=0$ and therefore a logarithmic dependence on the radius is absent, consistently with the absence of a conformal anomaly. We therefore remain with the sum

The remaining sum can be evaluated in terms of derivatives of the Hurwitz zeta function

$$\zeta(s;q)=\sum_{k=0}^{\infty}(k+q)^{-s}\ .$$

For $q=1$ it coincides with the definition of the ordinary zeta function. We find that formally

$$F_{scalar}=-\frac{1}{2}\frac{d}{ds}\left[2\zeta(s-2,\frac{1}{2})+\frac{1}{2}\zeta(s,\frac{1}{2}))\right]=\frac{1}{16}\left(2\log 2-\frac{3\zeta(3)}{\pi^2}\right)\approx 0.0638$$

One can perform a similar computation for a free massless Dirac fermion field

$$F_{fermion}=\frac{\log 2}{4}+\frac{3\zeta(3)}{8\pi^2}\approx 0.219$$

The absolute value of the partition function of a massless Majorana fermion is just a half of the result above.

Chern-Simons theory associated to some gauge group G is

$$S=\frac{k}{4\pi}\int_M Tr\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)\ ,$$

where k is called the level. This theory has no second derivatives hence it has no propagating degrees of freedom. The field equation is

$$0 = F = dA + A \wedge A$$

This means that the "curvature of the gauge field" vanishes everywhere. These are called flat connections. The space of flat connection on the manifold M is fixed completely by topological properties of the manifold.

Surprisingly (!), even though there are no degrees of freedom associated to Chern-Simons theory, it also enters into the F-theorem. The partition function of CS theory on the three sphere has been computed by Witten in his famous paper about Jones Polynomials. He found that for $U(1)$ CS theory the answer is $\frac{1}{2}\log k$ while for $U(N)$ it is

$$F_{CS}(k,N) = \frac{N}{2}\log(k+N) - \sum_{j=1}^{N-1}(N-j)\log\left(2\sin\frac{\pi j}{k+N}\right)$$

So not only CS theory contributes to F, it actually contributes to it with a very large coefficient when the level is large!

One can check several simple flows. One can start from the conformal field theory described by $U(1)_k$ CS theory coupled to N_f Dirac fermions of charge 1. This is a conformal field theory because the Lagrangian has no coupling constant that can run. The CS coefficient is discrete because it is topological in nature. This conformal field theory is weakly coupled when $k >> 1$. Hence, the F coefficient is

$$F_{UV} \approx \frac{1}{2}\log k + N_f\left(\frac{\log 2}{4} + \frac{3\zeta(3)}{8\pi^2}\right)$$

Let us now deform this by a mass term. The fermions disappear, but there is a pure CS term in the infrared with a shifted level $k \pm N_f/2$, where the sign depends on the sign of the mass term. Hence,

$$F_{IR} \approx \frac{1}{2}\log\left(k \pm N_f/2\right) \ ,$$

and it follows that in the regime where our analysis

$$F_{UV} > F_{IR}$$

holds.

There are many more complicated examples that have been checked, all consistent with the conjecture. There is no proof of this conjecture. For the special class of $\mathcal{N}=2$ supersymmetric theories in three dimensions, one can develop some confidence in this conjecture. This is discussed towards the end of these notes.

3. Four-Dimensional Theories

We saw that in two dimensions the natural monotonic property of the RG evolution was tightly related to the trace anomaly in two dimensions. In three dimensions the conjecture is directly about some sphere partition function (there are no trace anomalies in three dimensions).

In four dimensions there are two trace anomalies and the monotonic property of flows concerns again with these anomalies. The anomalous correlation function is now

$$\langle T_{\mu\nu}(q) T_{\rho\sigma}(p) T_{\gamma\delta}(-q-p)\rangle$$

And again, like in our analysis in two dimensions, there are contact-terms which are necessarily inconsistent with $T^\mu_\mu = 0$. In four dimensions it turns out that there are two independent trace anomalies. Introducing a background metric field we have

$$T^\mu_\mu = aE_4 - cW^2 \ ,$$

where $E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$ is the Euler density and $W^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2$ is the Weyl tensor squared. These are called the a- and c-anomalies.

Cardy has conjectured in the 80's that if the conformal field theory of the ultraviolet CFT_{uv} flows to some CFT_{ir} then

$$a_{uv} > a_{ir} \ .$$

The c-anomaly does not satisfy such an inequality and also the more intuitive concept of dofs, the free energy, does not satisfy any general inequality.

SInce the four sphere is conformally flat the partition function on S^4 selects only the a-anomaly. Indeed,

$$\partial_{\log a} \log Z_{S^4} = -\int_{S^4} \sqrt{g}\langle T^\mu_\mu\rangle = -a\int_{S^4}\sqrt{g}E_4 = -64\pi^2 a$$

So we see that the conjecture in n dimensions is about the partition function of the n-sphere.

Real scalars contribute to the anomalies $(a,c) = \frac{1}{90(8\pi)^2}(1,3)$, Weyl fermion: $(a,c) = \frac{1}{90(8\pi)^2}(11/2,9)$, gauge field: $(a,c) = \frac{1}{90(8\pi)^2}(62,36)$.

We will now present a new proof of the two-dimensional case and then address four-dimensional theories. The techniques below are not directly as powerful in odd-dimensional flows since they rely strongly on the existence of anomalies.

The main idea is to promote various coupling constants to background fields.

Imagine any renormalizable QFT (in any number of dimensions) and set all the mass parameters to zero. The extended symmetry includes the full conformal group. If the number of space-time dimensions is even then the conformal group has trace anomalies. If the number of space-time dimensions is of the form $4k+2$, there may be gravitational anomalies. We will completely ignore gravitational anomalies here.

Upon introducing the mass terms, one violates conformal symmetry *explicitly*. Thus, in general, the conformal symmetry is violated both by trace anomalies and by an operatorial violation of the equation $T^\mu_\mu = 0$ in flat space-time. The latter violation can always be removed by letting the coupling constants transform. Indeed, replace every mass scale M (either in the Lagrangian or associated to some cutoff) by $Me^{-\tau(x)}$, where $\tau(x)$ is some background field (i.e. a function of space-time). Then the conformal symmetry of the Lagrangian is restored if we accompany the ordinary conformal transformation of the fields by a transformation of τ. To linear order, $\tau(x)$ always appears in the Lagrangian as $\sim \int d^dx\ \tau T^\mu_\mu$. Setting $\tau = 0$ one is back to the original theory, but we can also let τ be some general function of space-time. The variation of the path integral under such a conformal transformation that also acts on $\tau(x)$ is thus fixed by the anomaly of the conformal theory in the ultraviolet. This procedure allows us to study some questions about general RG flows using the constraints of conformal symmetry.

Consider integrating out all the high energy modes and flow to the deep infrared. Since we do not integrate out the massless particles, the dependence on τ is regular and local. As we have explained, the dependence on τ is tightly constrained by the conformal symmetry. Since in even dimensions the conformal group has trace anomalies, these must be reproduced by the low energy theory. The conformal field theory at long distances, CFT$_{IR}$, contributes to the trace anomalies, but to match to the defining theory, the dilaton functional has to compensate precisely for the difference between the anomalies of the conformal field theory at short distances, CFT$_{uv}$, and the conformal field theory at long distances, CFT$_{ir}$.

3.1. Warm Up

Let us see how these ideas are borne out in two-dimensional renormalization group flows. Let us study the constraints imposed by conformal symmetry on the action of τ (which is a background field). An easy way to analyze these constraints is to introduce a fiducial metric $g_{\mu\nu}$ into the system. Weyl transformations act on the dilaton and metric according to $\tau \to \tau + \sigma$, $g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu}$. If the Lagrangian for the dilaton and metric is Weyl invariant, upon setting the metric to be flat, one finds a conformal invariant theory for the dilaton. Hence, the task is to classify local diff$\times$ Weyl invariant Lagrangians for the dilaton and metric background fields.

It is convenient to define $\widehat{g}_{\mu\nu} = e^{-2\tau} g_{\mu\nu}$, which is Weyl invariant. At the level of two derivatives, there is only one diff$\times$Weyl invariant term: $\int \sqrt{\widehat{g}}\widehat{R}$. However, this is a topological term, and so it is insensitive to local changes of $\tau(x)$. Therefore, if one starts from a diff$\times$Weyl invariant theory, upon setting $g_{\mu\nu} = \eta_{\mu\nu}$, the term $\int d^2x (\partial\tau)^2$ is absent.

The key is to recall that unitary two-dimensional theories have a trace anomaly

$$T^\mu_\mu = -\frac{c}{24\pi} R \ . \tag{3.1}$$

(In this convention a free scalar field has $c = 1$.) One must therefore allow the Lagrangian to break Weyl invariance, such that the Weyl variation of the action is consistent with (3.1). The action functional which reproduces the two-dimensional trace anomaly is

$$S_{WZ}[\tau, g_{\mu\nu}] = \frac{c}{24\pi} \int \sqrt{g} \left(\tau R + (\partial\tau)^2 \right) \ . \tag{3.2}$$

We see that even though the anomaly itself disappears in flat space (3.1), there is a two-derivative term for τ that survives even after the metric is taken to be flat. This is of course the familiar Wess-Zumino term for the two-dimensional conformal group.

Consider now some general two-dimensional RG flow from a CFT in the UV (with central charge c_{uv} and a CFT in the IR (with central change c_{ir}). Replace every mass scale according to $M \to M e^{-\tau(x)}$. We also couple the theory to some background metric. Performing a simultaneous Weyl transformation of the dynamical fields and the background field $\tau(x)$, the theory is non-invariant only because of the anomaly $\delta_\sigma S = \frac{c_{uv}}{24} \int d^2x \sqrt{g} \sigma R$. Since this is a property of the full quantum theory, it must be reproduced at all scales. An immediate consequence of this idea is that also in the deep infrared the effective action should reproduce the transformation $\delta_\sigma S = \frac{c_{uv}}{24} \int d^2x \sqrt{g} \sigma R$. At long distances, one obtains a contribution c_{ir} to the anomaly from CFT$_{ir}$, hence, the rest of the anomaly must come from an explicit Wess-Zumino functional (3.2) with coefficient $c_{uv} - c_{ir}$. In particular, setting the background metric to be flat, we conclude that the low energy theory must contain a term

$$\frac{c_{uv} - c_{ir}}{24\pi} \int d^2x (\partial\tau)^2 \ . \tag{3.3}$$

Note that the coefficient of this term is universally proportional to the difference between the anomalies and it does not depend on the details of the flow. Higher-derivative terms for the dilaton can be generated from local diff$\times$Weyl invariant terms, and there is no a priori reason for them to be universal (that is, they may depend on the details of the flow, and not just on the conformal field theories at short and long distances).

Zamolodchikov's theorem follows directly from (3.3). Indeed, we consider the partition function of the (*Euclidean*) theory in the presence of two insertions of the background $\tau(x)$, as in figure 1.

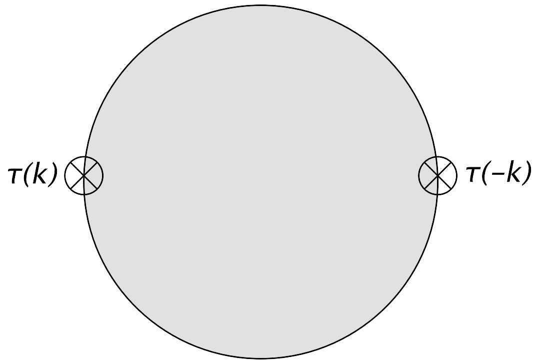

Fig.1: The partition function of the Euclidean theory with two insertions of the background field with momentum k.

From this general object we can extract $c_{uv} - c_{ir}$ by expanding around $k = 0$, reading out the term quadratic in momentum, and matching to (3.3). Reflection positivity thus immediately leads to

$$c_{uv} > c_{ir} \ . \tag{3.4}$$

We can be more explicit about what precisely goes into the calculation of figure 1. The coupling of τ to matter must take the form $\tau T^\mu_\mu + \cdots$, where the corrections have more τs. To extract the two-point function of τ with two derivatives we must use the insertion τT^μ_μ twice. (Terms containing τ^2 can be lowered once, but they do not contribute to the two-derivative term in the effective action of τ.) As a consequence, we find that

$$\begin{aligned} \left\langle e^{\int \tau T^\mu_\mu d^2x} \right\rangle &= \cdots + \frac{1}{2} \int\int \tau(x)\tau(y) \langle T^\mu_\mu(x) T^\mu_\mu(y)\rangle d^2x d^2y + \cdots \\ &= \cdots + \frac{1}{4}\int \tau(x)\partial_\rho\partial_\sigma \tau(x) \left(\int (y-x)^\rho (y-x)^\sigma \langle T^\mu_\mu(x) T^\mu_\mu(y)\rangle d^2y\right) d^2x + \cdots \ . \end{aligned} \tag{3.5}$$

In the final line of the equation above, we have concentrated entirely on the two-derivative term. It follows from translation invariance that the y integral is x-independent

$$\int (y-x)^\rho (y-x)^\sigma \langle T^\mu_\mu(x) T^\mu_\mu(y)\rangle d^2y = \frac{1}{2}\eta^{\rho\sigma}\int y^2 \langle T^\mu_\mu(0) T^\mu_\mu(y)\rangle d^2y \ . \tag{3.6}$$

To summarize, one finds the following contribution to the dilaton effective action at two derivatives

$$\frac{1}{8}\int d^2x \tau\Box\tau \int d^2y y^2 \langle T(y)T(0)\rangle \ . \tag{3.7}$$

According to (3.3), the expected coefficient of $\tau\Box\tau$ is $(c_{uv}-c_{ir})/24\pi$, and so by comparing we obtain

$$\Delta c = 3\pi \int d^2y y^2 \langle T(y)T(0)\rangle \ . \tag{3.8}$$

As we have already mentioned, $\Delta c > 0$ follows from reflection positivity (which is a property of unitary theories). Equation (3.8) agrees with the classic results about two-dimensional flows.

3.2. Back to Four Dimensions

One starts by classifying local diff$\times$Weyl invariant functionals of τ and a background metric $g_{\mu\nu}$. Again, we demand invariance under

$$g_{\mu\nu} \longrightarrow e^{2\sigma} g_{\mu\nu} \ , \qquad \tau \longrightarrow \tau + \sigma \ . \tag{3.9}$$

We will often denote $\widehat{g} = e^{-2\tau} g_{\mu\nu}$. The combination $\widehat{g}$ transforms as a metric under diffeomorphisms and is Weyl invariant.

The most general theory up to two derivatives is:

$$f^2 \int d^4x \sqrt{-\det \widehat{g}} \left(\Lambda + \frac{1}{6}\widehat{R}\right) \ , \tag{3.10}$$

where we have defined $\widehat{R} = \widehat{g}^{\mu\nu} R_{\mu\nu}[\widehat{g}]$.

Since we are ultimately interested in the Minkowskian theory, let us evaluate the kinetic term with $g_{\mu\nu} = \eta_{\mu\nu}$. Using integration by parts we get

$$S = f^2 \int d^4x e^{-2\tau} (\partial\tau)^2 \ . \tag{3.11}$$

One can use the field redefinition

$$\Psi = 1 - e^{-\tau} \tag{3.12}$$

to rewrite this as

$$S = f^2 \int d^4x \Psi \Box \Psi \ . \tag{3.13}$$

One can also study terms in the effective action with more derivatives. With four derivatives, one has three independent (dimensionless) coefficients

$$\int d^4x \sqrt{-\widehat{g}} \left(\kappa_1 \widehat{R}^2 + \kappa_2 \widehat{R}^2_{\mu\nu} + \kappa_3 \widehat{R}^2_{\mu\nu\rho\sigma} \right) \ . \tag{3.14}$$

It is implicit that indices are raised and lowered with $\widehat{g}$. Recall the expressions for the Euler density $\sqrt{-g}E_4$ and the Weyl tensor squared

$$E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2 \ , \qquad W^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2 \ . \tag{3.15}$$

We can thus choose instead of (3.14) a different parameterization

$$\int d^4x \sqrt{-\widehat{g}} \left(\kappa'_1 \widehat{R}^2 + \kappa'_2 \widehat{E}_4 + \kappa'_3 \widehat{W}^2_{\mu\nu\rho\sigma} \right) \ . \tag{3.16}$$

We immediately see that the κ'_2 term is a total derivative. If we set $g_{\mu\nu} = \eta_{\mu\nu}$, then $\widehat{g}_{\mu\nu} = e^{-2\tau}\eta_{\mu\nu}$ is conformal to the flat metric and hence also the κ'_3 term does not play any role as far as the dilaton interactions in flat space are concerned. Consequently, terms in the flat space limit arise solely from $\widehat{R}^2$. A straightforward calculation yields

$$\int d^4x \sqrt{-\widehat{g}} \widehat{R}^2 \Big|_{g_{\mu\nu}=\eta_{\mu\nu}} = 36 \int d^4x \left(\Box \, \tau - (\partial\tau)^2 \right)^2 \sim \int d^4x \frac{1}{(1-\Psi)^2} \left(\Box \Psi \right)^2 . \tag{3.17}$$

So far we have only discussed diff×Weyl invariant terms in four-dimensions, but from the two-dimensional examples we anticipate the importance of the anomalous functional.

The most general anomalous variation one needs to consider takes the form

$$\delta_\sigma S_{anomaly} = \int d^4x \sqrt{-g} \sigma \left(c W^2_{\mu\nu\rho\sigma} - a E_4 \right) \ . \tag{3.18}$$

The question is then how to write a functional $S_{anomaly}$ that reproduces this anomaly. (Note that $S_{anomaly}$ is only defined modulo diff×Weyl invariant terms.) Without the field τ one must resort to non-local expressions, but in the presence of the dilaton one has a local action.

It is a little tedious to solve (3.18), but the procedure is straightforward in principle. We first replace σ on the right-hand side of (3.18) with τ

$$S_{anomaly} = \int d^4x \sqrt{-g} \tau \left(c W^2_{\mu\nu\rho\sigma} - a E_4 \right) + \cdots \ . \tag{3.19}$$

While the variation of this includes the sought-after terms (3.18), as the $\cdots$ suggest, this cannot be the whole answer because the object in parenthesis is not Weyl invariant. Hence, we need to keep fixing this expression with more factors of τ until the procedure terminates. Note that $\sqrt{-g}W^2_{\mu\nu\rho\sigma}$, being the square of the Weyl tensor, is Weyl invariant, and hence we do not need to add any fixes proportional to the c-anomaly in (3.19). This makes the c-anomaly "Abelian" in some sense. The "non-Abelian" structure coming from the Weyl variation of E_4 is the key to our construction. The a-anomaly is therefore quite distinct algebraically from the c-anomaly.

The final expression for $S_{anomaly}$ is

$$
\begin{aligned}
S_{anomaly} = & -a\int d^4x\sqrt{-g}\Big(\tau E_4 + 4\big(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\big)\partial_\mu\tau\partial_\nu\tau - 4(\partial\tau)^2\Box\,\tau + 2(\partial\tau)^4\Big) \\
& + c\int d^4x\sqrt{-g}\tau W^2_{\mu\nu\rho\sigma}\ .
\end{aligned}
\tag{3.20}
$$

Note that even when the metric is flat, self-interactions of the dilaton survive. This is analogous to what happens in pion physics when the background gauge fields are set to zero and also to what we saw in two dimensions.

Setting the background metric to be flat we thus find that the non-anomalous terms in the dilaton generating functional are

$$
\int d^4x\left(\alpha_1 e^{-4\tau} + \alpha_2(\partial e^{-\tau})^2 + \alpha_3\left(\Box\tau - (\partial\tau)^2\right)^2\right)\ , \tag{3.21}
$$

where α_i are some real coefficients.

The a-anomaly has a Wess-Zumino term, leading to the additional contribution

$$
S_{WZ} = 2(a_{UV} - a_{IR})\int d^4x\left(2(\partial\tau)^2\Box\tau - (\partial\tau)^4\right)\ . \tag{3.22}
$$

The coefficient is universal because the total anomaly has to match.

We see that if one knew the four-derivative terms for the dilaton, by comparing (3.21) and (3.22), one could extract $a_{UV} - a_{IR}$. A more transparent way to discern the WZ term from the term proportional to α_3 in (3.21) is in terms of the variable $\Psi = 1 - e^{-\tau}$. Then (3.21) becomes

$$
\int d^4x\left(\alpha_1\Psi^4 + \alpha_2(\partial\Psi)^2 + \frac{\alpha_3}{(1-\Psi)^2}(\Box\Psi)^2\right)\ , \tag{3.23}
$$

while the WZ term (3.22) is

$$S_{WZ} = 2(a_{UV} - a_{IR}) \int d^4x \left(\frac{2(\partial\Psi)^2 \Box\Psi}{(1-\Psi)^3} + \frac{(\partial\Psi)^4}{(1-\Psi)^4} \right) . \tag{3.24}$$

We see that if we consider background fields Ψ which are null ($\Box\Psi = 0$) α_3 disappears and only the last term in (3.24) remains. Therefore, by computing the partition function of the QFT in the presence of four null insertions of Ψ one can extract directly $a_{UV} - a_{IR}$.

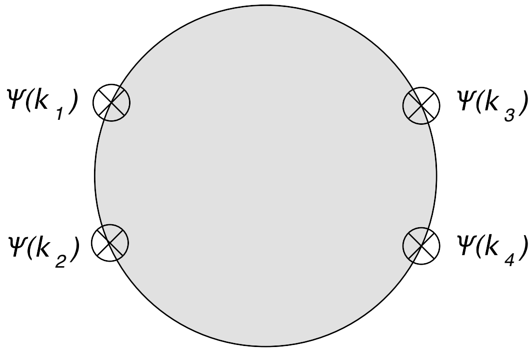

Fig.2: Four insertions of the background field Ψ with $\sum_i k_i = 0$ and $k_i^2 = 0$. The blob represents the quantum matter fields.

Indeed, consider all the diagrams with four insertions of a background Ψ with momenta k_i, such that $\sum_i k_i = 0$ and $k_i^2 = 0$ (see figure 2). Expanding this amplitude to fourth order in the momenta k_i, one finds that the momentum dependence takes the form $s^2 + t^2 + u^2$ with $s = 2k_1 \cdot k_2$, $t = 2k_1 \cdot k_3$, $u = 2k_1 \cdot k_4$. Our effective action analysis shows that the coefficient of $s^2 + t^2 + u^2$ is directly proportional to $a_{UV} - a_{IR}$.

In fact, one can even specialize to the so-called forward kinematics, choosing $k_1 = -k_3$ and $k_2 = -k_4$. Then the amplitude of figure 2 is only a function of $s = 2k_1 \cdot k_2$. $a_{UV} - a_{IR}$ can be extracted from the s^2 term in the expansion of the amplitude around $s = 0$. Continuing s to the complex plane, there is a branch cut for positive s (corresponding to physical states in the s-channel) and negative s (corresponding to physical states in the u-channel). There is a crossing symmetry $s \leftrightarrow -s$ so these branch cuts are identical.

To calculate the imaginary part associated to the branch cut we utilize the optical theorem. See figure 3. The imaginary part is manifestly positive definite.

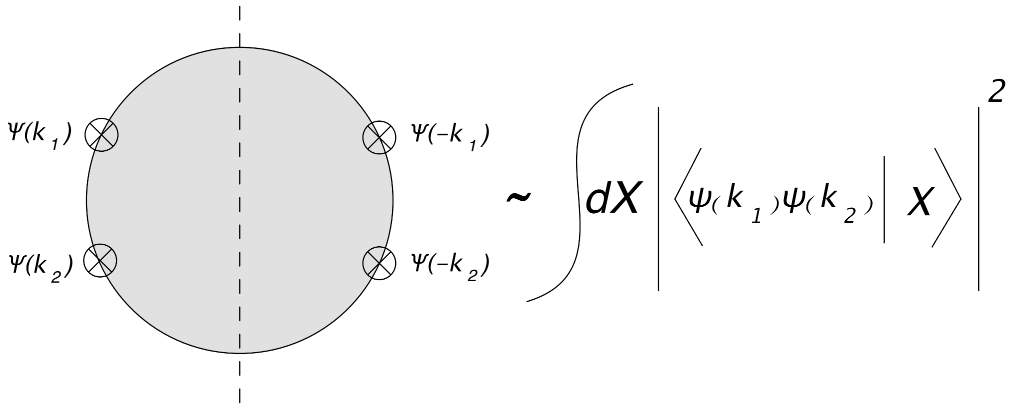

Fig.3: The imaginary part is given by calculating all the connected diagrams involving two insertions of the background field and any final state. One then squares the amplitude for the transition to this particular final state and sums over all possible final states.

Using Cauchy's theorem we can relate the low energy coefficient of s^2, $a_{UV} - a_{IR}$, to an integral over the branch cut. Fixing all the coefficients one finds

$$a_{UV} - a_{IR} = \frac{1}{4\pi} \int_{s>0} \frac{Im\mathcal{A}(s)}{s^3} \ . \tag{3.25}$$

As explained, the imaginary part $Im\mathcal{A}(s)$ can be evaluated by means of the optical theorem, figure 3, and hence it is manifestly positive. Since the integral converges by power counting (and thus no subtractions are needed), we conclude

$$a_{UV} > a_{IR} \ . \tag{3.26}$$

Note the difference between the ways positivity is established in two and four dimensions. In two dimensions, one invokes reflection positivity of a two-point function (reflection positivity is best understood in *Euclidean* space). In four dimensions, the Wess-Zumino term involves four dilatons, so the natural positivity constraint comes from the forward kinematics (and hence, it is inherently *Minkowskian*).

Let us say a few words about the physical relevance of (3.26). Such an inequality constrains severely the dynamics of quantum field theory, and in favorable cases can be used to establish that some symmetries must be broken or that some symmetries must be unbroken. In a similar fashion, if a system naively admits several possible dynamical scenarios one can use (3.26) as an additional handle.

4. Looking for a Single Overarching Principle

The conjecture in any d-dimensional field theory seems to be that the partition function on S^d in the deep UV vs the deep IR satisfies

$$-\log Z^{uv}_{S^d} > -\log Z^{ir}_{S^d} \ ,$$

where these partition functions are evaluated in the respective conformal field theories.

This proposal passes some elementary tests. For instance, for this conjecture to be able to even take off one needs to show that $Z^{CFT}_{S^d}$ is independent of exactly marginal couplings. Otherwise the conjecture is manifestly false.

Indeed, let λ be an exactly marginal coupling, coupling to an exactly marginal operator $\mathcal{O}$. Then we have a family (actually a manifold) of conformal field theories CFT_λ. We have

$$\partial_\lambda Z^{CFT_\lambda}_{S^d} = \int [d\Phi]\left(\int_{S^d} d^dx\sqrt{g}\mathcal{O}(x)\right) e^{-\int_{S^d} d^dx\sqrt{g}(\mathcal{L}_0[\Phi]+\lambda\mathcal{O})} \ .$$

This is an integrated one point function on the sphere. A conformal transformation maps this to a one point function of this exactly marginal operator in flat space. However, in a CFT the only operator with nonzero expectation value is the unit operator, with which a marginal operator cannot mix by dimensional analysis. Thus,

$$\partial_\lambda Z^{CFT_\lambda}_{S^d} = 0 \ .$$

This fundamental invariance property is NOT shared by the free energy of QFT. Indeed, the free energy depends on exactly marginal operators (see $\mathcal{N}=4$ at strong vs weak 't Hooft coupling, where one finds the famous 3/4). Therefore the free energy is not a good candidate in general.

Another issue is that the partition function over the sphere lacks obvious intuitive meaning. We will now review a curious duality between the partition function and a certain entanglement entropy. The precise statement is that for a CFT in d dimensions the entanglement entropy (EE) across an S^{d-2} contains the same data about the continuum theory as the partition function over S^d. In particular, if d is even, the EE contains $a\log(R\Lambda_{UV})$, where a is the a-anomaly, and in odd dimensions it contains a finite piece F.

Due to lack of time I am not going prove this curious duality, instead, I will define the EE and discuss some basic properties of it. Hopefully, by the end of this the statement of the duality will be clear.

A quantum system can be either in a pure state with a wave function Ψ or we may be ignorant about its wave function and it may be, with probability p_j, in a state Ψ_j (this is called a mixed state). Then the appropriate description is with the density matrix

$$\rho = \sum_j p_j |\Psi_j\rangle\langle\Psi_j| \ .$$

The entropy associated with a mixed state is

$$S = -\sum p_j \log(p_j) = -Tr(\rho \log \rho)$$

The entropy of a mixed state is therefore strictly positive while that of a pure state vanishes.

Sometimes we don't really have a choice and we must only describe the system with a density matrix. Consider a product Hilbert space $H = H_A \otimes H_B$ and assume that the wave function in this large Hilbert space is fixed. We may be ignorant about the state of the B degrees of freedom and consequently (entanglement) we don't know the state of A either. In general

$$\rho_A = Tr_B(\rho_{AB})$$

For instance, if the full system is in a pure state Ψ then $\rho_A = Tr_B(|\Psi\rangle\langle\Psi|)$. The reduced density matrix therefore has entropy S_A. This is the famous entanglement entropy. It is easy to prove that if the full system is in a pure state then $S_A = S_B$.

Entanglement entropy has proven to be very useful as topological order parameter in various applications to condensed matter systems (in fact there is a lot of interest in this quantity nowadays). We would like to apply this for the quantum vacuum $|VAC\rangle$ (the quantum vacuum is a pure state). At $t = 0$ we divide our space R^{d-1} into two complementary regions $A, B = A^c$. The Hilbert space should look like $H_A \otimes H_B$. We define

$$\rho_A = Tr_B(|VAC\rangle\langle VAC|)$$

and cross our fingers that this makes sense. In fact because of the ultraviolet divergences in the theory, across the boundary ∂A there are divergences sensitive to the short distance cutoff. In general the answer would look like

$$S_A = \Lambda_{UV}^{d-2} Vol(\partial A) + \Lambda_{UV}^{d-4} L^{d-4} + ...$$

Note the emergent area law – people have tried to connect it to BH entropy. Here L is a typical scale of the region A. In even dimensions there is a logarithmic term $\kappa(A)\log(\Lambda_{UV} L)$

while in odd dimensions there is just a finite term $\mu(A)$. $\kappa(A)$ and $\mu(A)$ can be interpreted as parts of the continuum theory IF the the underlying QFT is conformal. This is because no rescaling of the cutoff can mix with the coefficient of the logarithm in even dimensions, and similarly in odd dimensions. (In massive theories the data really associated to the continuum theory is more subtle.) Note the superficial similarity to the partition function.

The claim is that (for conformal field theories) if $A = S^{d-2}$ then in even dimensions $\kappa(S^{d-2}) \sim a$ and in odd dimensions $\mu(S^{d-2}) \sim F$. This can be proven by a sequence of conformal transformations.

Then one could hope to prove the monotonicity of the RG evolution via properties of the EE. Indeed, there is an alternative proof of the Zamolodchikov theorem in two dimensions using one of the simple inequalities the entanglement entropy satisfies (in information theory this inequality is called strong subadditive inequality) for any two regions A, B

$$S(A) + S(B) \geq S(A \cup B) + S(A \cap B) \ .$$

Perhaps one can use such general properties of the EE to establish the theorem in any d.

5. Localization

Localization is the phenomenon that a complicated integral (even a path integral) can be evaluated by considering only a very small subset of the integration domain. We will now develop some intuition by studying a simple ordinary integral where this takes place. The setup is a matrix model with fermions

5.1. Toy Model

$$Z = \int dx d\psi_1 d\psi_2 e^{-S(x,\psi_{1,2})} \tag{5.1}$$

where the action is just the most general conceivable one

$$S(x, \psi_{1,2}) = S_0(x) - \psi_1\psi_2 S_1(x)$$

with the usual rule for integration of fermionic variables

$$\int d\psi_1 d\psi_2 \psi_1 \psi_2 = 1$$

We can integrate over the fermions and find

$$Z = \int dx S_1(x) e^{-S_0} \ . \tag{5.2}$$

This would generally obtain contributions from all x's.

Consider the special case

$$S(x, \psi_{1,2}) = \frac{1}{2}(\partial h)^2 - \psi_1 \psi_2 \partial^2 h \ , \tag{5.3}$$

with $h(x)$ any function. Now the integral (5.2) becomes

$$Z = \int dx \partial^2 h e^{-\frac{1}{2}(\partial h)^2}$$

To guarantee convergence we assume $|\partial h| \to \infty$ as $x \to \pm\infty$.

This integral can actually be done. It is almost a total derivative. **Exercise:** show that

$$Z = \sqrt{2\pi} \sum_{\{x_i \big| \partial h(x_i) = 0\}} sign[\partial^2 h(x_i)] \ . \tag{5.4}$$

We say in this case that the integral localizes on the critical points.

We will now show how to solve for the partition function of (5.3) in a more general way, namely a way that generalizes vastly beyond the simple example above (for which one could evaluate the "path integral" directly). We slightly rewrite (5.3) as

$$S = \frac{1}{2}H^2 + iH\partial h - \psi_1 \psi_2 \partial^2 h$$

This does not change the partition function in an important way since the H integral is Gaussian. Now we can find an off-shell SUSY

$$\delta x = \epsilon(\psi_1 + \psi_2) \ , \qquad \delta \psi_{1,2} = \pm \epsilon i H \ , \qquad \delta H = 0 \ . \tag{5.5}$$

One easily finds $\delta^2 = 0$. It turns out that the action is Q-exact. An easy calculation reveals

$$S = \frac{1}{4}\{Q, (\psi_1 - \psi_2)(2\partial h + iH\} \equiv \{Q, V\}$$

Hence, the path integral is

$$Z = \int dx e^{-\{Q,V\}} \tag{5.6}$$

Now add an arbitrary real (positive) parameter t and consider

$$Z_t = \int dx d\psi_1 d\psi_2 e^{-t\{Q,V\}} \tag{5.7}$$

This is clearly independent of t. Indeed a derivative with respect to t gives Q of something and this vanishes as usual. Hence we can take $t \to \infty$. At $t \to \infty$ the only contributions arise from

$$\{Q, V\} = 0 \tag{5.8}$$

which is given by (integrating out H)

$$\partial h = \psi_{1,2} = 0$$

Hence, we see manifestly that the integral localizes to the critical points.

It only remains to expand $S = \frac{1}{2}(\partial h)^2 - \psi_1\psi_2\partial^2 h$ around these critical points to second order. We find $S = \frac{1}{2}(\partial^2 h(x_i))^2(x - x_i)^2 - \psi_1\psi_2\partial^2 h(x_i)$ The Gaussian integral around the zeros of $\{Q, V\}$ therefore looks like

$$\begin{aligned}&\int dx d\psi_1 d\psi_2 e^{-t\left(\frac{1}{2}(\partial^2 h(x_i))^2(x-x_i)^2 - \psi_1\psi_2\partial^2 h(x_i)\right)} = t\partial^2 h(x_i)\int dx e^{-t\frac{1}{2}(\partial^2 h(x_i))^2(x-x_i)^2}\\ &= t\partial^2 h(x_i)\int dx e^{-t\frac{1}{2}(\partial^2 h(x_i))^2(x-x_i)^2} \sim \sqrt{t}\frac{\partial^2 h(x_i)}{|\partial^2 h(x_i)|}\end{aligned} \tag{5.9}$$

The $\sqrt{t}$ cancels from the H integral and in total we thus get (5.4).

5.2. General Story

One constructs an action S where there is a nilpotent symmetry $\delta^2 = 0$. Then, one can always modify

$$S' = S + t\{Q, V\}\ . \tag{5.10}$$

The partition function does not depend on t. Furthermore, correlation functions of δ closed operators, $\delta\mathcal{O} = 0$, are independent of t.

If $\{Q, v\}$ is positive definite we can take $t \to \infty$ and then the path integral reduces to integrating over the locus $\{Q, V\} = 0$.

In general the Lagrangian may or may not be itself Q-exact. Namely, there or may not exist a Ψ such that $\mathcal{L} = \{Q, \Psi\}$. Is such a Ψ exist then also the the EM tensor would be Q-exact and the theory in the Q-closed sector would be actually topological. In particular the partition function does not depend on the metric

$$\frac{\delta}{\delta h_{ab}} Z = \langle T_{ab}\rangle = \langle\{Q, \frac{\delta}{\delta h_{ab}}\Psi\}\rangle = 0 \tag{5.11}$$

However, there are many non-topological theories which can be localized. In the next section we will develop some of the necessary formalism.

5.3. Localization in QFT

It turns out that for certain supersymmetric QFTs we can calculate the partition function on various curved manifolds. The path integral reduces to a matrix integral. There are many examples of this phenomenon. In four dimensions we find $\mathcal{N}=2$ theories on S^4, $\mathcal{N}=1$ theories on $S^3\times S^1$. In three dimensions $\mathcal{N}=2$ theories can be localized on S^3 and various squashings of this S^3. There could be many more examples, nobody has yet to classify the various possibilities.

The case of 3d is of special interest in the context of these lectures because the partition function on S^3 is exactly what appears in the context of the F-theorem. Hence, one could learn things about RG flows by studying theories that can be localized!

Let us begin by explaining how supersymmetric theories can be put on curved spaces (often while preserving supersymmetries).

Let us start from $\mathcal{N}=2$ theories in flat $2+1$ dimensional space. It turns out that the interesting class of theories consists of those that have an R-symmetry.

$\mathcal{N}=2$ is defined by the algebra

$$\{Q_\alpha, Q_\beta\} = \{\overline{Q}_\alpha, \overline{Q}_\beta\} = 0 \ , \qquad \{Q_\alpha, \overline{Q}_\beta\} = 2\sigma^\mu_{\alpha\beta}P_\mu + 2i\epsilon_{\alpha\beta}Z \ .$$

The σ^μ can be chosen real and symmetric. Z is a real central term (which in the dimensional reduction corresponds to the momentum P_3 in the reduced direction). The central term will not play a direct role in our discussions. As in four dimensions, the automorphism of this algebra is $U(1)_R$, rotating the supercharges. We will assume that this automorphism is a symmetry of the Lagrangian.

$\mathcal{N}=2$ in three dimensions is very similar to $\mathcal{N}=1$ in four dimensions, they have the same amount of supersmmetry and the same superspace. So writing Lagrangians for $\mathcal{N}=2$ in three dimensions in flat space is quite straightforward.

The theories under discussion don't have to be conformal. Normally coupling to curve space means deforming the theory by $\int d^3x T_{\mu\nu}h^{\mu\nu}+\cdots$. To do this for supersymmetric theories we need to embed the EM tensor into a supersymmetric multiplet. For theories with an R-symmetry the natural candidate is called the R-multiplet. It is defined by

$$\overline{D}^\beta \mathcal{R}_{\alpha\beta} = \chi_\alpha \ , \qquad \overline{D}_\alpha \chi_\beta = 0 \ , \qquad D^\alpha \chi_\alpha = -\overline{D}^\alpha \overline{\chi}_\alpha \ . \tag{5.12}$$

Here $\mathcal{R}_{\alpha\beta} = -2\gamma^\mu_{\alpha\beta}\mathcal{R}_\mu$ is the symmetric bi-spinor corresponding to $\mathcal{R}_\mu$. We can express χ_α in terms of a real linear superfield $\mathcal{J}^{(Z)}$,

$$\chi_\alpha = -4i\overline{D}_\alpha \mathcal{J}^{(Z)} \ , \qquad D^2\mathcal{J}^{(Z)} = \overline{D}^2\mathcal{J}^{(Z)} = 0 \ . \tag{5.13}$$

In components,

$$
\begin{aligned}
\mathcal{R}_\mu &= j_\mu^{(R)} - i\theta S_\mu - i\overline{\theta}\overline{S}_\mu - \left(\theta\gamma^\nu\overline{\theta}\right)\left(2T_{\mu\nu} + i\epsilon_{\mu\nu\rho}\partial^\rho J^{(Z)}\right) \\
&\quad - i\theta\overline{\theta}\left(2j_\mu^{(Z)} + i\epsilon_{\mu\nu\rho}\partial^\nu j^{(R)\rho}\right) + \cdots \ , \\
\mathcal{J}^{(Z)} &= J^{(Z)} - \frac{1}{2}\theta\gamma^\mu S_\mu + \frac{1}{2}\overline{\theta}\gamma^\mu\overline{S}_\mu + \left(\theta\gamma^\mu\overline{\theta}\right) j_\mu^{(Z)} - i\theta\overline{\theta}\, T^\mu{}_\mu + \cdots \ .
\end{aligned}
\tag{5.14}
$$

Here $j_\mu^{(R)}$ is the R-current, $S_{\alpha\mu}$ is the supersymmetry current, $T_{\mu\nu}$ is the symmetric energy-momentum tensor, and $j_\mu^{(Z)}$ is the current associated with the central charge Z in the supersymmetry algebra. The scalar $J^{(Z)}$ gives rise to a string current $\epsilon_{\mu\nu\rho}\partial^\rho J^{(Z)}$. All of these currents are conserved.

It is convenient to express the real linear superfield $\mathcal{J}^{(Z)}$ as a field strength,

$$
\begin{aligned}
\mathcal{J}^{(Z)} &= \frac{i}{2}\overline{D}D\mathcal{U} \ , \\
\mathcal{U} &= \cdots + \left(\theta\gamma^\mu\overline{\theta}\right) u_\mu - i\theta\overline{\theta} J^{(Z)} - \frac{i}{2}\theta^2\overline{\theta}\gamma^\mu\overline{S}_\mu + \frac{i}{2}\overline{\theta}^2\theta\gamma^\mu\overline{S}_\mu + \frac{1}{2}\theta^2\overline{\theta}^2 T^\mu{}_\mu \ .
\end{aligned}
\tag{5.15}
$$

Unlike $\mathcal{J}^{(Z)}$, the superfield $\mathcal{U}$ is possibly not well defined. It can be shifted by gauge transformations, $\mathcal{U} \to \mathcal{U} + \Omega + \overline{\Omega}$, where Ω is chiral.

If the theory is superconformal the R-multiplet can be improved to satisfy (5.12) with $\chi_\alpha = 0$.

Having the supercurrent multiplet, we can couple it to a supergravity background. We now describe what one finds at the level of linearized supergravity for simplicity.

The idea is to couple the $\mathcal{R}$-multiplet to the metric superfield $\mathcal{H}_\mu$,

$$
\delta S = -2\int d^3x \int d^4\theta\, \mathcal{R}_\mu \mathcal{H}^\mu \ . \tag{5.16}
$$

The supergravity gauge transformations are embedded in a superfield L_α,

$$
\delta\mathcal{H}_{\alpha\beta} = \frac{1}{2}\left(D_\alpha\overline{L}_\beta - \overline{D}_\beta L_\alpha\right) + (\alpha \leftrightarrow \beta) \ . \tag{5.17}
$$

Demanding gauge invariance of (5.16) leads to constraints,

$$
D^\alpha\overline{D}^2 L_\alpha + \overline{D}^\alpha D^2\overline{L}_\alpha = 0 \ . \tag{5.18}
$$

In Wess-Zumino gauge, the metric superfield takes the form

$$
\mathcal{H}_\mu = \frac{1}{2}\left(\theta\gamma^\nu\overline{\theta}\right)\left(h_{\mu\nu} - iB_{\mu\nu}\right) - \frac{1}{2}\theta\overline{\theta}C_\mu - \frac{i}{2}\theta^2\overline{\theta}\overline{\psi}_\mu + \frac{i}{2}\overline{\theta}^2\theta\psi_\mu + \frac{1}{2}\theta^2\overline{\theta}^2\left(A_\mu - V_\mu\right) \ . \tag{5.19}
$$

Here $h_{\mu\nu}$ is the linearized metric, so that $g_{\mu\nu} = \delta_{\mu\nu} + 2h_{\mu\nu}$. The vectors C_μ and A_μ are Abelian gauge fields, and $B_{\mu\nu}$ is a two-form gauge field. The gravitino is $\psi_{\mu\alpha}$.

For convenience we denote

$$\begin{aligned} V_\mu &= -\epsilon_{\mu\nu\rho}\partial^\nu C^\rho \ , \qquad \partial^\mu V_\mu = 0 \\ H &= \frac{1}{2}\epsilon_{\mu\nu\rho}\partial^\mu B^{\nu\rho} \ . \end{aligned} \tag{5.20}$$

The fields V_μ and H are *real* in a unitary theory. We can now express the coupling (5.16) in components,

$$\delta S = -\int d^3x \left(T_{\mu\nu}h^{\mu\nu} - j_\mu^{(R)}\big(A^\mu - \frac{3}{2}V^\mu\big) + u_\mu V^\mu - J^{(Z)}H \right) + (\text{fermions}) \ . \tag{5.21}$$

Since the gauge field A^μ couples to the R-current, we see that the gauge transformations (5.17) include local R-transformations. This supergravity theory is the three-dimensional analog of $\mathcal{N} = 1$ new minimal supergravity in four dimensions. As far as I know nobody has really constructed the full nonlinear theory in three dimensions.

We can turn on various combinations of the background fields h, H, V, A. These describe general curved manifolds with extra fluxes, gauge bundles etc. Only a subset of the possible fluxes preserves some amount of supersymmetry. The amount of supersymmetries preserved is inferred from the number of spinors leaving the gravitino null.

If we turn on

$$g_{\mu\nu} = \frac{4r^2}{(r^2 + x^2)^2}\delta_{\mu\nu} \ , \qquad H = -\frac{i}{r} \ , \tag{5.22}$$

then we have four real supercharges preserved. This is a supersymmetric round three-sphere. In addition to the metric which defines the ordinary sphere, here there is also flux through the sphere.

One can readily find many of the couplings that need to be introduced on this sphere to preserve supersymmetry via (5.21). Especially note the terms which go like $1/r$ due to the coupling to the three form flux H.

It is especially interesting that H is imaginary. Hence, the resulting theories on S^3 are not unitary. This is just the Euclidean version of the familiar statement that there is no SUSY in dS space.

So why is it interesting to study them? First we observe that H is the only source for non-unitarity on the three sphere. H couples to the operator $J^{(Z)}$ in the R-multiplet. Hence when we put a superconformal field theory on the three sphere while preserving

SUSY, there won't be violation of unitarity. The idea is then to compute the partition function of superconformal field theories. In doing this computation we will need to introduce the regulator (actually a localizing term $\int d^3\sqrt{g}\{Q,V\}$) and this would break unitarity because this would not be conformal. But we are not afraid from unitarity violation in the regulator, after all, PV works quite well. More precisely, the violations of unitary would have to be accounted for by counterterms, and we can always tune those to restore unitarity.

Let us consider the simplest example of what has been discussed so far. We study a single, free, chiral multiplet on S^3. To linear order in $1/r$ we have explained how to find the Lagrangian via our analysis of linearized supergravity. There are several approaches to completing the construction

1. The Noether procedure: Add corrections which die off as $1/r^2$, whlie requiring that one can preserve a consistent SUSY algebra in curved space. This has been done for S^3 but not for more general manifolds.
2. Start from full nonlinear supergravity with auxiliary fields and look for solutions where the graviton fluctuations and gravitino can be set to zero consistently. This also has not been fully explored yet. Especially that there is not yet an available nonlinear new-minimal formalism in 3d.

For the free chiral multiplet, denote the R-charge of the bottom component, ϕ, by R. Then one finds the action

$$S = \int d^3x\sqrt{g}\left((\partial_\mu\phi)^2 + i\psi^\dagger\sigma^\mu\partial_\mu\psi + F^\dagger F + \frac{R-\frac{1}{2}}{r}\psi^\dagger\psi + \frac{R(2-R)}{r^2}\phi^\dagger\phi\right) .$$

This is invariant under the transformation rules

$$\delta\phi = 0 \ , \qquad \delta\phi^\dagger = \psi^\dagger\epsilon \ , \qquad \delta\psi = \left(-i\sigma^\mu\partial_\mu\phi + \frac{R}{r}\phi\right)\epsilon \ ,$$

$$\delta\psi^\dagger = \epsilon^T F^\dagger \ , \qquad \delta F = \epsilon^T\left(-i\sigma^\mu\partial_\mu\psi + \frac{\frac{1}{2}-R}{r}\psi\right) \ , \qquad \delta F^\dagger = 0 \ .$$

As appropriate to curved space, here ϵ is a kind of a Killing spinor

$$\nabla_\mu\epsilon = \frac{i}{2}\gamma_\mu\epsilon$$

and $\epsilon^\dagger\epsilon = 1$. Note that for $R = 1/2$ the term linear in $1/r$ disappears. This term breaks unitarity because it is missing an i. This is the coupling to H we mentioned.

One can add to this free chiral field any superpotential, such as $W = \phi^4$. This does not modify the transformation rules and affects the Lagrangian only in the usual way (no new r suppressed terms). In general, adding a superpotential constrains the choice of R-symmetry. With $W = \phi^4$ we must choose $R = \frac{1}{2}$.

The theory above has an R-symmetry and it also has a global symmetry. Under the global symmetry the chiral multiplet is assigned charge 1. It is very natural to couple this theory to the background gauge multiplet – this can be viewed as a generating functional for correlation functions of the current multiplet. A gauge multiplet in three dimensions consists of real scalar σ, gauging λ, and gauge field A_μ, and a scalar field D.

The action generalizes to

$$S = \int d^3x\sqrt{g}\Bigg((D_\mu\phi)^2 + i\psi^\dagger\sigma^\mu D_\mu\psi + F^\dagger F + \frac{R-\frac{1}{2}}{r}\psi^\dagger\psi + \frac{R(2-R)}{r^2}\phi^\dagger\phi$$

$$+\phi^\dagger(\sigma)^2\phi + i\phi^\dagger D\phi - i\psi^\dagger\sigma\psi + \frac{2i}{r}\left(R-\frac{1}{2}\right)\phi^\dagger\sigma\phi + \text{gaugino terms}\Bigg) \ .$$

And the transformation rules

$$\delta\phi = 0 \ , \qquad \delta\phi^\dagger = \psi^\dagger\epsilon \ , \qquad \delta\psi = \left(-i\sigma^\mu D_\mu\phi - i\sigma\phi + \frac{R}{r}\phi\right)\epsilon \ ,$$

$$\delta\psi^\dagger = \epsilon^T F^\dagger \ , \qquad \delta F = \epsilon^T\left(-i\sigma^\mu D_\mu\psi + i\sigma\psi + \frac{\frac{1}{2}-R}{r}\psi + \text{gaugino}\right) \ , \qquad \delta F^\dagger = 0 \ .$$

An important thing to realize now is that if the background field σ obtains a VEV m, then the fermions and bosons get a mass m. This VEV for σ does not break supersymmetry, in flat space. But in curved space the variation of the gauging is $\delta\lambda = \left(-D - \frac{\sigma}{m} + \cdots\right)\epsilon$, so when we turn on a VEV for σ we must also have a VEV for D, $D = -\sigma/r \equiv m/r$. This preserves supersymmetry on the sphere. Such a mass is called a real mass term in three dimensions.

We can compute the partition function of such supersymmetric theories on curved space. Since the manifold is compact, the partition function is IR finite. The main claim about such partition functions is that they are holomorphic in $<\sigma> + i\frac{R}{r}$

$$Z_{S^3} = Z_{S^3}\left(\langle\sigma\rangle + i\frac{R}{r}\right) \ .$$

This is quite bizarre: there is holomorphy in the real masses and R charges. It is not understood why this is the case from first principles. One can actually obtain analytic forms for the partition function via localization, namely, there is some deformation

$$S_t = S + t \int \{\delta, V\} \ .$$

that allows to localize on rather simple field configuration (e.g. in the ϕ^4 theory mentioned above).

Let us now study the consequences of this holomorphy. This means that

$$\partial_m Z = -ir\partial_R Z \ .$$

Now consider a conformal field theory, and choose the R-charge to be the superconformal R-charge. Then $\partial_m Z$ is a one point function in this conformal field theory. Since S^3 is stereographically equivalent to R^3, a one point function maps to a one point function in R^3 . In conformal field theories in flat space, all one-point functions besides that of the unit operator vanish (the identity operator has a real VEV). This means that

$$\partial_m \log Z = 0$$

Actually there is a subtlety in this equation because of some mixed flavor-gravitational CS terms and the correct equation is

$$\partial_m \log Z = real$$

Thus by the holomorphy equation

$$\partial_R |Z|^2 = 0$$

We can write $Z = e^{-F}$ then,

$$\partial_{R_i} |F| = 0 \tag{5.23}$$

It is also conjectured that

$$\partial_{R_j} \partial_{R_i} |F| < 0 \ . \tag{5.24}$$

There is not yet a compelling argument for that.

In general there is the superconformal R-symmetry but there are many $U(1)$ global symmetries too. Then we can build many effective R-symmetries. The equation above means that the partition function is stationary w.r.t. derivatives in all these directions.

Now imagine any flow. If there are no accidental symmetries in the IR conformal field theory, then the amount of $U(1)$ symmetries in the IR CFT is smaller than the number of $U(1)$ symmetries in the UV. Then, we are minimizing over a smaller space in the IR, which means that

$$F_{UV} > F_{IR}$$

This argument has many flows but it applies in some class of theories:

A. It is not been proven that (5.24) is true at a conformal fixed point, but checked empirically in many examples.

B. We are only talking about local minima, and for the argument above to hold we need, for instance, the respective local minima to be actually global.

C. Accidental symmetries are commonplace and there is no a priori way to classify cases when they arise and when they do not.