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Fundamentals of GPS Navigation and Receiver Processing

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Fundamentals of GPS Navigation and Receiver Processing

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Presented to
Workshop on Science Application of GNSS in Developing Countries
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Overview

What we plan to cover over the next two days

1. GPS Navigation Solutions
2. Differential GPS
3. GNSS Receiver Design
4. Kalman Filtering and Inertial Navigation Systems

A Few Notes Before We Begin

- I will not present every slide in detail
  - Some we will discuss in detail
  - Some we will quickly go over
  - Some are included for background purposes and will not be covered
- Interaction is good!
  - Ask questions if you don't understand
- A little about me…

GPS Receiver Measurements

What does the receiver measure?

GPS Measurements (Overview)

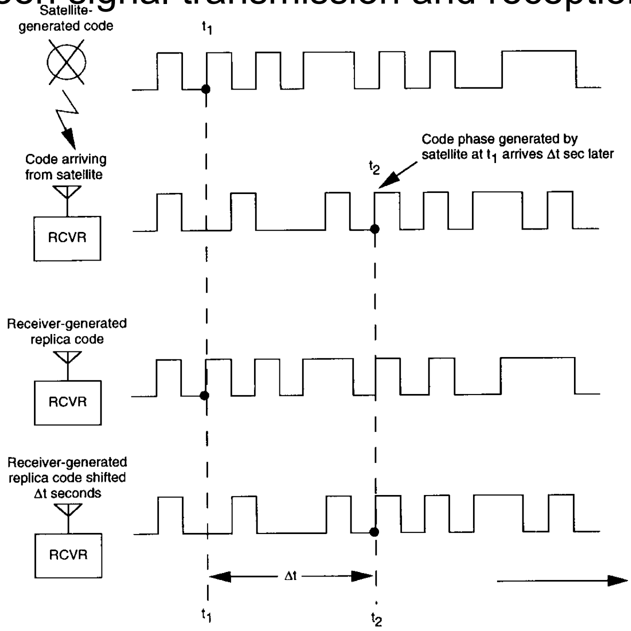
- Each separate tracking loop typically can give 4 different measurement outputs
  - Pseudorange measurement
  - Carrier-phase measurement (sometimes called integrated Doppler)
  - Doppler measurement
  - Carrier-to-noise density C/N_0
- Actual output varies depending upon receiver
  - Ashtech Z-surveyor (or Z-12) gives them all
  - RCVR-3A gives just C/N_0
- Note: We're talking here about *raw measurements*
  - Almost all receivers generate navigation processor outputs (position, velocity, heading, etc.)

Measurement Rates and Timing

- Most receivers take measurements on all channels/tracking loops simultaneously
  - Measurements time-tagged with the receiver clock (receiver time)
  - The time at which a set of measurements is made is called a data epoch.
- The data rate varies depending upon receiver/application. Typical data rates:
  - Static surveying: One measurement every 30 seconds (120 measurements per hour)
  - Typical air, land, and marine navigation: 0.5-2 measurement per second (most common)
  - Specialized high-dynamic applications: Up to 50 measurements per second (recent development)

GPS Pseudorange Measurement

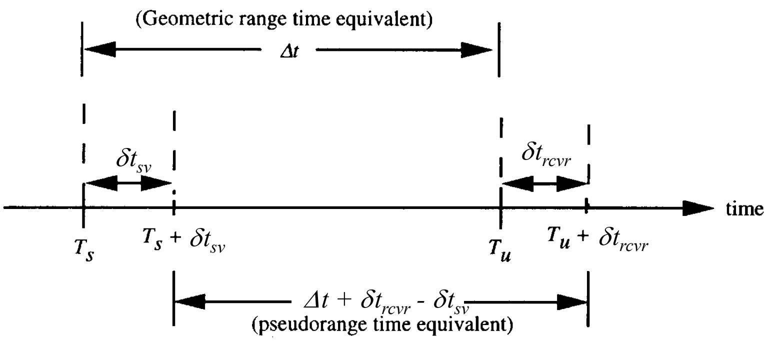
- Pseudorange is a measure of the difference in time between signal transmission and reception



Kaplan (ed.), *Understanding GPS: Principles and Applications*, Artech House, 1996

Effect of Clock Errors on Pseudorange

- Since pseudorange is based on time difference, any clock errors will fold directly into pseudorange



- Small clock errors can result in large pseudorange errors (since clock errors are multiplied by speed of light)
- Satellite clock errors (δt_{sv}) are very small
  - Satellites have atomic time standards
  - Satellite clock corrections transmitted in navigation message
- Receiver clock (δt_{rcvr}) is dominant error

Kaplan (ed.), *Understanding GPS: Principles and Applications*, Artech House, 1996

Doppler Shift

- For electromagnetic waves (which travel at the speed of light), the received frequency f_R is approximated using the standard Doppler equation

$$f_R = f_T\left(1 - \frac{(\boldsymbol{v}_r \cdot \boldsymbol{a})}{c}\right)$$

$$\begin{aligned} f_R &= \text{received frequency (Hz)} \\ f_T &= \text{transmitted frequency (Hz)} \\ \boldsymbol{v}_r &= \text{satellite-to-user relative velocity vector (m/s)} \\ \boldsymbol{a} &= \text{unit vector pointing along} \\ &\quad\ \text{line-of-sight from user to SV} \\ c &= \text{speed of light (m/s)} \end{aligned}$$

  - Note that $\boldsymbol{v}_r$ is the (vector) velocity difference

$$\begin{aligned} \boldsymbol{v}_r &= \boldsymbol{v} - \dot{\boldsymbol{u}} \\ \boldsymbol{v} &= \text{velocity vector for satellite (m/s)} \\ \dot{\boldsymbol{u}} &= \text{velocity vector for user (m/s)} \end{aligned}$$

- The Doppler shift Δf is then

$$\Delta f = f_R - f_T \quad \text{(Hz)}$$

Doppler Measurement

- The GPS receiver locks onto the carrier of the GPS signal and measures the received signal frequency
  - Relationship between true and measured received signal frequency: $f_{R_{meas}}$

$$f_R = f_{R_{meas}}(1 + \delta\dot{t}_{rcvr})$$

$$\begin{aligned} f_R &= \text{true received signal frequency (Hz)} \\ f_{R_{meas}} &= \text{measured received signal frequency (Hz)} \\ \delta\dot{t}_{rcvr} &= \text{receiver clock drift rate (sec/sec)} \end{aligned}$$

  - Doppler measurement formed by differencing the measured received frequency and the transmit frequency:

$$\Delta f_{meas} = f_{R_{meas}} - f_T$$

  - Note: transmit frequency is calculated using information about SV clock drift rate given in navigation message

Doppler Measurement Sign Convention

- Sign convention based on Doppler definition
  - A satellite moving away from the receiver (neglecting clock errors) will have a *negative* Doppler shift

$$f_{R_{meas}} < f_T$$

$$\Delta f_{meas} = f_{R_{meas}} - f_T < 0$$

  - Sign convention used for NovAtel (and possibly other) receivers
- Sign convention based on relationship between Doppler and pseudorange
  - Doppler is essentially a measurement of the rate of change of the pseudorange
  - A satellite moving away from the receiver (neglecting clock errors) will have a *positive* Doppler measurement value
  - More common sign convention for GPS receivers (Ashtech, Trimble, and others)
- Carrier-phase measurement follows same convention as Doppler measurement (normally)

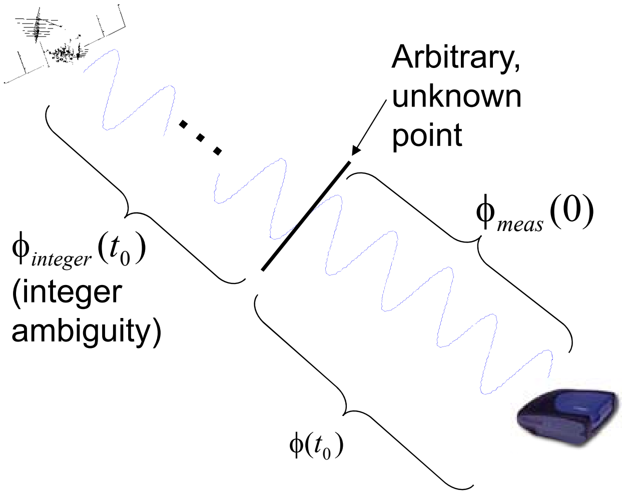
Carrier-Phase (Integrated Doppler) Measurement

- The carrier-phase measurement $\phi_{meas}(t)$ is calculated by integrating the Doppler measurements

$$\text{range}(t) = \underbrace{\int_{t_o}^{t} \Delta f_{meas}(t)dt + \phi(t_0)}_{\substack{\phi_{meas}(t) \\ \text{(can be measured by receiver)}}} + \phi_{integer}(t_0) + \text{clock error} + \text{other errors}$$

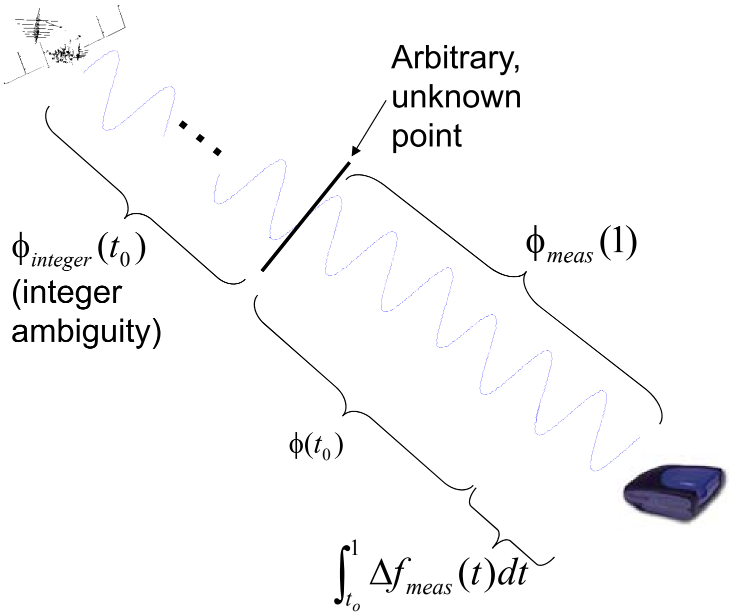
- The integer portion of the initial carrier-phase at the start of the integration ($\phi_{integer}(t_0)$) is known as the “carrier-phase integer ambiguity”
  - Because of this ambiguity, the carrier-phase measurement is not an absolute measurement of position
  - Advanced processing techniques can be used to resolve these carrier-phase ambiguites (carrier-phase ambiguity resolution)
- Alternative way of thinking: carrier-phase measurement is the “beat frequency” between the incoming carrier signal and receiver generated carrier.

Phase Tracking Example
At Start of Phase Lock (Time = 0 seconds)



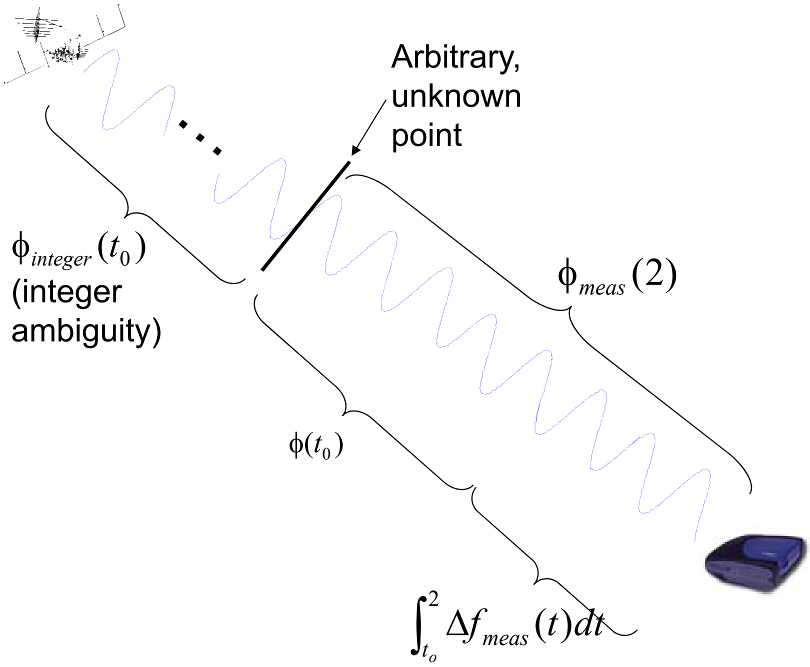
Ignoring clock and other errors

Phase Tracking Example
After Movement (for 1 Second)



Ignoring clock and other errors

Phase Tracking Example After Movement (for 2 Seconds)



Ignoring clock and other errors

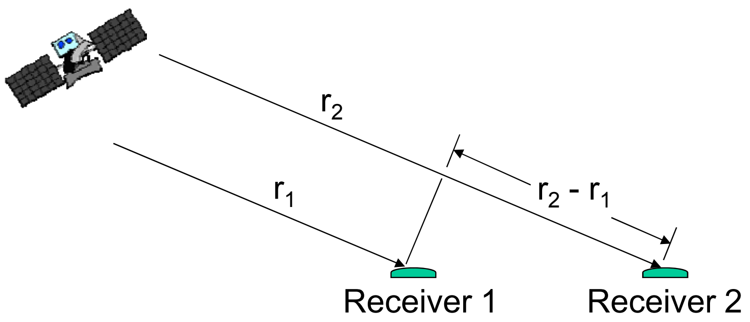
Comparison Between Pseudorange and Carrier-Phase Measurements

	Pseudorange	Carrier-Phase
Type of measurement	Range (absolute)	Range (ambiguous)
Measurement precision	~1 m	~0.01 m
Robustness	More robust	Less robust (cycle slips possible)

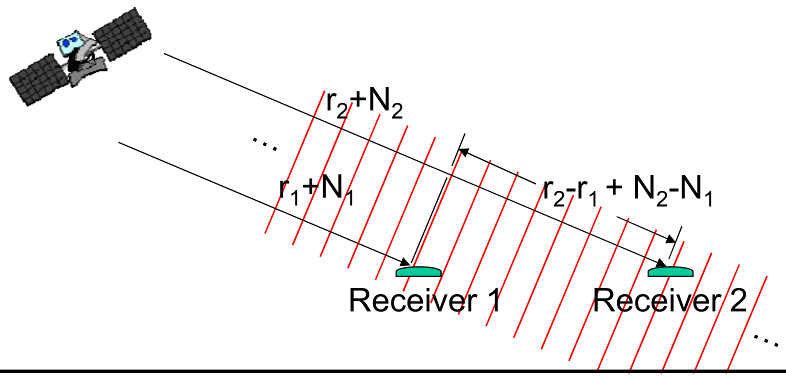
Necessary for high precision GPS

What Does a DGPS Measurement Tell You?

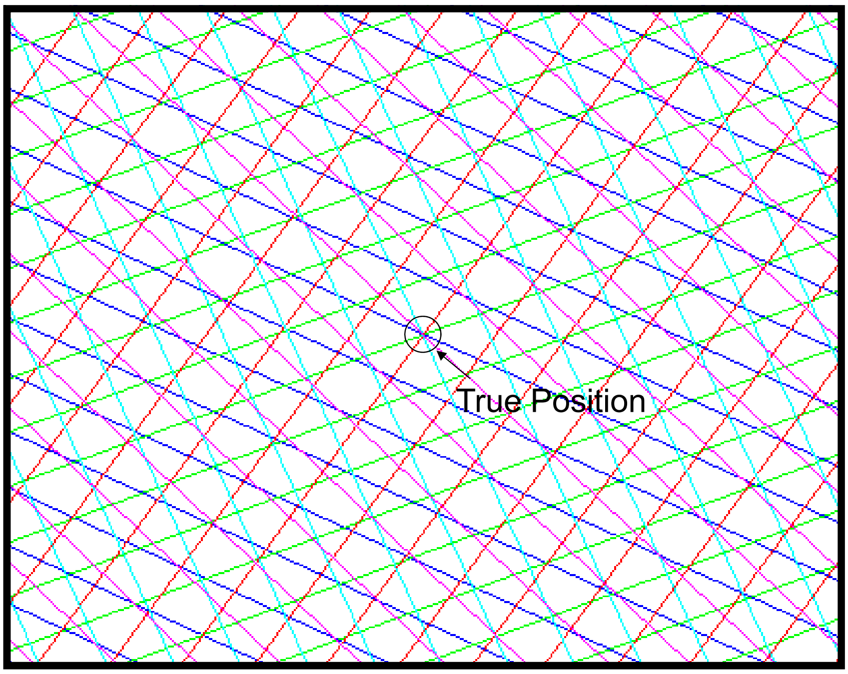
DGPS using pseudorange measurements



DGPS using carrier-phase measurements



Five Satellite Carrier-Phase DGPS Example



Carrier-to-Noise Density (C/N_0)

- The carrier-to-noise density is a measure of signal strength
  - The higher the C/N_0, the stronger the signal (and the better the measurements)
  - Units are dB-Hz
  - General rules-of-thumb:
    - $C/N_0 > 40$: Very strong signal
    - $32 < C/N_0 < 40$: Marginal signal
    - $C/N_0 < 32$: Probably losing lock
- C/N_0 tends to be receiver-dependent
  - Can be calculated many different ways
  - Absolute comparisons between receivers not very meaningful
  - Relative comparisons between measurements in a single receiver are very meaningful

GPS Navigation Solution

"OK, so I have all of these pseudorange measurements. Where in the world are we?"

Pseudorange Equation

- The pseudorange is the sum of the true range plus the receiver clock error
  - We're assuming (for now) that the receiver clock error is the only remaining error
    - SV clock error has been corrected for
    - All other errors are deemed negligible (or have been corrected)

$$\rho_j = \sqrt{(x_j - x_u)^2 + (y_j - y_u)^2 + (z_j - z_u)^2} + c\delta t_u$$
$$= f(x_u, y_u, z_u, \delta t_u)$$

ρ_j = pseudorange measurement from satellite j (m)

x_j, y_j, z_j = ECEF position of satellite j (m)

x_u, y_u, z_u = ECEF position of user (m)

δt_u = receiver clock error (sec)

- For now, only use one type of pseudorange (L1 C/A, L1 P, or L2 P)

Statement of the Problem

- At a given measurement epoch, the GPS receiver generates n pseudorange measurements (from n different satellites)

$$\begin{aligned}
\rho_1 &= \sqrt{(x_1 - x_u)^2 + (y_1 - y_u)^2 + (z_1 - z_u)^2} + c\delta t_u \\
\rho_2 &= \sqrt{(x_2 - x_u)^2 + (y_2 - y_u)^2 + (z_2 - z_u)^2} + c\delta t_u \\
\rho_3 &= \sqrt{(x_3 - x_u)^2 + (y_3 - y_u)^2 + (z_3 - z_u)^2} + c\delta t_u \\
&\vdots \\
\rho_n &= \sqrt{(x_n - x_u)^2 + (y_n - y_u)^2 + (z_n - z_u)^2} + c\delta t_u
\end{aligned}$$

- Goal: Determine user position and clock error, expressed in state-vector form as

$$\mathbf{x} = \begin{bmatrix} x_u \\ y_u \\ z_u \\ c\delta t_u \end{bmatrix}$$

Solving the Pseudorange Equations

- The n pseudorange equations are non-linear (so no easy solution)
- Ways to solve
  - Closed form solutions
    - Complicated
    - May not give as much insight
  - Iterative techniques based on linearization
    - Often using least-squares estimation
    - Arguably the simplest approach
    - Approach covered in this course
  - Kalman filtering
    - Similar to least-squares approach, except with additional ability to handle measurements over a period of time
    - Will discuss briefly
- What is linearization?
  - Pick a nominal (or approximate) solution
  - Linearize about that point, resulting in a set of linear equations
  - Solve the linear equations
  - Will use Taylor series expansion for linearization

Taylor Series Expansion (1/2)

- Taylor series expansion (1 variable)

$$f(a+\Delta a) = f(a) + \Delta a \frac{df}{da} + \frac{(\Delta a)^2}{2!}\frac{d^2 f}{da^2} + \frac{(\Delta a)^3}{3!}\frac{d^3 f}{da^3} + \dots$$

- This can be used to linearize about a certain value of the independent variable a.
  - Example: the function $f(t) = 2 + 3t - 6t^2$ is a non-linear function in t
  - Suppose we want to linearize about the point $\hat{t} = 2$
  - The complete Taylor series expression is

$$f(\hat{t}+\Delta t) = f(\hat{t}) + \Delta t \frac{df}{dt} + \frac{(\Delta t)^2}{2!}\frac{d^2 f}{dt^2}$$

$$= 2 + 3\hat{t} - 6\hat{t}^2 + \Delta t(3 - 12\hat{t}) + \frac{(\Delta t)^2}{2}(-12)$$

  - To linearize, we set $\hat{t} = 2$ and neglect higher order (non-linear) terms of Δt
    - Valid for perturbations (i.e., small values of Δt)

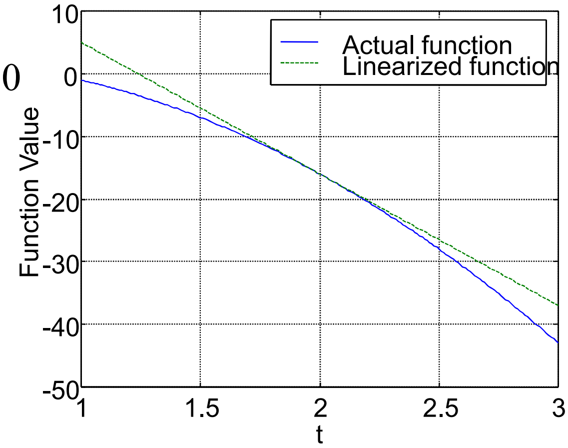
Taylor Series Expansion (2/2)

– (Continued example) Linearized form

$$f(\hat{t}+\Delta t)\Big|_{\hat{t}=2} = f(\hat{t}=2)+\Delta t\frac{df}{dt}\Big|_{\hat{t}=2} + \cancelto{0}{\frac{(\Delta t)^2}{2!}\frac{d^2 f}{dt^2}\Big|_{\hat{t}=2}}$$

$$= 2+3(2)-6(2^2)+\Delta t(3-12(2))$$

$$= -16-21\Delta t$$



- First order Taylor series for function in two variables:

$$f(\hat{a}+\Delta a,\hat{b}+\Delta b) = f(\hat{a},\hat{b})+\Delta a\frac{\partial f}{\partial a}\Big|_{\hat{a},\hat{b}} + \Delta b\frac{\partial f}{\partial b}\Big|_{\hat{a},\hat{b}} + \text{h.o.t.}$$

Linearization of Pseudorange Equations (1/5)

- First, define a nominal state (position and clock error) as

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{x}_u \\ \hat{y}_u \\ \hat{z}_u \\ c\delta\hat{t}_u \end{bmatrix} = \text{nominal (approximate) state}$$

- An approximate (or expected) pseudorange can then be calculated for satellite j

$$\hat{\rho}_j = \sqrt{(x_j-\hat{x}_u)^2+(y_j-\hat{y}_u)^2+(z_j-\hat{z}_u)^2} + c\delta\hat{t}_u$$

$$= f(\hat{x}_u,\hat{y}_u,\hat{z}_u,c\delta\hat{t}_u)$$

– This approximate (expected) pseudorange is the pseudorange that we would expect to have if our position and clock error were actually $\hat{x}_u$, $\hat{y}_u$, $\hat{z}_u$, and $c\delta\hat{t}_u$.

Linearization of Pseudorange Equations (2/5)

- Relationship between true and approximate position and time

$$x_u = \hat{x}_u + \Delta x_u$$
$$y_u = \hat{y}_u + \Delta y_u$$
$$z_u = \hat{z}_u + \Delta z_u$$
$$c\delta t_u = c\delta \hat{t}_u + \Delta c\delta t_u$$

  – Vector form:

$$\mathbf{x}_u = \hat{\mathbf{x}}_u + \Delta \mathbf{x}_u$$

- Based on these relations, we can write

$$f(x_u, y_u, z_u, c\delta t_u) = f(\hat{x}_u + \Delta x_u, \hat{y}_u + \Delta y_u, \hat{z}_u + \Delta z_u, c\delta \hat{t}_u + \Delta c\delta t_u)$$

- To linearize, right-hand side of equation can be evaluated using a first order Taylor series expansion

Linearization of Pseudorange Equations (3/5)

- First order Taylor series expansion of pseudorange function:

$$\begin{aligned} f(\hat{x}_u + \Delta x_u, \hat{y}_u + \Delta y_u, \hat{z}_u + \Delta z_u, c\delta \hat{t}_u + \Delta c\delta t_u) &= f(\hat{x}_u, \hat{y}_u, \hat{z}_u, c\delta \hat{t}_u) \\ &+ \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, c\delta \hat{t}_u)}{\partial \hat{x}_u}\Delta x_u + \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, c\delta \hat{t}_u)}{\partial \hat{y}_u}\Delta y_u \\ &+ \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, c\delta \hat{t}_u)}{\partial \hat{z}_u}\Delta z_u + \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, c\delta \hat{t}_u)}{\partial c\delta \hat{t}_u}\Delta c\delta t_u \\ &+ \text{h.o.t.} \end{aligned}$$

- The partial derivatives are

$$\frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \delta \hat{t}_u)}{\partial \hat{x}_u} = -\frac{x_j - \hat{x}_u}{\hat{r}_j} \qquad \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \delta \hat{t}_u)}{\partial \hat{y}_u} = -\frac{y_j - \hat{y}_u}{\hat{r}_j}$$

$$\frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \delta \hat{t}_u)}{\partial \hat{z}_u} = -\frac{z_j - \hat{z}_u}{\hat{r}_j} \qquad \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \delta \hat{t}_u)}{\partial c\delta \hat{t}_u} = 1$$

$$\hat{r}_j = \sqrt{(x_j - \hat{x}_u)^2 + (y_j - \hat{y}_u)^2 + (z_j - \hat{z}_u)^2}$$

Linearization of Pseudorange Equations (4/5)

- Using above results, linearized pseudorange equation is

$$\rho_j = \hat{\rho}_j - \frac{x_j - \hat{x}_u}{\hat{r}_j}\Delta x_u - \frac{y_j - \hat{y}_u}{\hat{r}_j}\Delta y_u - \frac{z_j - \hat{z}_u}{\hat{r}_j}\Delta z_u + \Delta c\delta t_u$$

- This can be simplified to $\Delta\rho_j = a_{xj}\Delta x_u + a_{yj}\Delta y_u + a_{zj}\Delta z_u - \Delta c\delta t_u$ where

$$\Delta\rho_j = \hat{\rho}_j - \rho_j$$

$$a_{xj} = \frac{x_j - \hat{x}_u}{\hat{r}_j}, \qquad a_{yj} = \frac{y_j - \hat{y}_u}{\hat{r}_j}, \qquad a_{zj} = \frac{z_j - \hat{z}_u}{\hat{r}_j}$$

Linearization of Pseudorange Equations (5/5)

- Original (nonlinear) equations for n measurements

$$\begin{aligned}
\rho_1 &= \sqrt{(x_1 - x_u)^2 + (y_1 - y_u)^2 + (z_1 - z_u)^2} + c\delta t_u \\
\rho_2 &= \sqrt{(x_2 - x_u)^2 + (y_1 - y_u)^2 + (z_1 - z_u)^2} + c\delta t_u \\
\rho_3 &= \sqrt{(x_3 - x_u)^2 + (y_3 - y_u)^2 + (z_3 - z_u)^2} + c\delta t_u \\
&\vdots \\
\rho_n &= \sqrt{(x_n - x_u)^2 + (y_n - y_u)^2 + (z_n - z_u)^2} + c\delta t_u
\end{aligned}$$

- Linearized (error) equations for the same n measurements

$$\begin{aligned}
\Delta\rho_1 &= a_{x1}\Delta x_u + a_{y1}\Delta y_u + a_{z1}\Delta z_u - \Delta c\delta t_u \\
\Delta\rho_2 &= a_{x2}\Delta x_u + a_{y2}\Delta y_u + a_{z2}\Delta z_u - \Delta c\delta t_u \\
\Delta\rho_3 &= a_{x3}\Delta x_u + a_{y3}\Delta y_u + a_{z3}\Delta z_u - \Delta c\delta t_u \\
&\vdots \\
\Delta\rho_n &= a_{xn}\Delta x_u + a_{yn}\Delta y_u + a_{zn}\Delta z_u - \Delta c\delta t_u
\end{aligned}$$

Solving the Linearized Pseudorange Equations Using Least-Squares (1/2)

- We can express the set of pseudorange equations in matrix form

$$\Delta\boldsymbol{\rho} = \mathbf{H}\Delta\mathbf{x}$$

$$\Delta\boldsymbol{\rho} = \begin{bmatrix} \Delta\rho_1 \\ \Delta\rho_2 \\ \Delta\rho_3 \\ \vdots \\ \Delta\rho_n \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} a_{x1} & a_{y1} & a_{z1} & -1 \\ a_{x2} & a_{y2} & a_{z2} & -1 \\ a_{x3} & a_{y3} & a_{z3} & -1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{xn} & a_{yn} & a_{zn} & -1 \end{bmatrix} \qquad \Delta\mathbf{x} = \begin{bmatrix} \Delta x_u \\ \Delta y_u \\ \Delta z_u \\ \Delta c\delta t_u \end{bmatrix}$$

- Three possible cases
  - n < 4: Underdetermined case
    - Cannot solve for $\Delta\mathbf{x}$
    - Is there still useable information?
  - n = 4: Uniquely determined case
    - One valid solution for $\Delta\mathbf{x}$ (generally)
    - Solved by calculating $\mathbf{H}^{-1}$ ($\Delta\mathbf{x} = \mathbf{H}^{-1}\Delta\boldsymbol{\rho}$)
  - n > 4: Overdetermined case
    - No solution that perfectly solves equation (generally)
    - Can use least-squares techniques (which pick solution that minimizes the square of the error)

Solving the Linearized Pseudorange Equations Using Least-Squares (2/2)

- Basic least-squares solution (no measurement weighting)

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$$

  - Reasonable approach for single-point positioning in presence of SA
- Solution with measurement weighting (weighted least-squares)
  - Useful when
    - Measurements have different error statistics
    - Measurement errors are correlated
  - Measurement error covariance matrix $\mathbf{C}_\rho$
    - Diagonal terms are measurement error variances
    - Off-diagonal terms show cross-correlation between measurement errors

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{C}_\rho^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{C}_\rho^{-1}\Delta\boldsymbol{\rho}$$

  - Note that this is identical to unweighted case if $\mathbf{C}_\rho = \mathbf{I}$ (identity matrix)

Measurement Residuals

- For overdetermined system, generally no valid solution for $\Delta\mathbf{x}$ that solves measurement equation, so

$$\Delta\rho \neq \mathbf{H}\Delta\mathbf{x}$$

- Measurement residuals ($\mathbf{v}$)
  - Corrections that, when applied to measurements, would result in solution of above equation
  - Least-squares minimizes the sum of squares of these residuals

$$\mathbf{v} = \Delta\rho - \mathbf{H}\Delta\mathbf{x}$$

$$\Delta\rho = \mathbf{H}\Delta\mathbf{x} + \mathbf{v}$$

Iterating the Nominal State

- Linearized equations (and resulting $\mathbf{H}$ matrix) calculated using nominal state $\hat{\mathbf{x}}_u$
- Linearization valid when
  - Nominal state is close to true state
  - $\Delta\mathbf{x}$ is "small"
- If $\hat{\mathbf{x}}_u$ is not very accurate (i.e., $\Delta\mathbf{x}$ is large), iteration is required
  - For each iteration, a new value of $\hat{\mathbf{x}}_u$ is calculated based upon the old value and the corrections $\Delta\mathbf{x}$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

  - This new value of $\hat{\mathbf{x}}_u$ is then used to recalculate the corrections $\Delta\mathbf{x}$ (which should be smaller this time)
- Solution must converge
  - For standard GPS positioning, not much of a problem (will generally converge with an initial guess at the center of the Earth)
  - For more non-linear situations (e.g., using pseudolites), this can be more of a problem

Correcting for Satellite Clock Error

- Single point positioning only estimates receiver clock error
  - Assumes all other errors are negligible
  - Requires correction of satellite clock error
- Clock correction (from IS-GPS-200D)

$$\rho_{corr} = \rho + c\Delta t_{sv}$$

$$\Delta t_{sv} = a_{f_0} + a_{f_1}(t - t_{0_c}) + a_{f_2}(t - t_{0_c})^2 + \Delta t_r$$

$$\Delta t_r = Fe\sqrt{a}\sin E_k$$

ρ_{corr} = pseudorange corrected for SV clock error
ρ = original (raw) pseudorange measurement
Δt_{sv} = SV clock correction
$a_{f_0}, a_{f_1}, a_{f_2}, t_{0_c}$ = SV clock correction parameters from nav message
Δt_r = relativity correction (since not circular orbit)
F = constant = $-4.442807633 \times 10^{-10}$ sec/(meter)$^{1/2}$
e = eccentricity from nav message
$\sqrt{a}$ = square root of semi-major axis from nav message
E_k = Eccentric anomaly (from SV position calculation)

Determining Signal Transmit Time (1/2)

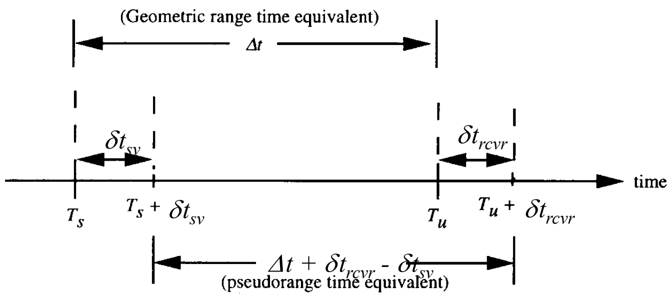
- For satellite position calculation, need true GPS transmit time of the signal (T_s)
  - Receiver provides time of reception according to the receiver clock ($T_u + \delta t_{rcvr}$)
  - From diagram below, if the pseudorange time equivalent is subtracted from the receive time, then the result is the true transmit time plus the satellite clock error

$$\underbrace{T_u + \delta t_{rcvr}}_{\text{receive time}} - \frac{PR}{c} = T_s + \delta t_{sv}$$

PR = pseudorange measurement (m)

$$\underbrace{T_u + \delta t_{rcvr}}_{\text{receive time}} - \frac{PR}{c} - \underbrace{\delta t_{sv}}_{} = T_s$$

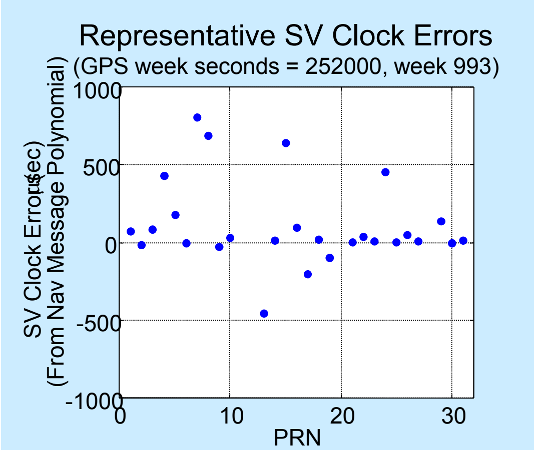
same as Δt_{sv} from the previous slide



Determining Signal Transmit Time (2/2)

- Effect of neglecting δt_{sv} for SV positioning[1]
  - Satellite clock error can grow to up to ~1 msec:
  - Typical satellite velocity is 3900 m/s
  - Worst-case position error from neglecting δt_{sv}

    3900 m/s x 0.001 s = **3.9 m**
  - Effect of neglecting δt_{sv}
    - Single point positioning: Can be significant (but not with SA)
    - Differential positioning: effectively cancelled out (acts like 3.9 m satellite position error)



[1]The SV clock error δt_{sv} will have a significant effect on the actual pseudorange measurement. This page only describes the impact of δt_{sv} on determining the position of the satellite.

Correcting for Satellite Group Delay

- Each satellite has a slight time bias between the L1 and the L2 signals
  - Not desired, but it's there nonetheless
  - Will affect dual-frequency users, unless it's accounted for
  - Can be measured and/or calibrated out
  - This calibration is accounted for when the control segment generates the satellite clock correction terms from broadcast nav message: $a_{f_0}, a_{f_1}, a_{f_2}$, and t_{0_c}
  - However, this is all designed for the dual-frequency user! Single frequency users need to remove the effect of this dual-frequency correction on their Δt_{sv} value
- Single frequency users must apply the group delay term (TGD) from the nav message to their SV clock correction term (from p. 90 of ICD-GPS-200C)

$$(\Delta t_{sv})_{L1} = \Delta t_{sv} - T_{GD}$$

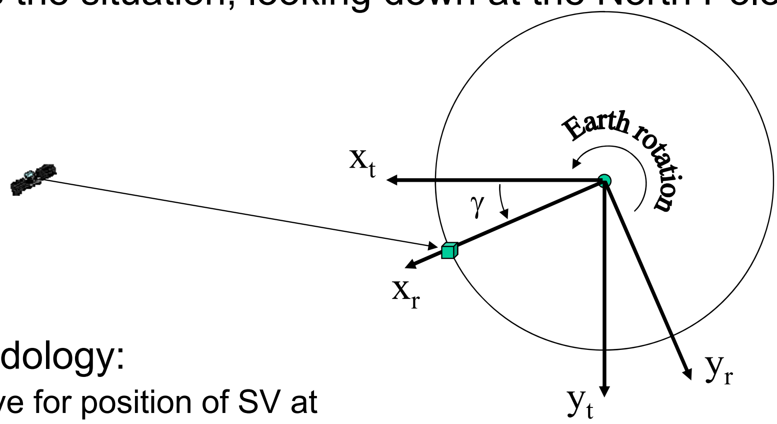
$$(\Delta t_{sv})_{L2} = \Delta t_{sv} - \left(\frac{77}{60}\right)^2 T_{GD}$$

Accounting for Signal Travel Time (1/3)

- Signal arrives at receiver ***after*** it is transmitted (due to signal travel time)
  - Transmit time: Time the signal was transmitted
  - Receive time: Time the signal was received
- Satellite position should be calculated based upon transmit time
  - When measuring a signal, we don't really care what happened after that signal was transmitted
  - Transmit time should be GPS system time (or as close to it as possible)
  - Very good approximate value of transmit time obtained by subtracting pseudorange (expressed in seconds) from the receive time as indicated by the receiver clock
    - Why?
- What other considerations do we need to make for signal travel time?

Accounting for Signal Travel Time (2/3)

- Here's the situation, looking down at the North Pole



- Methodology:
  - Solve for position of SV at transmit time, in ECEF coordinates at transmit time (x_t, y_t, and z_t) using ICD-GPS-200 equations
  - Rotate into ECEF reference frame at the time of reception:

$$\begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix}$$

$$\gamma = \dot{\Omega}_e t_{prop}$$

t_{prop} = Signal propagation time

Accounting for Signal Travel Time (3/3)

- Neglecting atmospheric delay, the signal propagation time is calculated by

$$t_{prop} = \frac{\text{geometric range to satellite}}{\text{speed of light}}$$

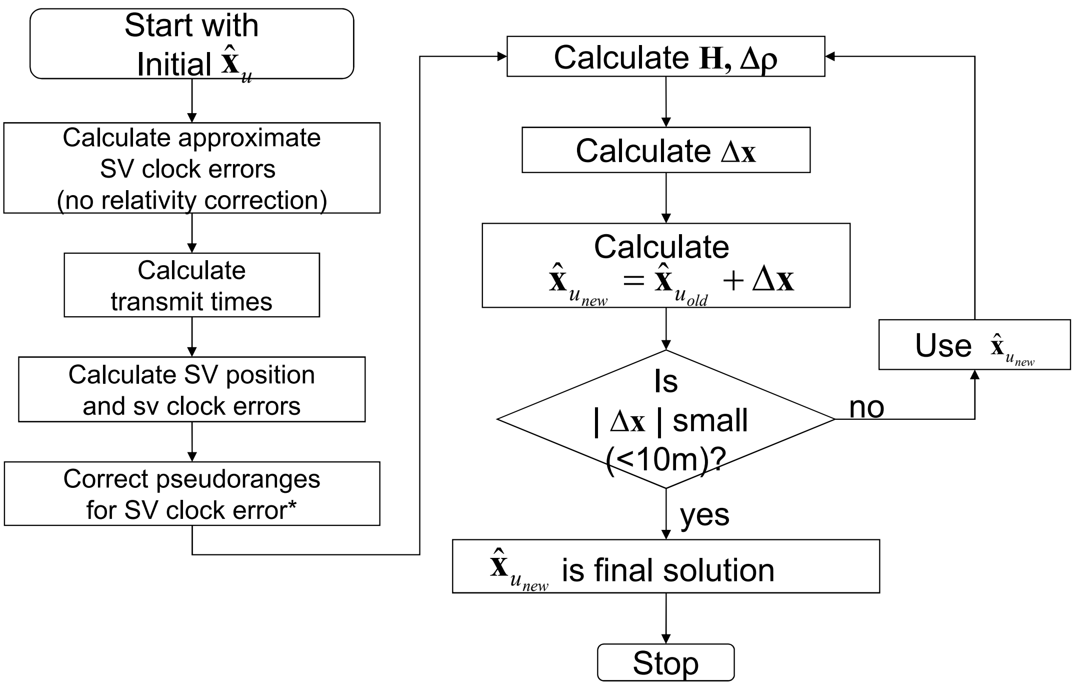
$$= \frac{\left| \boldsymbol{p}_{sv} - \boldsymbol{p}_{rcvr} \right|}{c}$$

$$\boldsymbol{p}_{sv} = \text{satellite ECEF position vector}$$

$$\boldsymbol{p}_{rcvr} = \text{receiver ECEF position vector}$$

- Note that the satellite position is needed to calculate t_{prop} (and vice-versa)
  - Satellite position in ECEF coordinates at transmit time is sufficiently accurate (x_t, y_t, and z_t)
  - Note that receiver position must be known
    - Can be approximate

Single Point Positioning Algorithm



*include group delay correction, if a single-frequency user

GPS Positioning Example

- We'll look at a single case to give an example
- Situation
  - Receiver measurement time (GPS week seconds): 220937
  - Initial $\hat{\mathbf{x}}_u$: $[\underbrace{506071.529 \quad -4882278.667 \quad 4109624.557}_{\text{Initial guess of position (in error by \textasciitilde 50 km)}} \quad 15.807]$

    15.807: Initial clock error expressed in m
  - Measurements:

PRN	Pseudorange
12	25022759.323
2	22075351.532
26	21929350.580
15	22677087.545
29	21039894.608
21	24757444.127
30	24032696.422

Example: Calculation of Transmit Time

Start with Initial $\hat{\mathbf{x}}_u$
↓
Calculate approximate SV clock errors (no relativity correction)
↓
Calculate transmit times
↓
Calculate SV position and sv clock errors
↓
Correct pseudoranges for SV clock error*

$$\Delta t_{sv_{approx}} = a_{f_0} + a_{f_1}(t - t_{0_c}) + a_{f_2}(t - t_{0_c})^2$$

(Receive time = 220937)

PRN	PR/c (s)	Approx SV Clock Error (s)	Transmit Time (s)
12	0.083466941	-0.000359172	220936.916892231
2	0.073635447	0.000199964	220936.926164589
26	0.073148440	0.000247391	220936.926604169
15	0.075642622	-0.000107130	220936.924464508
29	0.070181534	-0.000043329	220936.929861795
21	0.082581944	0.000065081	220936.917352974
30	0.080164446	0.000075192	220936.919760362

$$\underbrace{T_u + \delta t_{rcvr}}_{\text{receive time}} - \frac{PR}{c} - \underbrace{\delta t_{sv}} = T_s$$

Example: SV Position and Clock Error and Pseudorange Correction

Start with Initial $\hat{\mathbf{x}}_u$

Calculate approximate SV clock errors (no relativity correction)

Calculate transmit times

Calculate SV position and sv clock errors

Correct pseudoranges for SV clock error*

PRN	SV ECEF X (m)	SV ECEF Y (m)	SV ECEF Z (m)	SV Clock Err (s)
12	-9924909.896	-22418412.217	-10238600.462	-0.000359164
2	19519446.654	-12864169.870	12106498.214	0.000199964
26	9518973.908	-24465469.002	-347289.566	0.000247426
15	6420995.137	-25601178.700	-2907089.329	-0.000107131
29	-8265550.815	-16497554.935	19234406.500	-0.000043326
21	-22299549.612	-6458590.524	13615554.839	0.000065090
30	-18044425.181	-19566072.431	-289953.964	0.000075168

Orbital calculations + time-of-transit rotation correction

$$\Delta t_{sv} = a_{f_0} + a_{f_1}(t - t_{0_c}) + a_{f_2}(t - t_{0_c})^2 + \Delta t_r$$

$$\rho_{corr} = \rho + c\left(\Delta t_{sv} - T_{GD}\right)$$

PRN	Original PR (m)	Corrected PR (m)	Group Delay Tgd (ns)
12	25022759.323	24915088.264	-12.107193470
2	22075351.532	22135304.471	-17.229467630
26	21929350.580	22003529.125	-6.519258022
15	22677087.545	22644973.532	-10.244548320
29	21039894.608	21026908.513	-8.847564459
21	24757444.127	24776961.101	-11.641532180
30	24032696.422	24055233.876	-8.381903172

*include group delay correction, if a single-frequency user

Example: H Matrix (Iteration 1)

Calculate $\mathbf{H}, \Delta\rho$

Calculate $\Delta\mathbf{x}$

Calculate $\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$

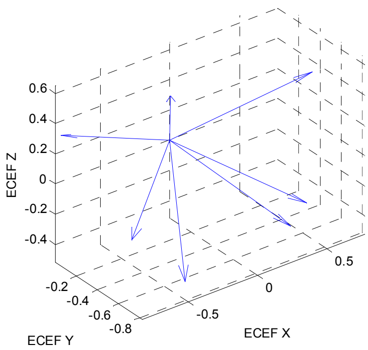
Is $|\Delta\mathbf{x}|$ small (<10m)?

yes

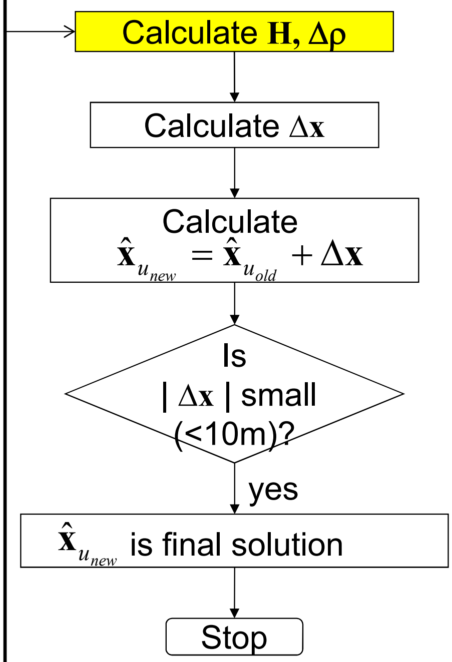
$\hat{\mathbf{x}}_{u_{new}}$ is final solution

Stop

$$\mathbf{H} = \begin{bmatrix} -0.4182 & -0.7030 & -0.5752 & -1 \\ 0.8597 & -0.3609 & 0.3616 & -1 \\ 0.4094 & -0.8896 & -0.2025 & -1 \\ 0.2610 & -0.9143 & -0.3096 & -1 \\ -0.4179 & -0.5533 & 0.7205 & -1 \\ -0.9212 & -0.0637 & 0.3840 & -1 \\ -0.7709 & -0.6102 & -0.1828 & -1 \end{bmatrix}$$



Example: Δρ (Iteration 1)

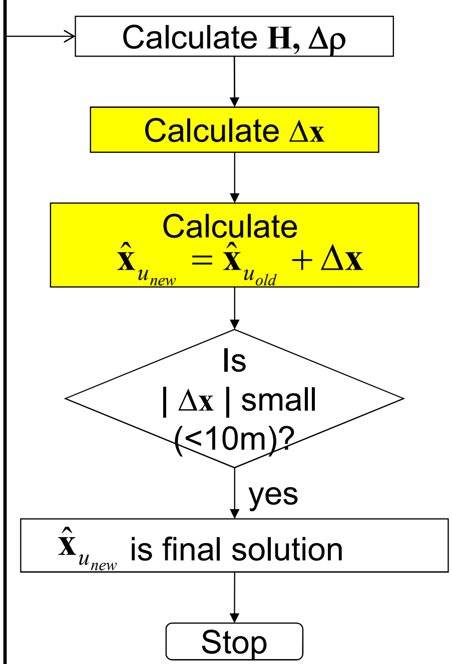


Calculated Measured (corrected)

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24943810.919	24915088.264	28722.655
2	22117181.292	22135304.471	-18123.179
26	22013598.807	22003529.125	10069.682
15	22660408.867	22644973.532	15435.335
29	20990847.857	21026908.513	-36060.657
21	24757718.148	24776961.101	-19242.953
30	24064325.866	24055233.876	9091.990

Example: Solution and Residuals (Iteration 1)



$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\rho$$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506068.143	506071.529	-3.386
-4882283.665	-4882278.667	-4.998
4059632.252	4109624.557	-49992.305
63.927	15.807	48.120

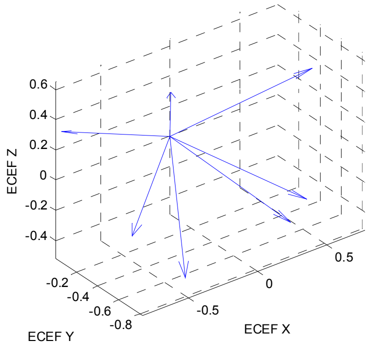
Residuals: $\mathbf{v} = \Delta\rho - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\rho$	$\mathbf{H}\Delta\mathbf{x}$
12	9.162	28722.655	28713.493
2	1.699	-18123.179	-18124.878
26	-6.800	10069.682	10076.482
15	-0.178	15435.335	15435.513
29	4.853	-36060.657	-36065.510
21	-3.299	-19242.953	-19239.654
30	-5.436	9091.990	9097.426

Example: H Matrix (Iterations 1 and 2)

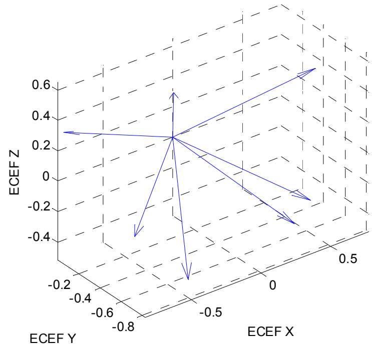
Iteration 1

$$\mathbf{H} = \begin{bmatrix} -0.4182 & -0.7030 & -0.5752 & -1 \\ 0.8597 & -0.3609 & 0.3616 & -1 \\ 0.4094 & -0.8896 & -0.2025 & -1 \\ 0.2610 & -0.9143 & -0.3096 & -1 \\ -0.4179 & -0.5533 & 0.7205 & -1 \\ -0.9212 & -0.0637 & 0.3840 & -1 \\ -0.7709 & -0.6102 & -0.1828 & -1 \end{bmatrix}$$



Iteration 2

$$\mathbf{H} = \begin{bmatrix} -0.4187 & -0.7038 & -0.5739 & -1 \\ 0.8590 & -0.3606 & 0.3635 & -1 \\ 0.4096 & -0.8900 & -0.2003 & -1 \\ 0.2612 & -0.9149 & -0.3076 & -1 \\ -0.4172 & -0.5524 & 0.7217 & -1 \\ -0.9204 & -0.0636 & 0.3857 & -1 \\ -0.7712 & -0.6104 & -0.1808 & -1 \end{bmatrix}$$



Example: Δρ (Iterations 1 and 2)

Iteration 1

Calculated

Measured (corrected)

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24943810.919	24915088.264	28722.655
2	22117181.292	22135304.471	-18123.179
26	22013598.807	22003529.125	10069.682
15	22660408.867	22644973.532	15435.335
29	20990847.857	21026908.513	-36060.657
21	24757718.148	24776961.101	-19242.953
30	24064325.866	24055233.876	9091.990

Iteration 2

Calculated

Measured (corrected)

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24915130.980	24915088.264	42.716
2	22135355.242	22135304.471	50.771
26	22003576.788	22003529.125	47.662
15	22645023.243	22644973.532	49.711
29	21026941.948	21026908.513	33.435
21	24777000.804	24776961.101	39.703
30	24055278.650	24055233.876	44.773

Example: Solution and Residuals (Iterations 1 and 2)

Iteration 1 $\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\rho$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506068.143	506071.529	-3.386
-4882283.665	-4882278.667	-4.998
4059632.252	4109624.557	-49992.305
63.927	15.807	48.120

Residuals: $\mathbf{v} = \Delta\rho - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\rho$	$\mathbf{H}\Delta\mathbf{x}$
12	9.162	28722.655	28713.493
2	1.699	-18123.179	-18124.878
26	-6.800	10069.682	10076.482
15	-0.178	15435.335	15435.513
29	4.853	-36060.657	-36065.510
21	-3.299	-19242.953	-19239.654
30	-5.436	9091.990	9097.426

Iteration 2 $\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\rho$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506075.869	506068.143	7.726
-4882274.608	-4882283.665	9.057
4059622.275	4059632.252	-9.977
13.120	63.927	-50.807

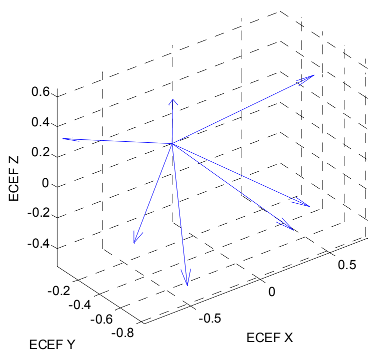
Residuals: $\mathbf{v} = \Delta\rho - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\rho$	$\mathbf{H}\Delta\mathbf{x}$
12	-4.208	42.716	46.924
2	0.220	50.771	50.551
26	-0.248	47.662	47.910
15	2.103	49.711	47.609
29	-1.946	33.435	35.381
21	0.431	39.703	39.272
30	3.648	44.773	41.125

Example: H Matrix (Iterations 2 and 3)

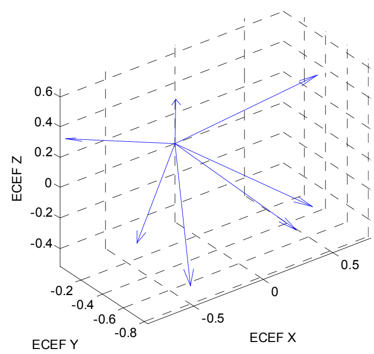
Iteration 2

$$\mathbf{H} = \begin{bmatrix} -0.4187 & -0.7038 & -0.5739 & -1 \\ 0.8590 & -0.3606 & 0.3635 & -1 \\ 0.4096 & -0.8900 & -0.2003 & -1 \\ 0.2612 & -0.9149 & -0.3076 & -1 \\ -0.4172 & -0.5524 & 0.7217 & -1 \\ -0.9204 & -0.0636 & 0.3857 & -1 \\ -0.7712 & -0.6104 & -0.1808 & -1 \end{bmatrix}$$



Iteration 3

$$\mathbf{H} = \begin{bmatrix} -0.4187 & -0.7038 & -0.5739 & -1 \\ 0.8590 & -0.3606 & 0.3635 & -1 \\ 0.4096 & -0.8900 & -0.2003 & -1 \\ 0.2612 & -0.9149 & -0.3076 & -1 \\ -0.4172 & -0.5524 & 0.7217 & -1 \\ -0.9204 & -0.0636 & 0.3857 & -1 \\ -0.7712 & -0.6104 & -0.1808 & -1 \end{bmatrix}$$



Example: Δρ (Iterations 2 and 3)

Iteration 2

Calculated

Measured (corrected)

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24915130.980	24915088.264	42.716
2	22135355.242	22135304.471	50.771
26	22003576.788	22003529.125	47.662
15	22645023.243	22644973.532	49.711
29	21026941.948	21026908.513	33.435
21	24777000.804	24776961.101	39.703
30	24055278.650	24055233.876	44.773

Iteration 3

Calculated

Measured (corrected)

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24915084.055	24915088.264	-4.208
2	22135304.691	22135304.471	0.220
26	22003528.878	22003529.125	-0.248
15	22644975.634	22644973.532	2.103
29	21026906.567	21026908.513	-1.946
21	24776961.532	24776961.101	0.431
30	24055237.525	24055233.876	3.648

Example: Solution and Residuals (Iterations 2 and 3)

Iteration 2

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506075.869	506068.143	7.726
-4882274.608	-4882283.665	9.057
4059622.275	4059632.252	-9.977
13.120	63.927	-50.807

Residuals: $\mathbf{v} = \Delta\boldsymbol{\rho} - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\boldsymbol{\rho}$	$\mathbf{H}\Delta\mathbf{x}$
12	-4.208	42.716	46.924
2	0.220	50.771	50.551
26	-0.248	47.662	47.910
15	2.103	49.711	47.609
29	-1.946	33.435	35.381
21	0.431	39.703	39.272
30	3.648	44.773	41.125

Iteration 3

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506075.869	506075.869	0.000
-4882274.608	-4882274.608	0.000
4059622.275	4059622.275	0.000
13.120	13.120	0.000

On order of 10^{-6}

Residuals: $\mathbf{v} = \Delta\boldsymbol{\rho} - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\boldsymbol{\rho}$	$\mathbf{H}\Delta\mathbf{x}$
12	-4.208	42.716	46.924
2	0.220	50.771	50.551
26	-0.248	47.662	47.910
15	2.103	49.711	47.609
29	-1.946	33.435	35.381
21	0.431	39.703	39.272
30	3.648	44.773	41.125

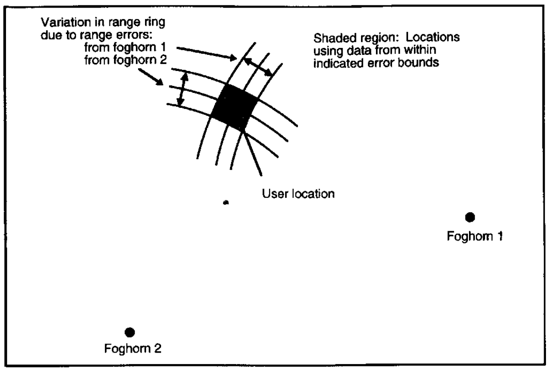
Convergence

- Practically speaking, getting the system to converge with GNSS is easy
  - Example showed case where initial guess was 50 km in error
  - Can start with the center of the Earth as a guess, and it would only add an iteration or two
  - Normally, a receiver will use its last solution as a starting point, so only a single iteration is necessary
- Nonlinearities (which drive the need for iteration) are more severe when dealing with pseudolites
  - Much closer to receiver than satellite
  - H matrix varies more quickly as a function of position

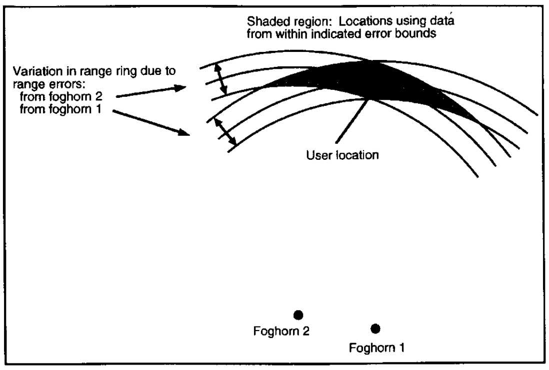
Effect of Geometry on Positioning Accuracy (Foghorn Example)

Consider the foghorn example, except allow for a measurement error

Good Geometry Example

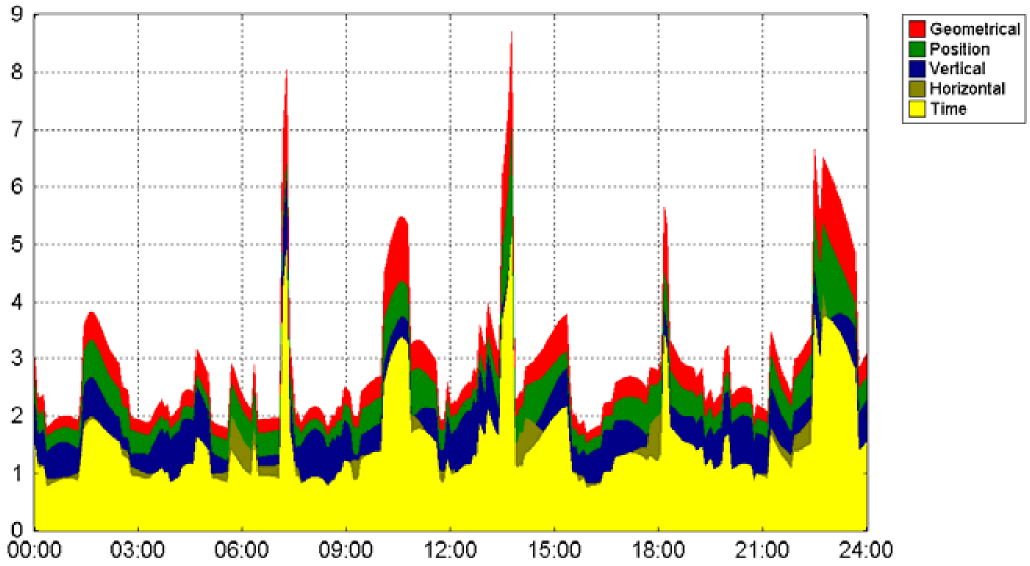


Poor Geometry Example



Typical DOP Plot

Dayton Ohio – 24 Apr 2003 – All Visible SVs (above 10° elevation)

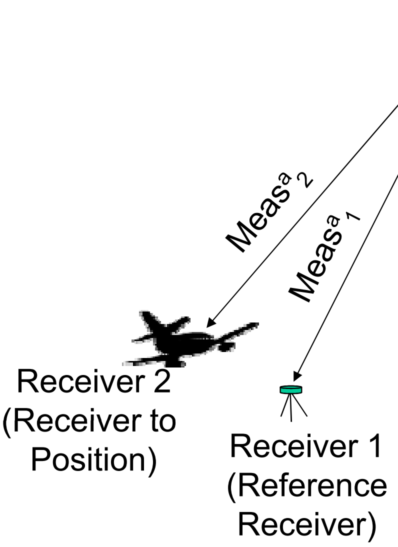


Overview

<u>What we plan to cover over the next two days</u>

1. GPS Navigation Solutions
2. Differential GPS
3. GNSS Receiver Design
4. Kalman Filtering and Inertial Navigation Systems

Differential GPS Concept



Error Type	Comparison between Meas_1^a and Meas_2^a	DGPS Effect on Error
Satellite clock error	Same	Removed
Receiver clock error	Different (uncorrelated)	Added
Ephemeris error[1]	Very similar[2]	Reduced[2]
SA – dither	Same	Removed
SA – epsilon	Very similar[2]	Reduced[2]
Ionospheric delay	Very similar[2]	Reduced[2]
Tropospheric delay	Very similar[2]	Reduced[2]
Multipath	Different (uncorrelated)	Added (and amplified)
Measurement noise	Different (uncorrelated)	Added (and amplified)

[1]Effect of ephemeris error on positioning (actually only affects the calculated range, not the actual measurement)
[2]Errors grow as the separation distance between receivers 1 and 2 increases. (The errors are the same and are removed for very short baseline distances).

DGPS Variations

- DGPS is a broad term, and there are many different ways DGPS can be applied.
  - Measurements used
    - Code only
    - Carrier-smoothed code
    - Carrier-phase
  - Application type
    - Positioning
    - Attitude
  - Position domain vs. measurement domain
  - Post-processing vs. real-time
  - Type of correction
  - Number of reference receivers
  - Area of coverage
    - LADGPS
    - RADGPS
    - WADGPS
  - Differencing method used
    - Single-differencing
    - Double-differencing

576 possible combinations!

- Each of these will be covered in the slides that follow.

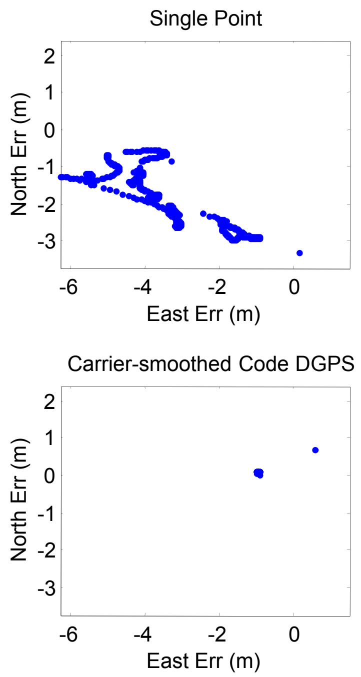
DGPS - Measurements Used (1/5)

- The type of measurements is one of the primary distinguishing factors between different DGPS implementations
  - Code only
    - Simplest to implement
    - Based purely on pseudorange measurements
    - In best case (short baseline), errors include code multipath and noise
    - Typical accuracy: 2-4 m
  - Carrier-smoothed code
    - Carrier-phase measurement is very precise (~1 cm), but it is not an absolute measurement (due to unknown integer ambiguity).
    - Code (pseudorange) measurement is absolute, but it is much less precise (~1-2 m).
    - A filter can be used to combine the carrier-phase and the code measurements to take advantage of their respective strengths.
      - Filter time constant limited by code-carrier ionospheric divergence (due to different signs on ionospheric error term)
    - Carrier-phase smoothing of the code essentially removes most of the code multipath and noise
    - Typical carrier-smoothed code DGPS accuracy: 0.1-0.5 m
    - Relatively easy to implement

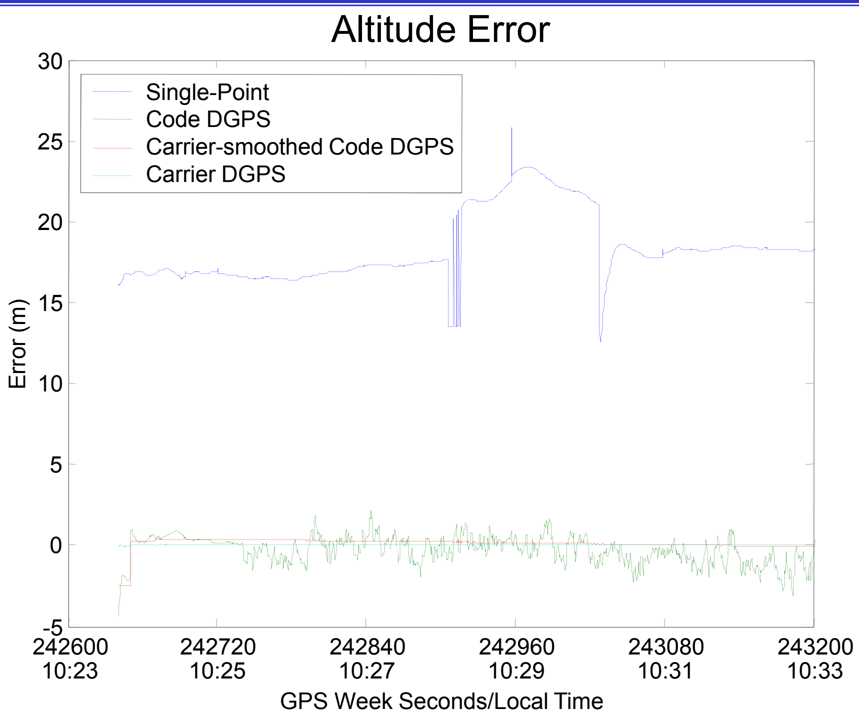
DGPS - Measurements Used (2/5)

- Type of measurements (continued)
  - Carrier-phase
    - GPS receiver can track exact phase of incoming GPS carrier
      - Can determine “where” in the cycle
      - Cannot determine “which” cycle
      - Results in an unknown integer ambiguity
    - If carrier-phase integer ambiguities can be determined, then the carrier-phase measurement will yield the most precise (and accurate) positioning possible
    - Fairly complex to implement
    - Difficult to resolve integer ambiguities over long reference/mobile receiver baselines
    - Normally requires some period of time to resolve ambiguities
      - 1-3 minutes typical
      - Depends upon baseline distance, algorithm
    - Extremely sensitive to loss of carrier-lock (or cycle slips)
    - Often, code measurements will be used to initially aid in determining the integer ambiguities
      - Final solution normally based primarily on carrier-phase measurements
    - Typical accuracy: 0.01-0.05 m

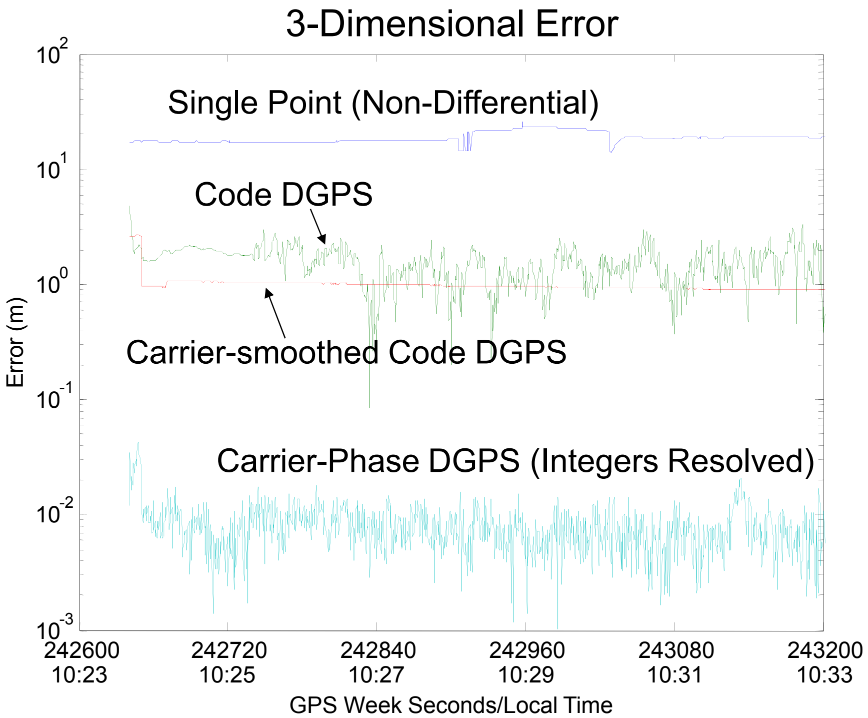
DGPS - Measurements Used (3/5)
Sample Comparison of Horizontal Error



DGPS - Measurements Used (4/5)
Sample Comparison of Altitude Error

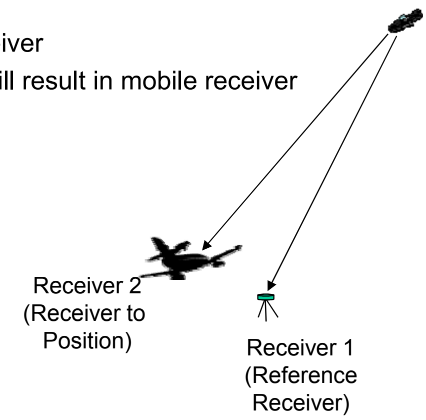


DGPS - Measurements Used (5/5)
Sample Comparison of 3-D Error



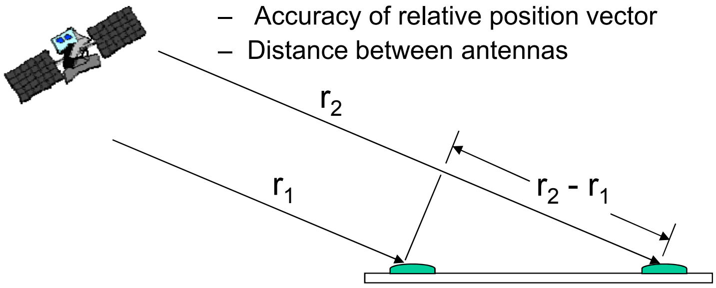
DGPS - Application Type (1/2)

- DGPS gives *relative* position between two receivers
  - Can be expressed as a 3-D vector
- This relative positioning information can be used in two ways
  - Positioning (most common)
    - Know position of reference receiver
    - Can calculate position of “mobile” receiver
    - Errors in reference receiver position will result in mobile receiver positioning errors



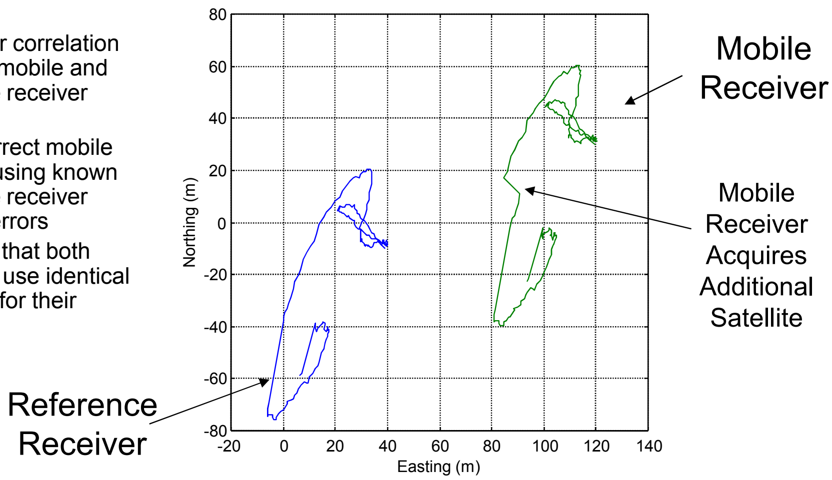
DGPS - Application Type (2/2)

- This relative positioning information can be used in two ways (continued)
  - Attitude determination
    - Antennas are in fixed, known configuration relative to defined “body” axes
    - Relative position vector between antennas is function of attitude of body
    - Can calculate attitude using relative position vector
      - Two antennas → two attitude axes (e.g., yaw and pitch)
      - Three or more antennas → complete attitude
    - Normally based on carrier-phase differential techniques with integer ambiguity resolution for most precise results
      - Relatively easy to resolve integer ambiguities in this case
    - Attitude accuracy depends upon
      - Accuracy of relative position vector
      - Distance between antennas



DGPS - Position vs. Measurement Domain (1/2)

- Position Domain
  - Reference receiver at known point (origin of plot)
  - Mobile receiver located to the northeast
  - Horizontal position of both receivers plotted on local coordinate system
    - Note error correlation between mobile and reference receiver errors
    - Could correct mobile receiver using known reference receiver position errors
    - Requires that both receivers use identical satellites for their solutions!



DGPS - Position vs. Measurement Domain (2/2)

- Measurement domain
  - Differential corrections are given for each *measurement*
  - These corrections are then applied to the mobile receiver measurements
    - Results in corrected measurements
    - Position calculated using corrected measurements
  - Advantages
    - Doesn't require same satellite coverage at mobile and reference receivers
      - Reference receiver can only generate corrections for measurements that it can see
    - Standardized formats are defined
      - RTCM SC-104 messages
    - Makes it possible to detect individual measurement errors
  - Disadvantages
    - Requires that more data be transmitted to mobile user than position domain approach
    - Not generally a large problem with modern radio data modems
    - Insignificant for non-real-time applications

DGPS - Post-Processing vs. Real-Time

- Post-processing
  - Data is collected separately by each receiver
  - Later, data is combined and processed
  - Advantages
    - No data latency (can correlate times exactly)
    - Does not require real-time data link
    - Easier to implement (both hardware and software)
    - Can study and fix anomalies
    - Allows for use of other data and tools that may not be available real-time
      - Precise orbits
      - Ionospheric grid data
- Real-time
  - Differential corrections are sent to mobile receiver as soon as possible (i.e., near real-time)
    - Hard-wire (close applications)
    - Ground radio data link (10s of km)
    - Satellite data link (large areas)
  - Advantages
    - Many applications require real-time positioning!
    - Reduces data turn-around time, enables field checking

DGPS - Type of Correction

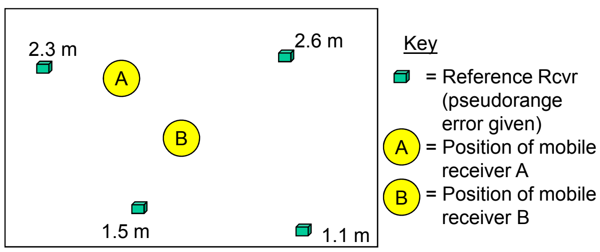
- Two ways to give corrections in measurement domain
  - Corrections to measurements
    - Actual correction values to be applied to each individual measurement
    - Simple, easy to implement
  - Explicit representation of errors
    - DGPS corrections describe all of the errors in a particular measurement
    - Sometimes, error functions or data are transmitted
      - Different error sources can then be combined to generate a correction for a single measurement
      - Example
        - » Precise ephemeris (to remove satellite position error)
        - » Ionospheric grid (to remove ionospheric error)
        - » Tropospheric model parameters (to improve tropospheric model)
    - Advantages
      - Generally valid for wider area of coverage
      - More flexible
    - Disadvantages
      - More complex
      - Requires more differential data to be transmitted

DGPS - Number of Reference Receivers (1/4)

- Single reference receiver is simplest and most common case
  - Errors grow as distance between reference and mobile receivers grows
  - Motivates need for multiple reference stations for some applications
- Multiple reference receivers using code measurements
  - Can involve anywhere from two to hundreds of reference receivers
  - Normally, different error sources are explicitly estimated (satellite position, ionosphere, etc.)
  - Alternatively, individual measurement corrections can be generated for each reference station, and a linear combination of these corrections can be used to generate corrections for a specific point
    - Based upon relative positions between specific point and reference receivers

Example:

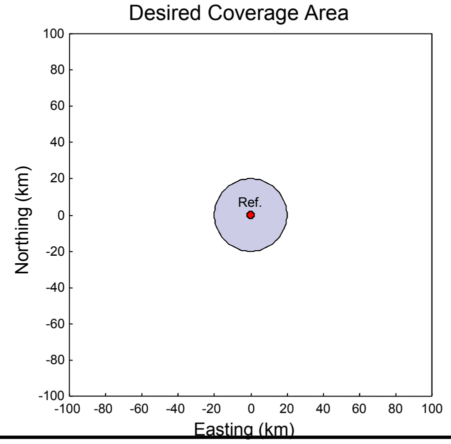
What should be the error at
- location A?
- location B?



DGPS - Number of Reference Receivers (2/4)

- Multiple reference receivers using carrier-phase measurements
  – More difficult than code approaches because of integer ambiguities
- Motivation

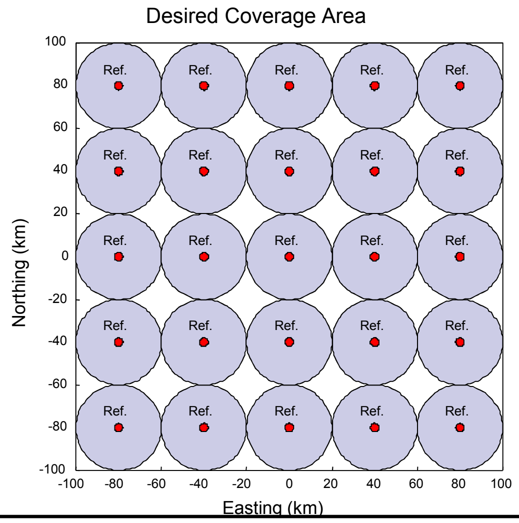
Single Reference Receiver
Not Enough Coverage



DGPS - Number of Reference Receivers (3/4)

- Motivation (continued)

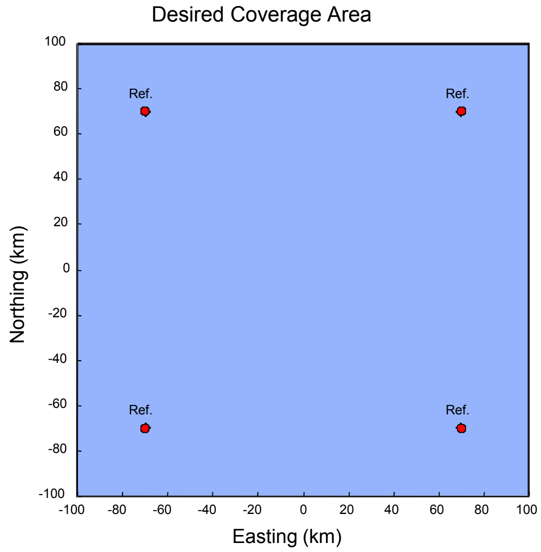
Independent Ref. Receivers
Not Efficient



DGPS - Number of Reference Receivers (4/4)

- Motivation (continued)

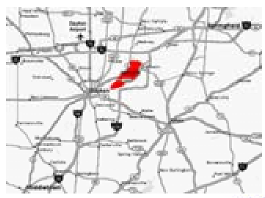
Reference Receiver Network
Efficiently Covers Large Area

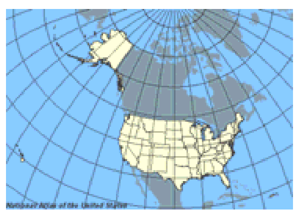


- One method: NetAdjust

DGPS - Area of Coverage

- DGPS is deployed on three different scales
  - Local Area Differential GPS (LADGPS)
    - Covers tens of km
    - Typically involves single reference station
    - Accuracy varies between ~ 0.01-2 m
    - Example: aircraft landing, surveying
  - Regional Area Differential GPS (RADGPS)
    - Covers hundreds of km
    - Involves multiple reference receivers
    - Can achieve decimeter (or sometimes centimeter) level accuracy
    - Example: Norwegian reference receiver network
  - Wide Area Differential GPS (WADGPS)
    - Covers thousands of km
    - Involves multiple reference receivers
    - Not as accurate as RADGPS or LADGPS (typically 1-2 m accuracy)
    - Example: Wide Area Augmentation System (WAAS) for non-precision (and Cat I) approach





DGPS - Differencing Methods - Pseudorange Measurement Errors

- Two types of differencing methods are common
  - Single differencing
  - Double differencing
- Choice of method depends upon application. Typically
  - Code differential → single differencing
  - Carrier-phase differential → double differencing
- Pseudorange errors
  - Original representation

$$\rho = r + c(\delta t_u - \delta t_{sv} + \delta t_D)$$

neglect

$$\delta t_D = \delta t_{trop} + \delta t_{iono} + \delta t_{noise\&res} + \delta t_{mp} + \cancel{\delta t_{hw}} + \delta t_{SA}$$

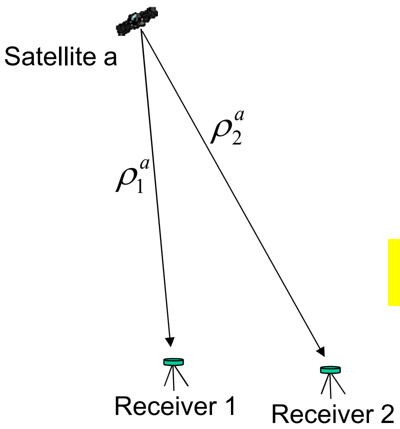
  - Simplification

$$\rho = r + c\delta t_u - c\delta t_{sv} + c\delta t_{trop} + c\delta t_{iono} + c\delta t_{noise\&res} + c\delta t_{mp} + c\delta t_{SA}$$

$$\rho = r + c\delta t_u - c\delta t_{sv} + T + I + \upsilon + m + SA$$

DGPS - Differencing Methods (1/2)

- Single differencing
  - Difference measurements between one satellite and two receivers



$$\Delta\rho_{12}^a \equiv \rho_1^a - \rho_2^a$$
$$= r_1^a + c\delta t_{u_1} - c\delta t_{sv}^a + T_1^a + I_1^a + \upsilon_1^a + m_1^a + SA^a$$
$$- r_2^a - c\delta t_{u_2} + c\delta t_{sv}^a - T_2^a - I_2^a - \upsilon_2^a - m_2^a - SA^a$$

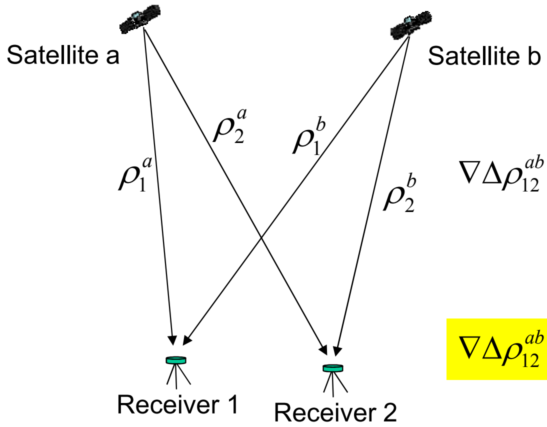
$$\Delta\rho_{12}^a = \Delta r_{12}^a + \Delta c\delta t_{u_{12}} + \Delta T_{12}^a + \Delta I_{12}^a + \Delta\upsilon_{12}^a + \Delta m_{12}^a$$

  - SV clock error and SA *cancelled*[1]
  - Tropospheric, ionospheric errors *reduced*
  - Multipath and noise *amplified* (by factor of $\sqrt{2}$)

[1]Assuming that only the dither portion of SA is utilized (if SA is on at all!)

DGPS - Differencing Methods (2/2)

- Double differencing
  - Difference between two single difference measurements



$$\nabla\Delta\rho_{12}^{ab} \equiv \Delta\rho_{12}^{a} - \Delta\rho_{12}^{b} = \rho_1^a - \rho_2^a - (\rho_1^b - \rho_2^b)$$

$$= \Delta r_{12}^{a} + \Delta c\delta t_{u_{12}} + \Delta T_{12}^{a} + \Delta I_{12}^{a} + \Delta\upsilon_{12}^{a} + \Delta m_{12}^{a}$$

$$- \Delta r_{12}^{b} - \Delta c\delta t_{u_{12}} - \Delta T_{12}^{b} - \Delta I_{12}^{b} - \Delta\upsilon_{12}^{b} - \Delta m_{12}^{b}$$

$$\nabla\Delta\rho_{12}^{ab} = \nabla\Delta r_{12}^{ab} + \nabla\Delta T_{12}^{ab} + \nabla\Delta I_{12}^{ab} + \nabla\Delta\upsilon_{12}^{ab} + \nabla\Delta m_{12}^{ab}$$

  - SV clock error, rcvr clock error, and SA *cancelled*[1]
  - Tropospheric, ionospheric errors *reduced*
  - Multipath and noise *amplified* (by factor of 2)

[1]Assuming that only the dither portion of SA is utilized (if SA is on at all!)

DGPS Errors

"So now I know what DGPS is and how it is applied. What kinds of errors can I expect, and how do those errors grow with baseline distance?"

DGPS Errors

- Errors completely cancelled by DGPS
  - Receiver clock error
  - Satellite clock error
  - SA[1]
- DGPS errors can be grouped into two classes
  - Uncorrelated errors
    - Errors that are not spatially related
    - Do not increase with reference/mobile baseline distance
    - Include multipath and measurement noise
    - DGPS actually increases these errors

Typical Multipath + Noise Error Standard Deviation Values

	Single Meas (non-DGPS)	Single Difference	Double Difference
Code	0.5-1.5 m	0.7-2.1 m	1-3 m
Carrier-Phase	0.2 - 1 cm	0.3 - 1.4 cm	0.4 - 2 cm

  - Correlated errors
    - Are spatially related
    - Increase with baseline distance
    - Include satellite position (ephemeris), ionospheric, and tropospheric errors

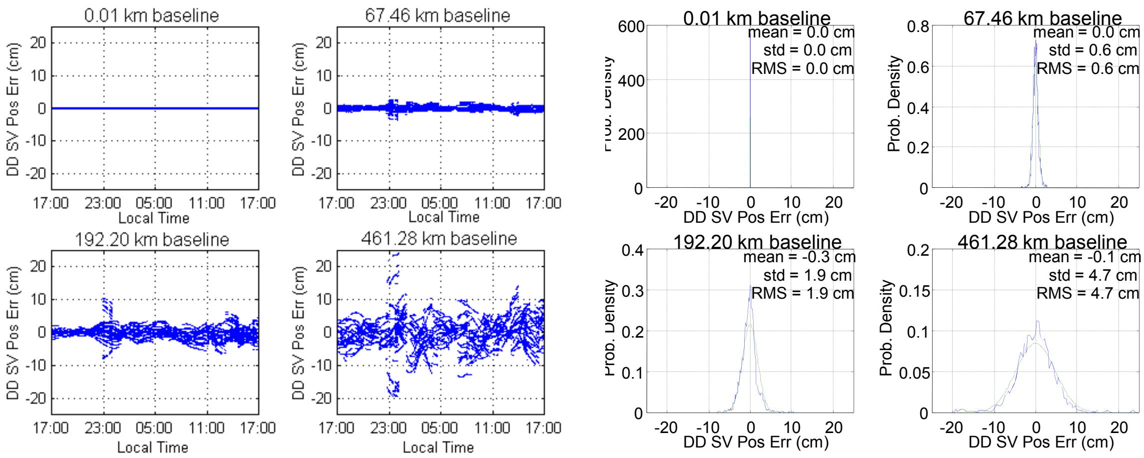
[1]Assuming that only the dither portion of SA is utilized (if SA is on at all!)

Differential Satellite Position Errors

- Satellite position errors are errors in ephemeris that cause calculated SV position to differ from true SV position
  - Absolute (non-DGPS) error
    - Zenith: ~1 m (1-σ)
    - Non-zenith axes: ~3 m (1-σ)
  - For a given measurement, it is the projection of the 3-D SV position error onto the measurement line-of-sight vector that counts
    - With DGPS, line-of-sight vectors converge as reference/mobile baseline distance goes to zero
  - Satellite position error can be determined using precise ephemeris as truth
    - Precise ephemeris accurate to ~10 cm
  - Differential satellite position errors typically less than 5 cm (1-σ), except for very long baselines (> 500 km)
    - True as long as same set of ephemeris is used for both reference and mobile receivers

Sample SV Position DGPS Error (Double Difference)

Data collected in Norway on Sep 30th, 1998

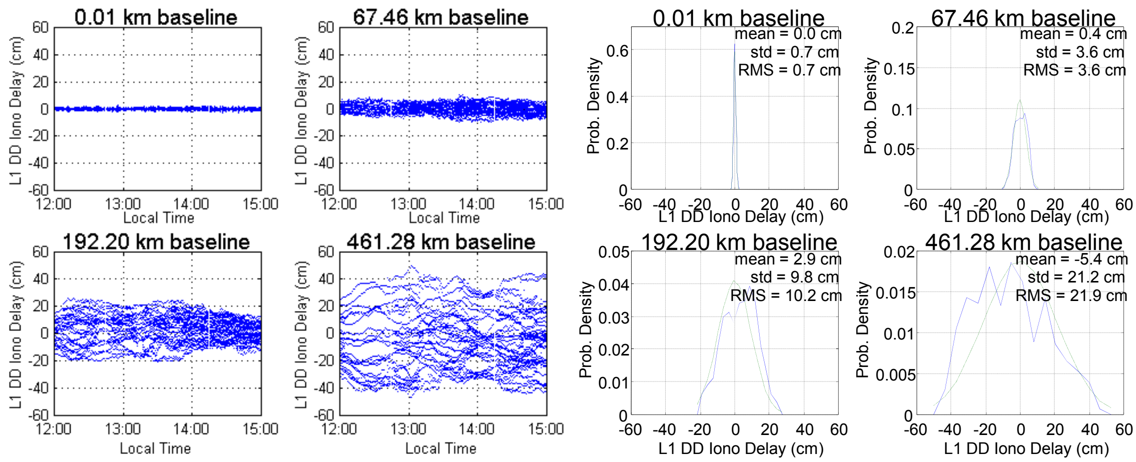


Differential Ionospheric Errors

- Ionospheric errors are spatially correlated
  - Signal from same satellite to two nearby receivers passes through approximately same ionosphere
    - Exception: scintillation
      - Highly local effects
      - Can affect one receiver but not another (unless receivers are collocated)
  - DGPS ionospheric error follows same general trends as overall (non-DGPS) ionospheric error
    - Maximum at ~14:00 local time
    - Minimum at night
    - Varies with solar cycle
  - Ionospheric delay (or phase advance) can be precisely measured using linear combination of phase measurements
    - Requires successful resolution of L1 and L2 carrier-phase ambiguities
    - Accurate to ~1 cm (includes effects of carrier-phase multipath and noise)

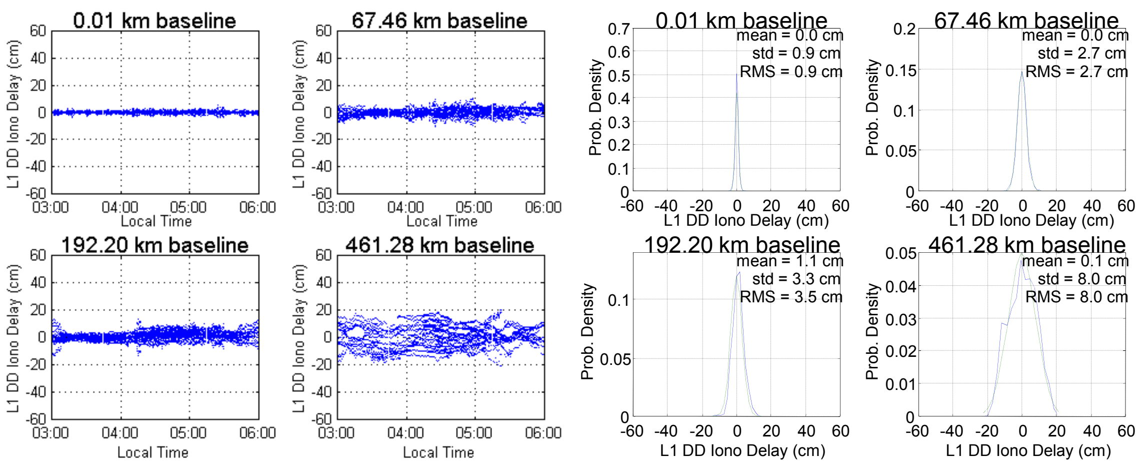
Sample Afternoon Ionospheric DGPS Errors (Double Difference)

Data collected in Norway on Sep 30th, 1998
(between minimum and mid-point of solar cycle)



Sample Nighttime Ionospheric DGPS Errors (Double Difference)

Data collected in Norway on Sep 30th, 1998
(between minimum and mid-point of solar cycle)

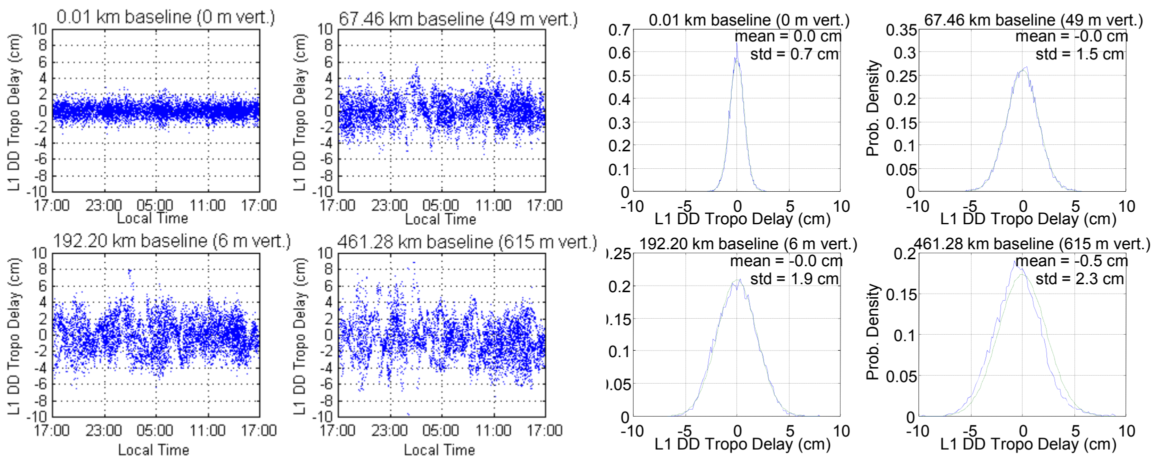


Differential Tropospheric Errors

- Tropospheric errors highly sensitive to altitude of receiver and elevation of satellite
  - Most of the error can be effectively modeled
  - Important to always apply tropospheric model for DGPS
    - If don't apply, then can introduce differential errors on order of meters for receivers at different altitudes
    - Should use same tropospheric model (if possible)
- With a good model, differential tropospheric errors are relatively small
  - Under normal conditions don't exceed ~3 cm (1-σ) for baselines < 500 km
  - Can be worse under extreme conditions (e.g., high humidity)
- Differential tropospheric error can be calculated from carrier-phase measurements
  - Use ionospheric-free combination with precise orbits to remove other errors
  - All that remains is tropospheric error (plus multipath and noise)

Sample Tropospheric DGPS Errors (Double Difference)

Data collected in Norway on Sep 30th, 1998
Modified Hopfield Tropospheric Error Model Applied



DGPS - Practical Considerations

- Reference/remote receiver baseline distance
  - Code DGPS
    - Multipath and Noise Dominates
    - Under normal conditions, other errors don't become significant until baseline reaches 100-200 km.
  - Carrier-phase DGPS (ambiguity resolution)
    - Ambiguity resolution process is based upon tests of measurement residuals
    - Errors (especially biases) in measurements cause significant problems for ambiguity resolution
    - Typical max baseline length to resolve ambiguities effectively (kinematic mode):
      - L1 only: 15-25 km
      - L1 and L2 (widelane): 40-60 km
- Data latency
  - Takes some amount of time for corrections to arrive at mobile receiver for real-time DGPS
  - SA was "fastest-moving" error (when it existed)
    - Max of 19 mm/s^2 acceleration and 2 m/s range rate
    - Data latency of 1 second could cause up to 2 m of DGPS error
  - Sometimes, corrections and time derivatives are transmitted

Overview

What we plan to cover over the next two days

1. GPS Navigation Solutions
2. Differential GPS
3. GNSS Receiver Design
4. Kalman Filtering and Inertial Navigation Systems

Course Overview

- GPS Signal Structure
- General Receiver Overview
- GNSS Signal Processing Overview
- Carrier Tracking Loops
- Code Tracking Loops
- Acquisition
- Tracking Loops
- Bit Synchronization/Frame Synchronization
- Measurement Generation

L1 and L2 Signal Breakdown

- Note: 50 bps navigation message modulated on all of the codes
- L1 signal
  - P(Y)-code
  - C/A-code modulated on carrier that is 90° out of phase from P-code carrier

$$s_{L1}(t) = \underbrace{A_{P_{L1}} Y(t) N(t) \cos(\omega_1 t)}_{\text{P(Y)-Code}} + \underbrace{A_{C/A} CA(t) N(t) \sin(\omega_1 t)}_{\text{C/A-Code}}$$

$$N(t) = \text{50 bps navigation message}$$

$$A_{P_{L1}} = \text{Amplitude of L1 P-code signal} \approx -163 \text{ dBW}$$

$$A_{C/A} = \text{Amplitude of C/A-code signal} \approx -160 \text{ dBW}$$

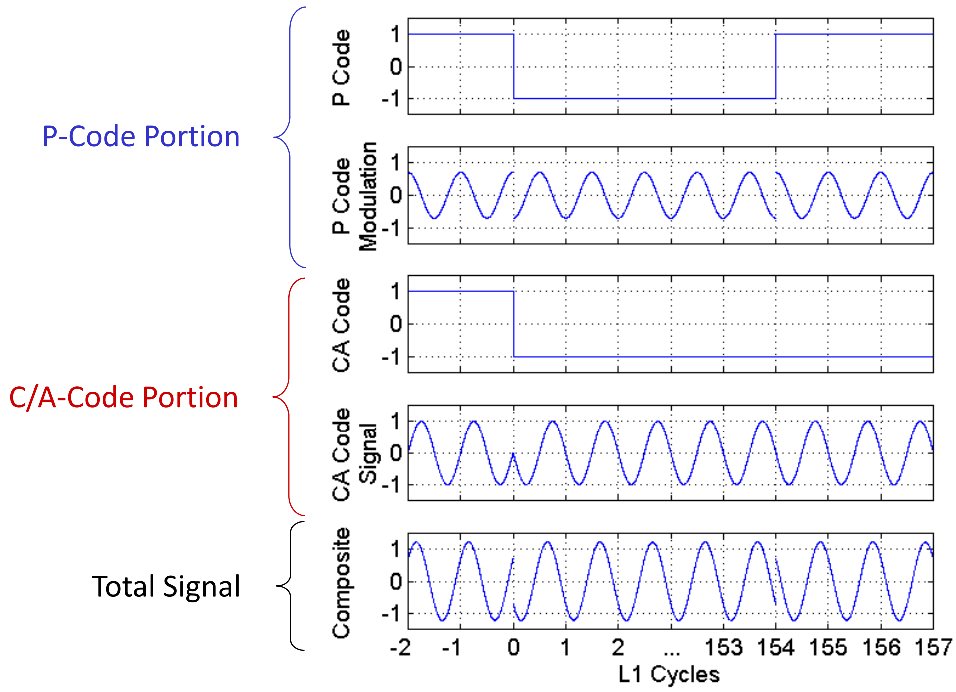
$$\omega_1 = 2\pi f_{L1}$$

- L2 signal
  - P-code only $s_{L2}(t) = \underbrace{A_{P_{L2}} Y(t) N(t) \cos(\omega_2 t)}_{\text{P(Y)-Code}}$

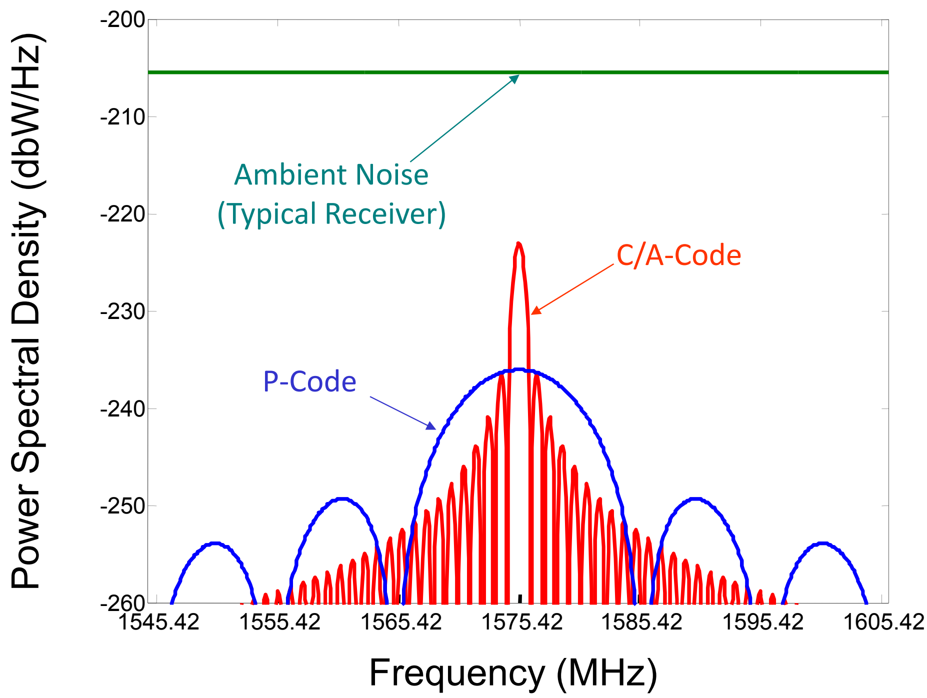
$$A_{P_{L2}} = \text{Amplitude of L2 P-code signal} \approx -166 \text{ dBW}$$

$$\omega_2 = 2\pi f_{L2}$$

Sample of How L1 Signal is Generated



Comparison of GPS C/A-Code and P-Code Power Spectral Densities with Noise

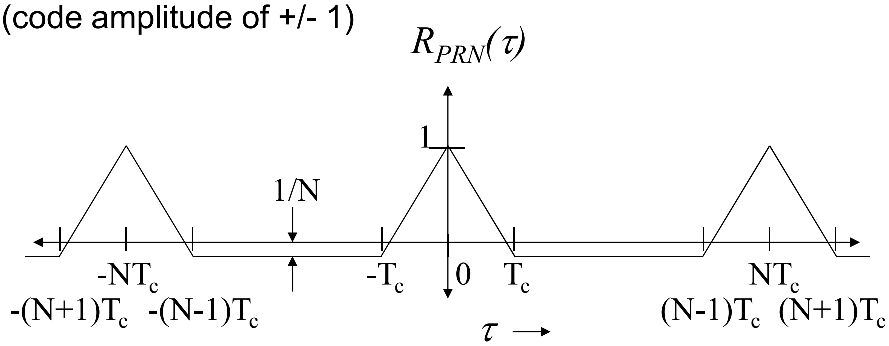


GPS Signal Autocorrelation

- Definition of autocorrelation for function $g(t)$:

$$R(\tau) = \int_{-\infty}^{\infty} g(t)g(t+\tau)dt$$

- Autocorrelation function for maximum length PRN sequence (code amplitude of +/- 1)



T_c = chipping period (1/chipping rate)
NT_c = code repeat period (i.e., repeats after N chips)
(N = 1023 for C/A-code)

GPS Receiver Design Acronyms (Reference)

AFC	Automatic Frequency Control
AGC	Automatic Gain Control
BPF	Band-Pass Filter
CW	Continuous Wave
DLL	Delay Locked Loop
FLL	Frequency Locked Loop
HPF	High Pass Filter
IF	Intermediate Frequency
LHCP	Left Hand Circularly Polarized
LNA	Low Noise Amplifier
LO	Local Oscillator
LPF	Low Pass Filter
NCO	Numerically Controlled Oscillator
PIT	Predetection Integration Time
PLL	Phase Locked Loop
RHCP	Right Hand Circularly Polarized

Definitions: Decibels vs. Ratio

- Definitions:

$$dB = 10\log_{10}(ratio)$$

$$ratio = 10^{\left(\frac{dB}{10}\right)}$$

  – For example, with GPS, power is represented with respect to 1 Watt

$$dBW = 10\log_{10}\left(\frac{power\text{ (Watts)}}{1\text{ Watt}}\right)$$

  – Another measure is dBm (power with respect to 1 mW)

$$dBm = 10\log_{10}\left(\frac{power\text{ (W)}}{0.001\text{ W}}\right)$$

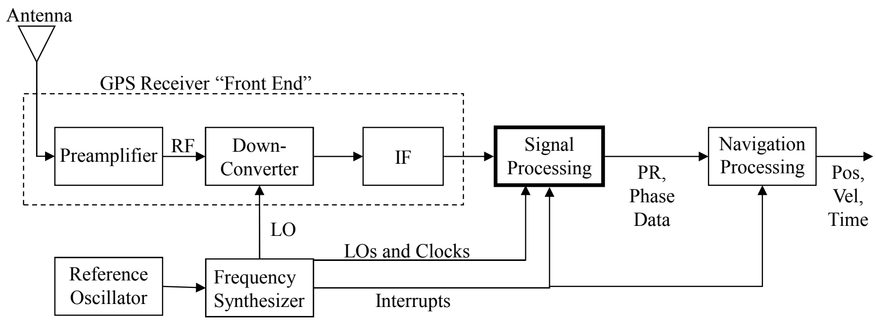
$$dBm = dBW + 30$$

- dB is commonly used
  – Turns multiplication problems into addition problems
  – Power more naturally considered on logarithmic scale
  – Be careful--sometimes a ratio is necessary!

Course Overview

- GPS Signal Structure
- ***General Receiver Overview***
- GNSS Signal Processing Overview
- Carrier Tracking Loops
- Code Tracking Loops
- Acquisition
- Tracking Loops
- Bit Synchronization/Frame Synchronization
- Measurement Generation

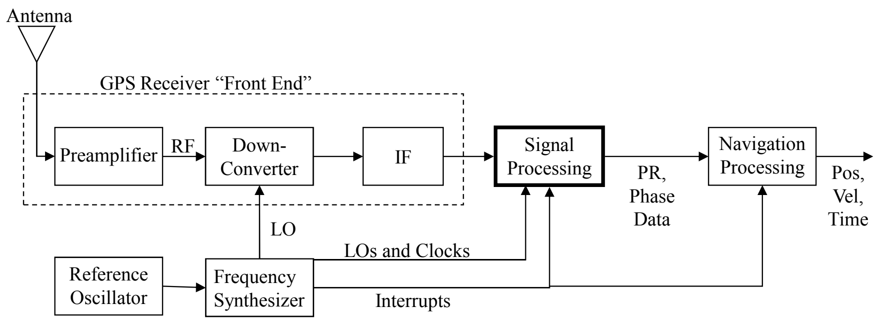
Generic GPS Receiver Functional Block Diagram (1/4)



- Antenna
  - Initial frequency filtering
  - Multipath mitigation
    - Reflected signal often LHCP
    - Steep gain cutoff at low elevations
  - Phase center stability could be important
- Preamplifier
  - Burnout protection
  - Filtering
  - Low Noise Amplifier

Adapted from A.J. Van Dierendonck, "Chap. 8: GPS Receivers," *Global Positioning System, Theory and Applications*, ed. Parkinson et al., AIAA, 1996.

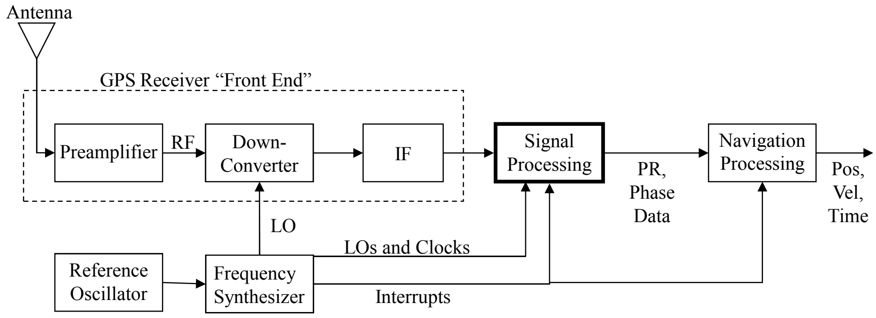
Generic GPS Receiver Functional Block Diagram (2/4)



- Downconverter
  - Converts from RF to some intermediate frequency (IF) that's easier to work with (from a signal processing point of view)
  - Can be single stage or multiple stages
  - Out-of-band filtering is also performed at this point
  - Techniques for rejecting CW interference also effective at this stage
- IF - Intermediate Frequency
  - The signal that is actually sampled and used in the signal processing portion of the receiver

Adapted from A.J. Van Dierendonck, "Chap. 8: GPS Receivers," *Global Positioning System, Theory and Applications*, ed. Parkinson et al., AIAA, 1996.

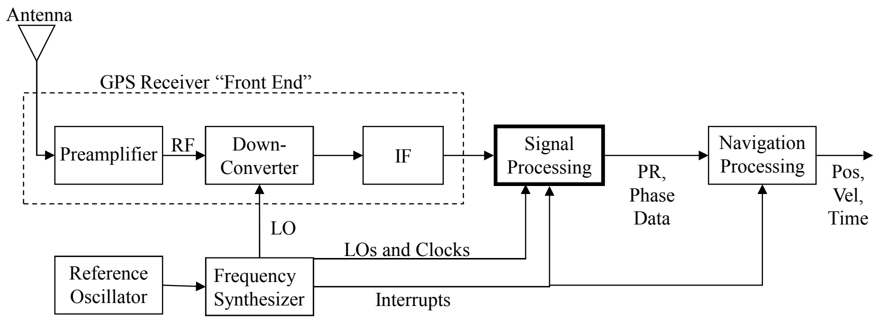
Generic GPS Receiver Functional Block Diagram (3/4)



- Reference oscillator
  - Provides signal at pre-set frequency that's used for basic unit of timing throughout receiver
  - Frequency stability is important
  - Many different types with different performance characteristics
- Frequency synthesizer
  - Generates signals at set frequencies, based upon reference oscillator
  - Implementation of receiver "frequency plan"

Adapted from A.J. Van Dierendonck, "Chap. 8: GPS Receivers," *Global Positioning System, Theory and Applications*, ed. Parkinson et al., AIAA, 1996.

Generic GPS Receiver Functional Block Diagram (4/4)



- Signal processing
  - Generates the pseudorange and phase measurements from the sampled IF signal
  - Normally a combination of hardware and software
  - The "heart" of the GPS receiver
    - Most demanding task of the receiver
    - Design changes here can drastically affect performance
- Navigation processing
  - Generation of position, velocity from pseudorange, phase, and/or Doppler measurements
  - Application dependent

Adapted from A.J. Van Dierendonck, "Chap. 8: GPS Receivers," *Global Positioning System, Theory and Applications*, ed. Parkinson et al., AIAA, 1996.

Course Overview

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- ***GNSS Signal Processing Overview***
- Carrier Tracking Loops
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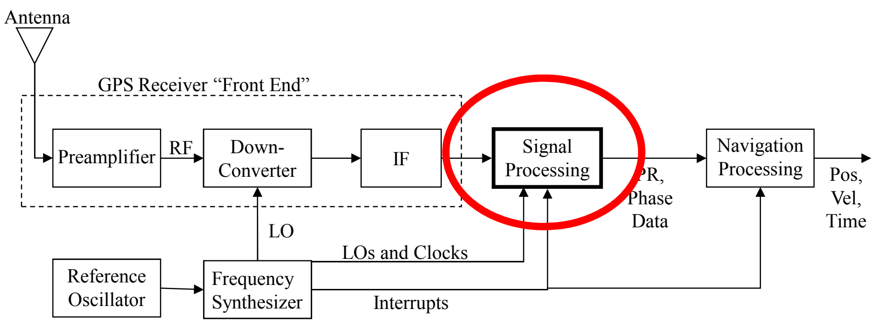
- Measurement Generation

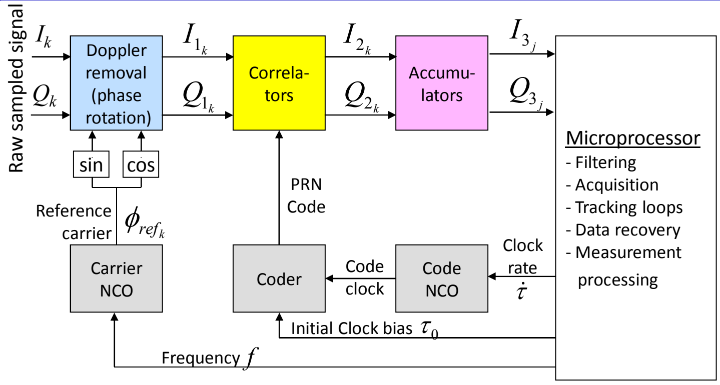
Signal Processing Section



- Goal is to generate measurements from sampled (baseband) signals
  - Pseudorange measurement
    - Calculation of the time that the signal was transmitted
      - "Lock" onto code
      - Once know where you are in code sequence, you know when it was transmitted
    - Pseudorange is current time minus transmission time (times speed of light)
  - Doppler measurement
    - Measure the actual frequency of incoming signal
    - Difference between measured and expected frequency is Doppler
  - Phase measurement
    - Integrate the Doppler measurements

Adapted from A.J. Van Dierendonck, "Chap. 8: GPS Receivers," *Global Positioning System, Theory and Applications*, ed. Parkinson et al., AIAA, 1996.

Signal Processing Section



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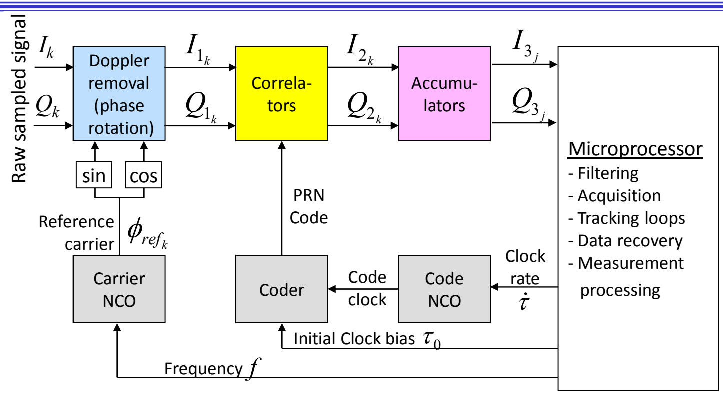
Signal Processing Section



- Two distinct closed loops
  - Phase tracking loop (through carrier NCO)
    - Either a frequency locked loop (FLL), phase locked loop (PLL) or some combination of the two
  - Code tracking loop (through coder)
    - Called the delay locked loop (DLL)
- *Both* loops must be working correctly for anything to work at all!

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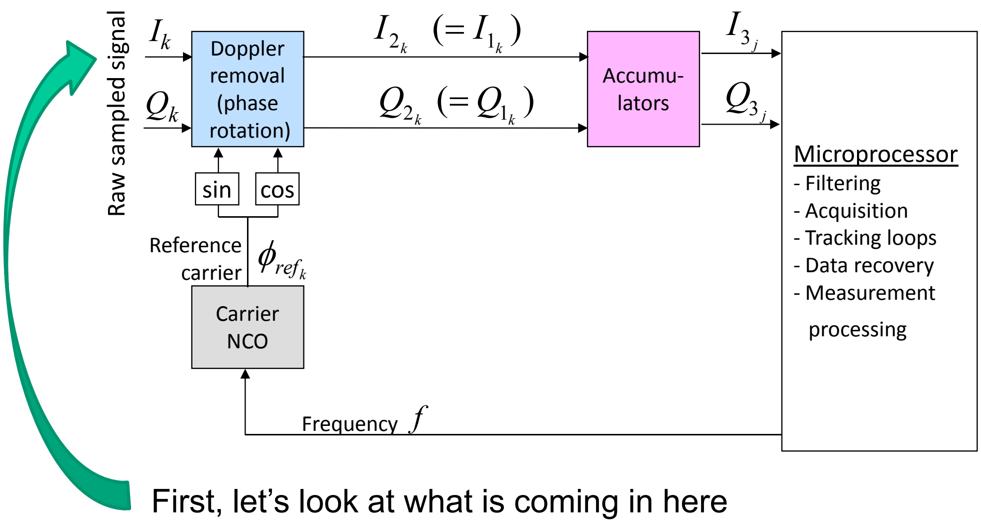
Signal Processing – Phase Tracking Loop

- For now, we'll ignore the code portion of the signal processing section:



First, let's look at what is coming in here

Description of Outputs from Front End

$$I_k = \frac{A}{\sqrt{2}} C_k D_k \cos\left(2\pi(f_B + \Delta f)t_k + \phi_0\right) \qquad Q_k = \frac{A}{\sqrt{2}} C_k D_k \sin\left(2\pi(f_B + \Delta f)t_k + \phi_0\right)$$

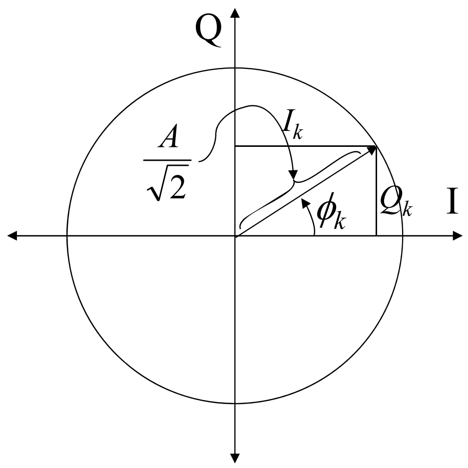
$$\boxed{I_k = \frac{A}{\sqrt{2}} C_k D_k \cos(\phi_k)} \qquad \boxed{Q_k = \frac{A}{\sqrt{2}} C_k D_k \sin(\phi_k)}$$

$$\phi_k = 2\pi(f_B + \Delta f)t_k + \phi_0$$

$A =$ Carrier amplitude
$C_k =$ PRN code samples
$D_k =$ Nav data bit samples
$f_B =$ Baseband frequency (Hz)
$\Delta f =$ Doppler shift (Hz)

- Note: Noise not (yet) represented
- I_k and Q_k have graphical representation

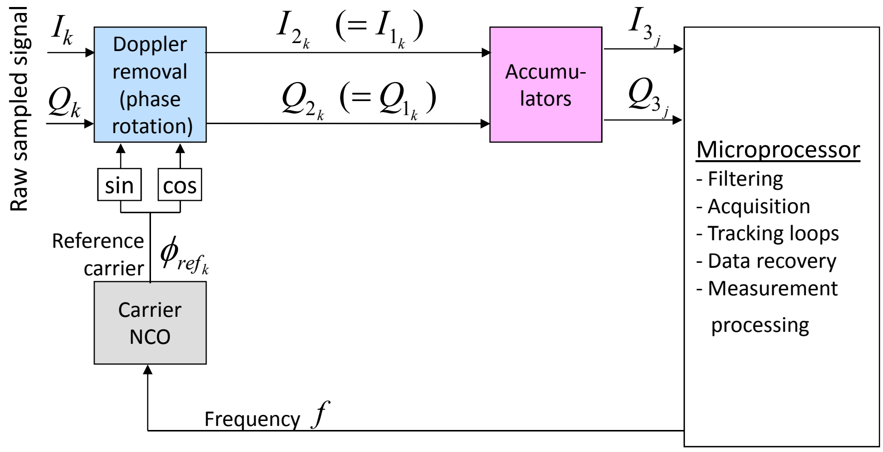
Phasor Diagram: Relationship Between I, Q, and Phase Angle



- How does frequency come into play in this diagram?
- Note: sometimes (often), this is represented as a complex plane
  - I-axis is real axis
  - Q-axis is complex axis
  - Compact representation: $e^{i\phi_k} = \underbrace{\cos\phi_k}_{\text{real}} + i \underbrace{\sin\phi_k}_{\text{imaginary}} = e^{i2\pi(f_B+\Delta f)t_k}$

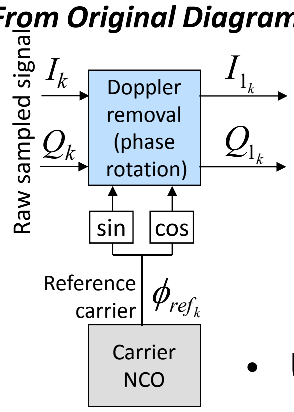
Component-By-Component Description

- Let's look at each component of the phase tracking loop separately
  - Carrier NCO
  - Doppler removal
  - Accumulators
- Later, we'll see how these all work together

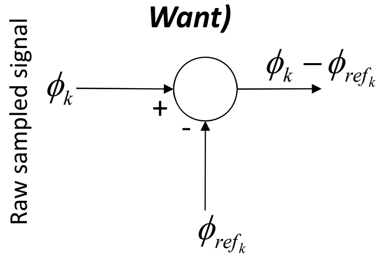


Doppler Removal (1/2)

I and Q Representation (From Original Diagram)



Phase Representation (Conceptually, What We Want)



- Using the above relationship:

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$I_{1_k} = \frac{A}{\sqrt{2}} C_k D_k \cos\left(\phi_k - \phi_{ref_k}\right)$$

$$= \underbrace{\frac{A}{\sqrt{2}} C_k D_k \cos\phi_k}_{I_k} \cos\phi_{ref_k} + \underbrace{\frac{A}{\sqrt{2}} C_k D_k \sin\phi_k}_{Q_k} \sin\phi_{ref_k}$$

$$\boxed{I_{1_k} = I_k \cos\phi_{ref_k} + Q_k \sin\phi_{ref_k}}$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$Q_{1_k} = \frac{A}{\sqrt{2}} C_k D_k \sin\left(\phi_k - \phi_{ref_k}\right)$$

$$= \underbrace{\frac{A}{\sqrt{2}} C_k D_k \sin\phi_k}_{Q_k} \cos\phi_{ref_k} - \underbrace{\frac{A}{\sqrt{2}} C_k D_k \cos\phi_k}_{I_k} \sin\phi_{ref_k}$$

$$\boxed{Q_{1_k} = Q_k \cos\phi_{ref_k} - I_k \sin\phi_{ref_k}}$$

Doppler Removal (2/2)

- Important note: If ϕ_{ref_k} is "tracking" ϕ_k , then

$$\phi_k = \phi_{ref_k}$$

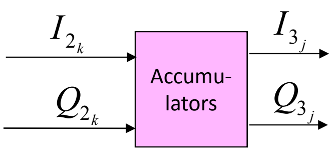
$$I_{1_k} = \frac{A}{\sqrt{2}} C_k D_k \cos(0) = \frac{A}{\sqrt{2}} C_k D_k \quad (+\text{noise})$$

$$Q_{1_k} = \frac{A}{\sqrt{2}} C_k D_k \sin(0) = 0 \quad (+\text{noise})$$

- This shows that I_{1_k} and Q_{1_k} may be useful for
  - Determining if there is phase lock
  - Keeping phase lock
  - – Problem: too much noise at this point
- The term "Doppler Removal" can be somewhat misleading
  - – In reality, its actually "carrier removal", i.e., it removes the entire carrier (Doppler plus carrier at baseband frequency)
  - – It's called Doppler removal because the Doppler is the important part
    - Baseband frequency does not change
    - Doppler changes with clock and vehicle dynamics
    - Doppler is used to generate measurements

John F. Raquet, 2012

Integrate and Dump Process



- Within the hardware, a total of M_E samples are accumulated over T seconds (where T is usually 1-20ms)

$$T_s = \text{Sample period} = \frac{1}{\text{sample rate}}$$

$$T = \text{Accumulation period (typically 1 ms)}$$

$$M_E = \text{Number of samples accumulated}$$
$$= T / T_s$$

- This can be represented mathematically as

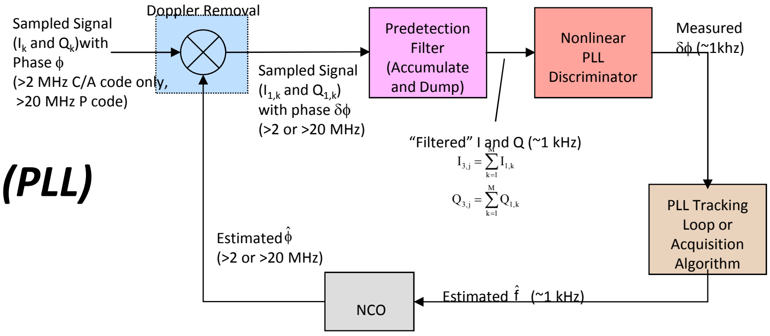
$$I_{3_j} = \sum_{k=(j-1)M_E+1}^{jM_E} I_{2_k} \qquad Q_{3_j} = \sum_{k=(j-1)M_E+1}^{jM_E} Q_{2_k}$$

- Note: This is very similar to an average:

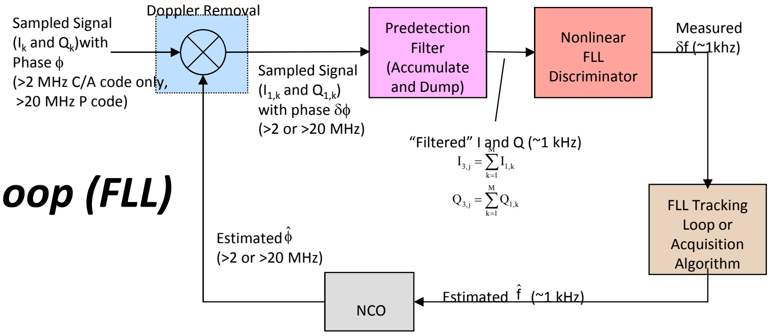
$$I_{3_{j\,average}} = \frac{1}{M_E}\left(\sum_{k=(j-1)M_E+1}^{jM_E} I_{2_k} \right) \qquad Q_{3_{j\,average}} = \frac{1}{M_E}\left(\sum_{k=(j-1)M_E+1}^{jM_E} Q_{2_k} \right)$$

The Phase Locked Loop and Frequency Locked Loop

Phase Locked Loop (PLL)



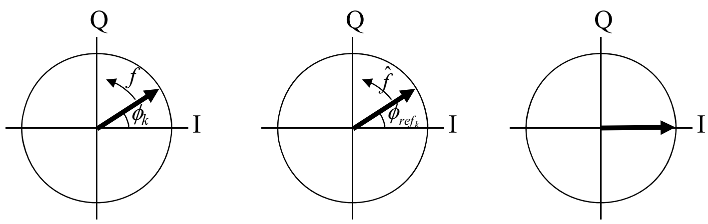
Frequency Locked Loop (FLL)



Phase Locked Loop (PLL)

- Goal is to have reference phase exactly track incoming signal phase

Incoming Signal ***Reference Signal*** ***After Phase Rotation***



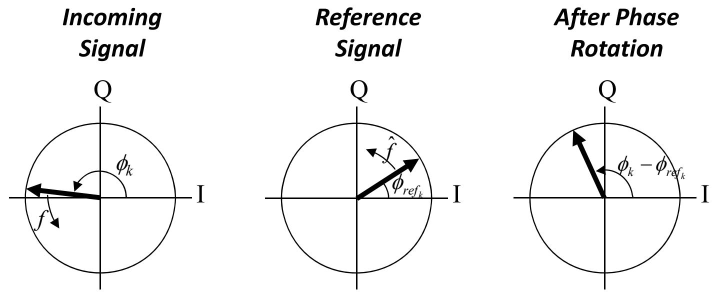
$$\phi_k - \phi_{ref_k} = 0$$

$$f - \hat{f} = 0$$

- Advantages of PLL
  - Precise, low-noise measurement
  - Carrier-phase ambiguities are integers
- Disadvantages of PLL
  - Sensitive to dynamic stress
  - Difficult to initially gain lock (relative to FLL)

Frequency Locked Loop (FLL)

- Goal is to have reference frequency exactly track incoming signal frequency
  - Can be a constant phase offset



$\phi_k - \phi_{ref_k}$ = constant

$f - \hat{f} = 0$

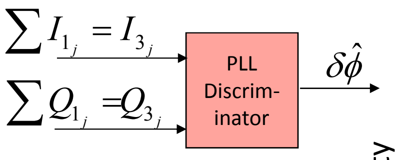
- Advantages of FLL
  - Good under high dynamics
  - Less sensitive to data bit transitions
  - Easier to acquire lock
- Disadvantages of FLL
  - Noisy

Nonlinear Discriminators

- Purpose: To convert from integrated I and Q values to quantities of interest for the tracking loops
  - Phase locked loop: Phase error
  - Frequency locked loop: Frequency error
  - Delay locked loop (will be covered later): Time error
- Computational complexity becoming less and less of an issue
- Noise effects can vary among discriminators
  - Generally, the more linear a discriminator is, the better the signal-to-noise ratio at the output

PLL Discriminators

- Want to use the accumulated I and Q samples to determine the phase offset



Increasing Complexity

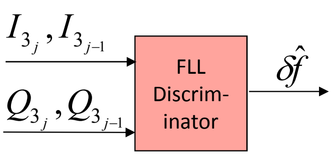
Discriminator Algorithm	Costas Loop*	Output Phase Error	Notes
$\mathrm{sign}(I)\cdot Q$	Yes	$\sin\phi$	Near optimal at high SNR. Slope proportional to signal amplitude A.
$I\cdot Q$	Yes	$\sin 2\phi$	Near optimal at low SNR. Slope proportional to signal amplitude squared A^2
Q/I	Yes	$\tan\phi$	Suboptimal, but good at high and low SNR. Slope not signal amplitude dependent.
$\mathrm{atan}(Q/I)$	Yes	ϕ	Optimal (maximum likelihood estimator) at high and low SNR. Slope not signal amplitude dependent.
$\mathrm{atan2}(Q,I)$	No	ϕ	Optimal (maximum likelihood estimator) at high and low SNR. Slope not signal amplitude dependent.

*Costas Loop: PLL discriminator that is insensitive to 180 degree phase reversals from navigation data. (Predetection integration must not straddle data bit transitions).

Adapted from Ward, "Satellite Signal Acquisition and Tracking," Chapter 5 of *Understanding GPS: Principles and Applications*, Kaplan (ed.), 1996.

FLL Discriminators

- Want to use the accumulated I and Q samples to determine the frequency offset



Translation: $I_{PS1} = I_{3_{j-1}}, I_{PS2} = I_{3_j}$

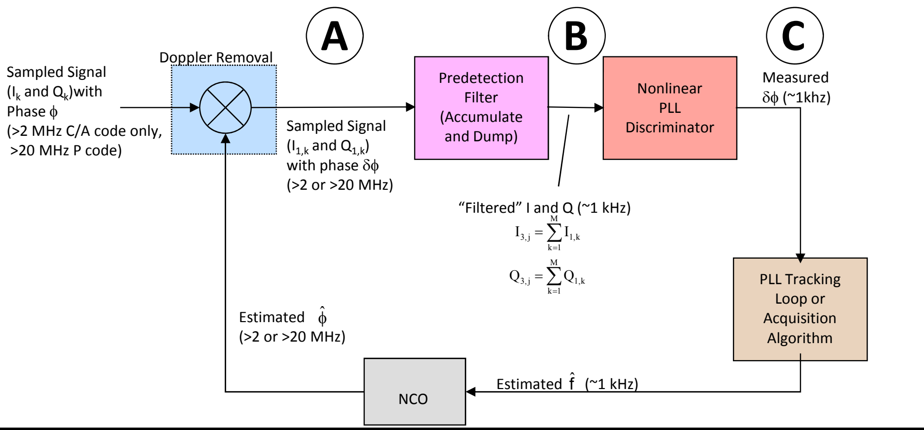
$Q_{PS1} = Q_{3_{j-1}}, Q_{PS2} = Q_{3_j}$

Discriminator Algorithm	*Output Frequency Error*	*Characteristics*
$\dfrac{\mathrm{sign(dot)cross}}{t_2 - t_1}$ where dot $= I_{PS1}\cdot I_{PS2} + Q_{PS1}\cdot Q_{PS2}$ cross $= I_{PS1}\cdot Q_{PS2} - I_{PS2}\cdot Q_{PS1}$	$\dfrac{\sin[2(\phi_2 - \phi_1)]}{t_2 - t_1}$	Near optimal at high SNR. Slope proportional to signal amplitude A. Moderate computational burden.
$\dfrac{\mathrm{cross}}{t_2 - t_1}$	$\dfrac{\sin[(\phi_2 - \phi_1)]}{t_2 - t_1}$	Near optimal at low SNR. Slope proportional to signal amplitude squared A^2. Least computational burden.
$\dfrac{\mathrm{ATAN2(cross,dot)}}{(t_2 - t_1)360}$	$\dfrac{\phi_2 - \phi_1}{(t_2 - t_1)360}$	Four-quadrant arctangent. Maximum likelihood estimator. Optimal at high and low SNR. Slope not signal amplitude dependent. Highest computational burden.

Ward, "Satellite Signal Acquisition and Tracking," Chapter 5 of *Understanding GPS: Principles and Applications*, Kaplan (ed.), 1996.

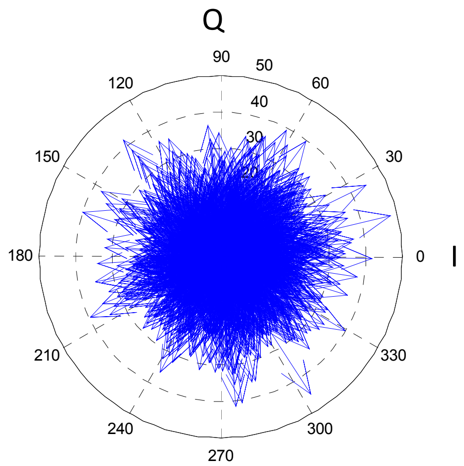
Simulated GPS Receiver Example

- The PL"L" of a GPS receiver was simulated using MATLAB
  - C/A code receiver
  - 5.714 MHz sampling rate
  - 2-bit quantization
  - 2 MHz (2-sided BW) front-end filter
  - NCO has perfect frequency
    - Not actually a tracking loop
    - Equivalent to tracking loop with zero tracking error
- Will examine I and Q values at "A", "B", and "C" to show how the receiver can maintain lock



Sampled Signal After Doppler Removal (Point "A")

Signal to Noise Ratio = -18 dB
1 ms of data (5714 samples)
No quantization



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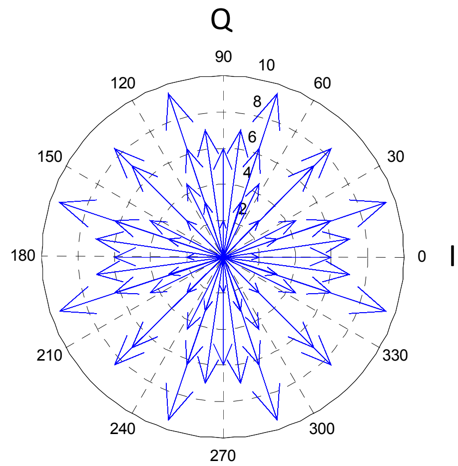
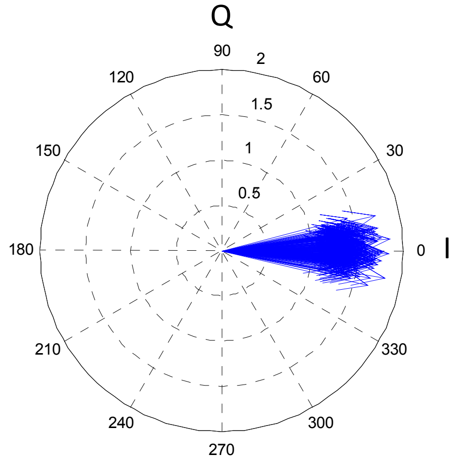


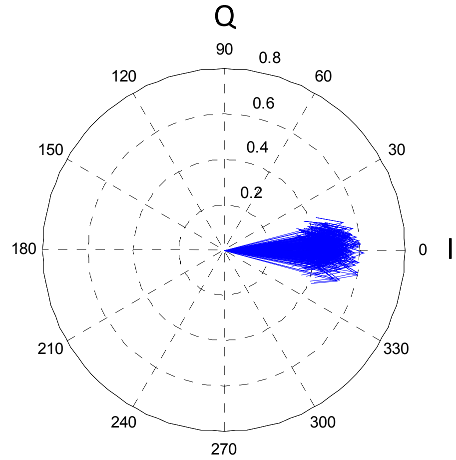
No obvious trends!

Signal After 1 ms Integrate and Dump (Point "B")

Signal to Noise Ratio = +15 dB
250 1 ms PIT* intervals
No quantization



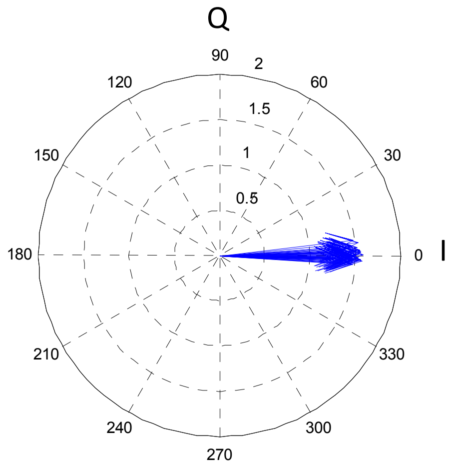
Signal to Noise Ratio = +15 dB
250 1 ms PIT* intervals
2-bit quantization



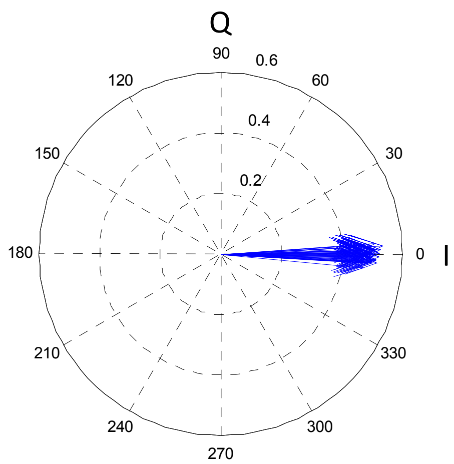
*PIT - Predetection Integration
Time (i.e., the integrate and dump

Signal After 5 ms Integrate and Dump (Point "B")

Signal to Noise Ratio = +22 dB
50 5 ms PIT intervals
No quantization



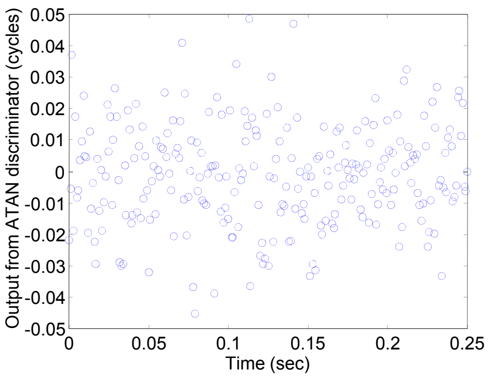
Signal to Noise Ratio = +22 dB
50 5 ms PIT intervals
2-bit quantization



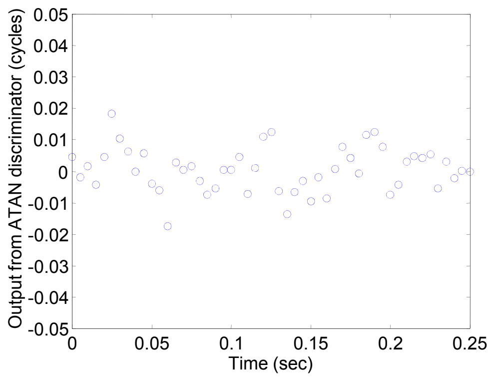
Output of ATAN Discriminator (Point "C")

- Since the PLL NCO is perfect, this should be zero
  - Any non-zero value is due to measurement error

SNR = +15 dB
1 ms PIT
2-bit quantization



SNR = +22 dB
5 ms PIT
2-bit quantization



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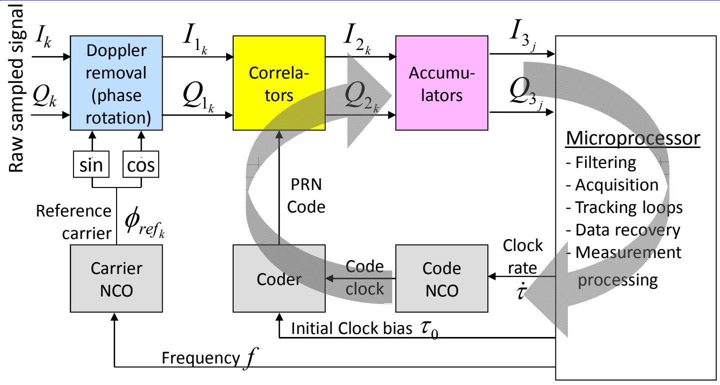
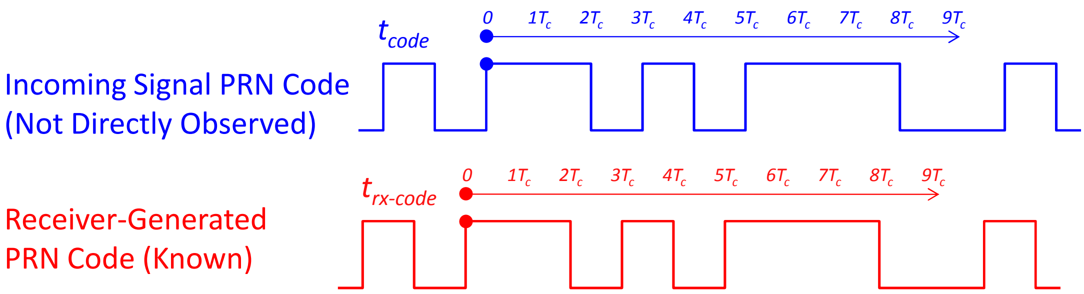
Delay Lock Loop



- Requires “reasonable” tracking from PLL/FLL
- Foundation for generation of pseudorange measurement

Fundamental Question: Code Alignment



- We want to know t_{code}
  - Can’t directly observe it (too much noise)
- We can generate a receiver generated PRN code at a known time ($t_{rx\text{-}code}$)
- If we can synchronize the receiver-generated PRN code to the incoming code, then

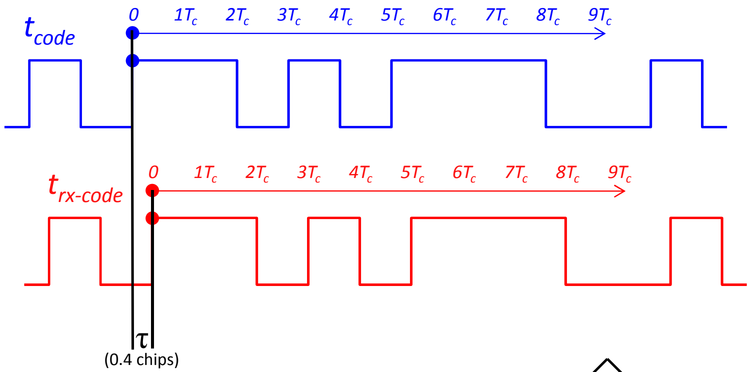
$t_{code} - t_{rx\text{-}code} = 0$ or, equivalently, $t_{code} = t_{rx\text{-}code}$

- What do we need in order to calculate $t_{code} - t_{rx\text{-}code}$?

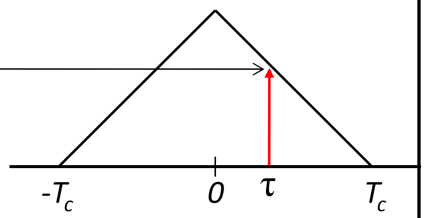
Autocorrelation

Incoming Signal PRN Code (Not Directly Observed)

Receiver-Generated PRN Code (Known)

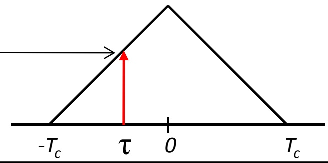


If we correlate the receiver generated code with the incoming code, we can measure the (normalized) magnitude of the correlation function (in this case, 0.6)

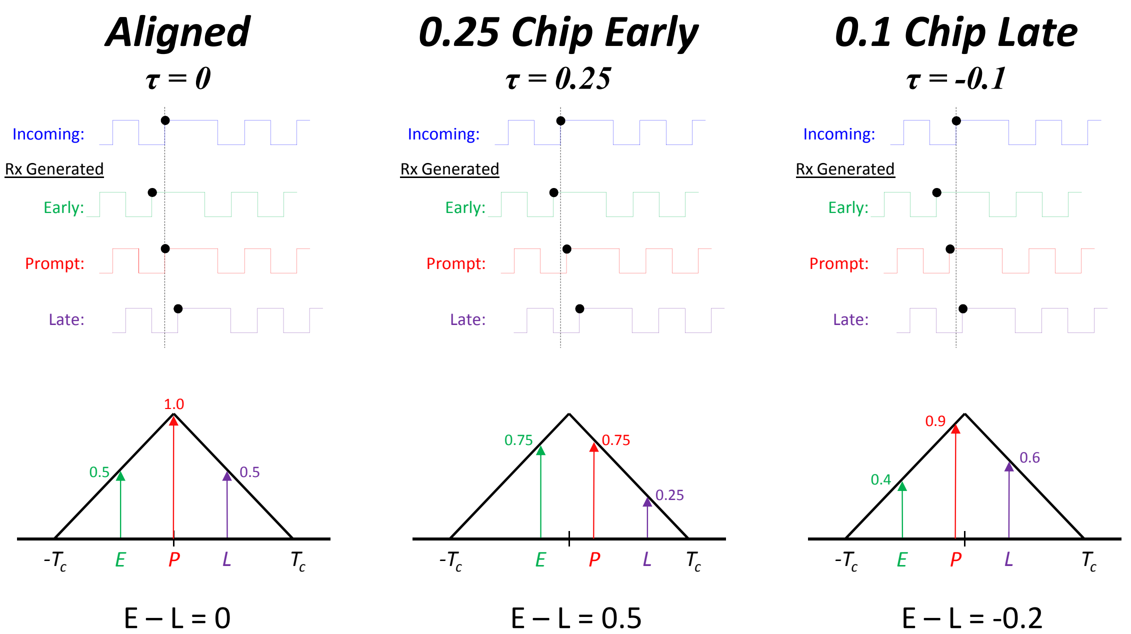


Does this tell us what τ is?

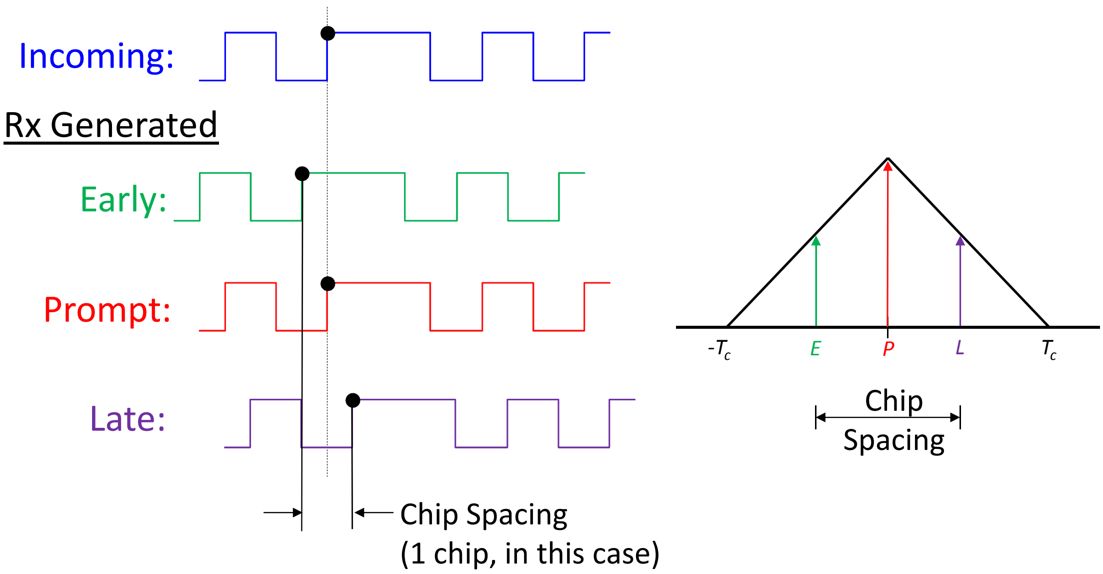
Not exactly. Here's another situation that will give a correlation of 0.6



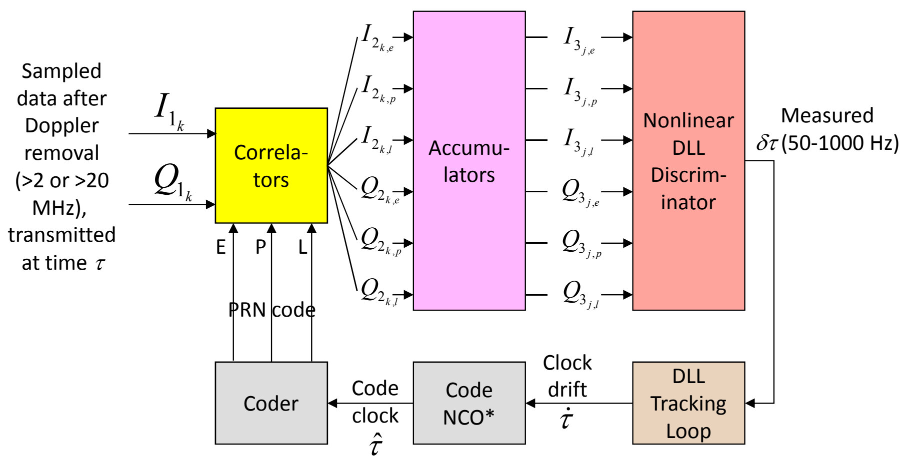
Use of Early, Prompt, and Late Correlator Outputs to Measure τ



Chip Spacing Definition



A More Detailed View of the Delay Lock Loop



*The code NCO outputs a discrete code clock pulse that's sent to the coder. It (or the microprocessor) also keeps track of the current time estimate $\hat{\tau}$.

More on coder implementation later…

DLL Discriminators (1/3)

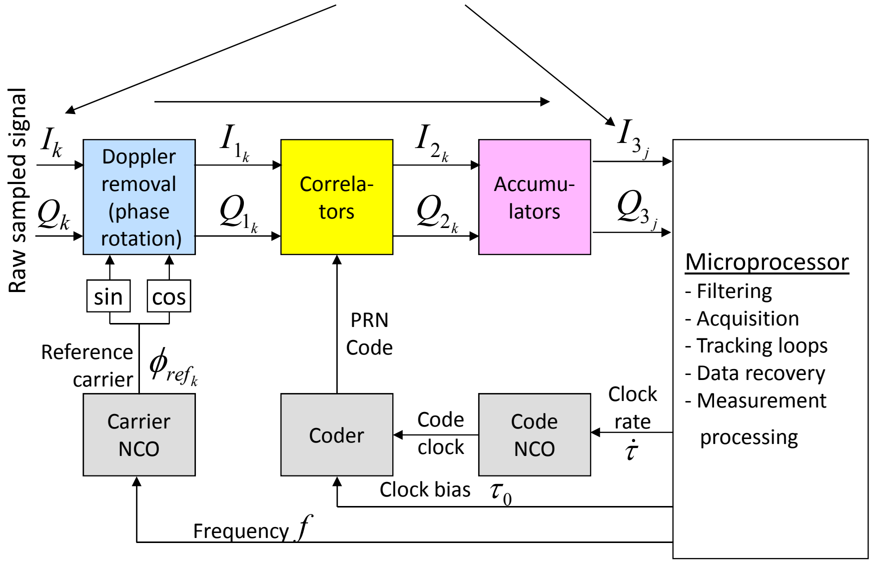
- Two types of DLL discriminators
  - Coherent - requires phase lock (PLL tracking)
    - All power in in-phase portion of signal
    - Can ignore quadrature portion
    - Sometimes used in simple receivers (to reduce number of correlators)
    - Also gives best S/N performance, so is good to use if in carrier lock
  - Non-coherent - does not require phase lock
    - Signal power can be in in-phase or quadrature portion of signal
    - Performance degrades as PLL (or FLL) frequency estimate becomes less accurate
- Samples (from accumulate and dump process)
  - I_E, Q_E: early samples (from integrate/dump)
    - Equivalent to early version of $I_{3_{j,e}}$, $Q_{3_{j,e}}$
  - I_P, Q_P: prompt samples (from integrate/dump)
    - Equivalent to $I_{3_{j,p}}$, $Q_{3_{j,p}}$
  - I_L, Q_L: late samples (from integrate/dump)
    - Equivalent to late version of $I_{3_{j,l}}$, $Q_{3_{j,l}}$
- Early and late samples typically +/- 0.5 chip off from prompt (1 chip spacing)

DLL Discriminators (2/3)

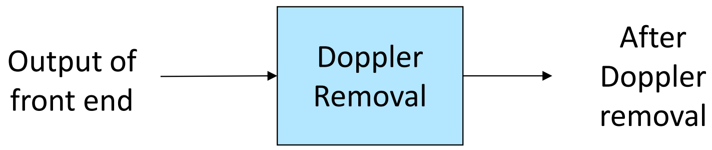
- Coherent Discriminator
  - I_E - I_L
  - Requires all power to be in in-phase part of signal
- Non-coherent discriminators
  - Dot product power → $(I_E - I_L)I_P + (Q_E - Q_L)Q_P$
    - Uses all three correlators
    - Lowest load
    - Some error, but pretty good within 0.5 chip
  - Early minus late power → $\frac{1}{2}\left[\left(I_E^2 + Q_E^2\right) - \left(I_L^2 + Q_L^2\right)\right]$
    - Good within 0.5 chip
    - Moderate computational load
  - Early minus late envelope → $\frac{1}{2}\left[\sqrt{I_E^2 + Q_E^2} - \sqrt{I_L^2 + Q_L^2}\right]$
    - Good within 0.5 chip
    - Higher computational load
  - Normalized early minus late envelope → $\dfrac{\sqrt{I_E^2 + Q_E^2} - \sqrt{I_L^2 + Q_L^2}}{\sqrt{I_E^2 + Q_E^2} + \sqrt{I_L^2 + Q_L^2}}$
    - Good within 1.5 chip (divide by zero at +/- 1.5 chip)
    - Highest computational load

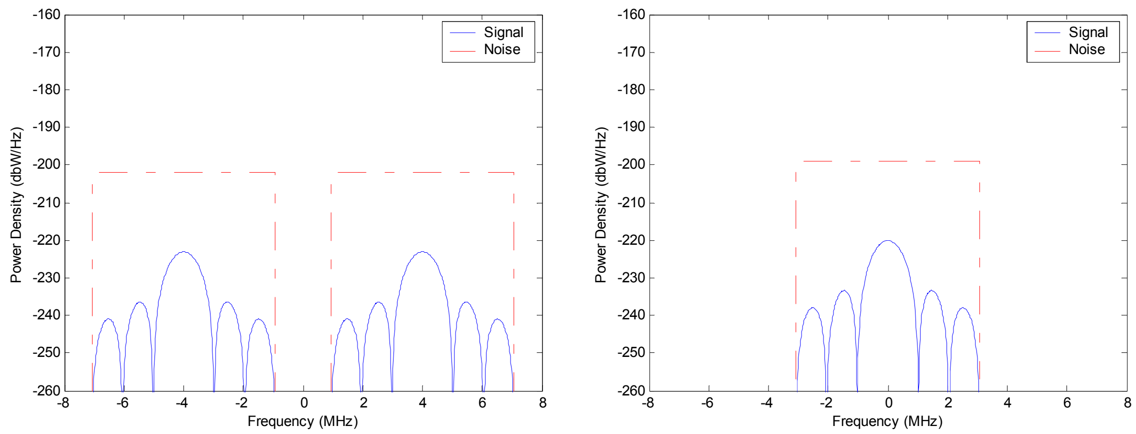
Frequency Spectrum Viewpoint

What is happening to the signal and noise as we move through the front end from here to here?



Signal Spectrum Perspective - Effect of Doppler Removal





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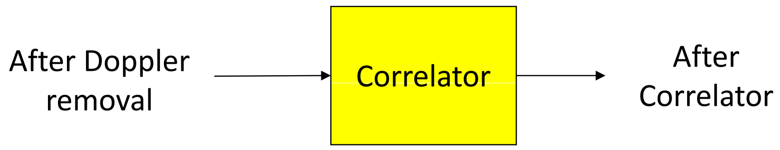
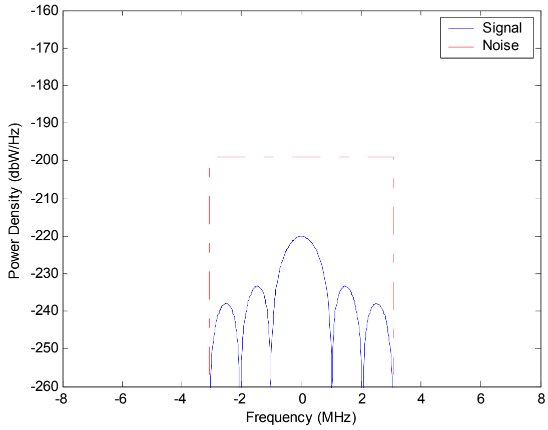
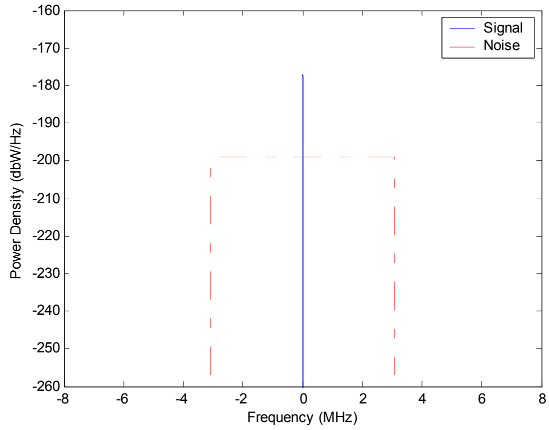
Signal Spectrum Perspective – Effect of Correlators







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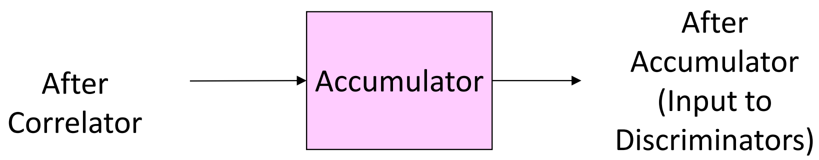
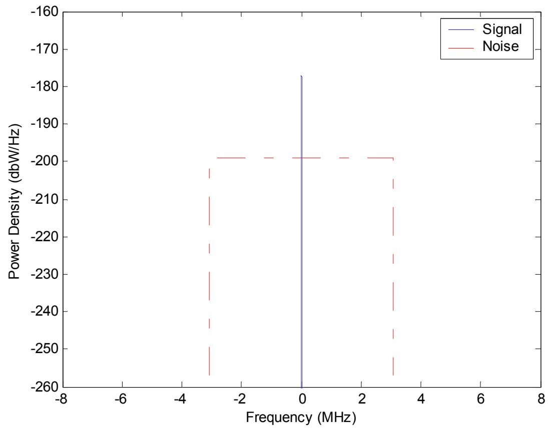
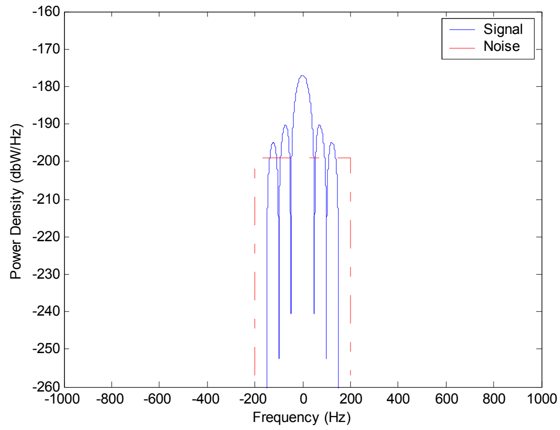
Signal Spectrum Perspective – Effect of Predetection Filter (Accumulate and Dump)







Course Overview

- GPS Signal Structure
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- Code Tracking Loops
- ***Acquisition***
- Tracking Loops
- Bit Synchronization/Frame Synchronization
- Measurement Generation

Signal Acquisition

- Recall that the microprocessor is getting the outputs from the integrate and dump process:

$$I_3 = \sqrt{\frac{2T_s}{M_E N_0}} (\underbrace{I_{3_{sj}}}_{\text{signal}} + \underbrace{I_{3_{nj}}}_{\text{noise}}) = \underbrace{\frac{\sin(\pi f_{error_j} T)}{(\pi \Delta f_{error_j} T)} \sqrt{2 \frac{S}{N_0}} T R(\tau_j) D_j \cos(\pi f_{error_j} T + \Delta\phi_j)}_{\text{signal}} + \underbrace{\eta_{I_j}}_{\text{noise}}$$

$$Q_3 = \sqrt{\frac{2T_s}{M_E N_0}} (\underbrace{Q_{3_{sj}}}_{\text{signal}} + \underbrace{Q_{3_{nj}}}_{\text{noise}}) = \underbrace{\frac{\sin(\pi f_{error_j} T)}{(\pi f_{error_j} T)} \sqrt{2 \frac{S}{N_0}} T R(\tau_j) D_j \sin(\pi f_{error_j} T + \Delta\phi_j)}_{\text{signal}} + \underbrace{\eta_{Q_j}}_{\text{noise}}$$

- Three variables determine the signal portions of the I_3 and Q_3 values
  - $\Delta\phi_j$ – phase error
    - Only determines distribution of signal between I and Q
    - Not a factor if we examine total signal power in both I and Q portions of signal
  - f_{error_j} – frequency error
  - τ_j – code timing error
- No signal will be detected if f_{errorj} or τ_j are not correct (or at least "close")

Signal Acquisition (cont'd)

- In acquisition, normally consider total signal power

$$I^2 + Q^2 \quad \text{or} \quad \sqrt{I^2 + Q^2}$$

  - Why?
- *A priori* knowledge of receiver position and satellite almanacs can greatly speed up acquisition process
  - Know what satellites to look for
  - Have a good prediction of Doppler
- Additional correlators can also speed up correlation process
- Precise time can also be useful
  - Critical for direct P-Code acquisition

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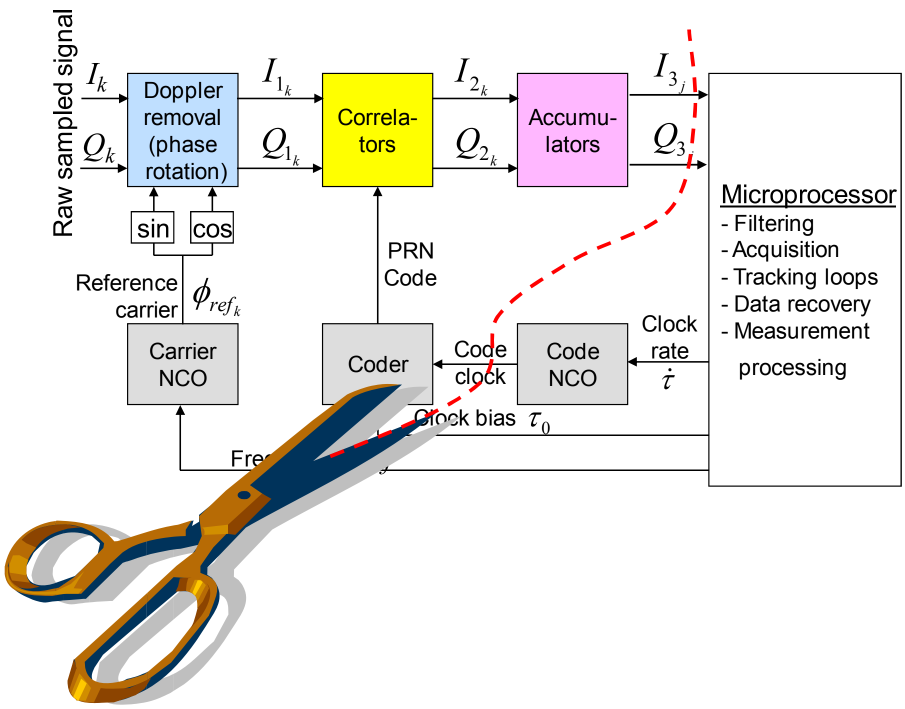
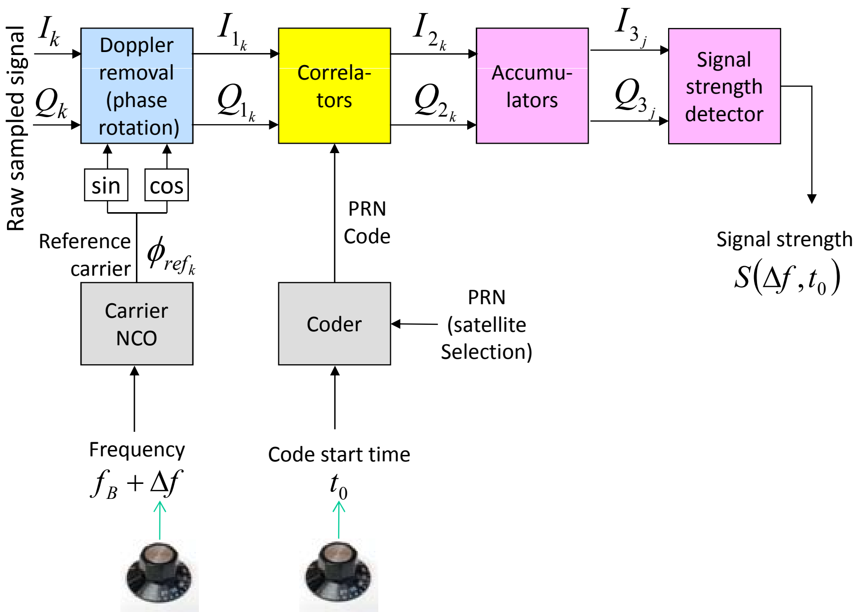
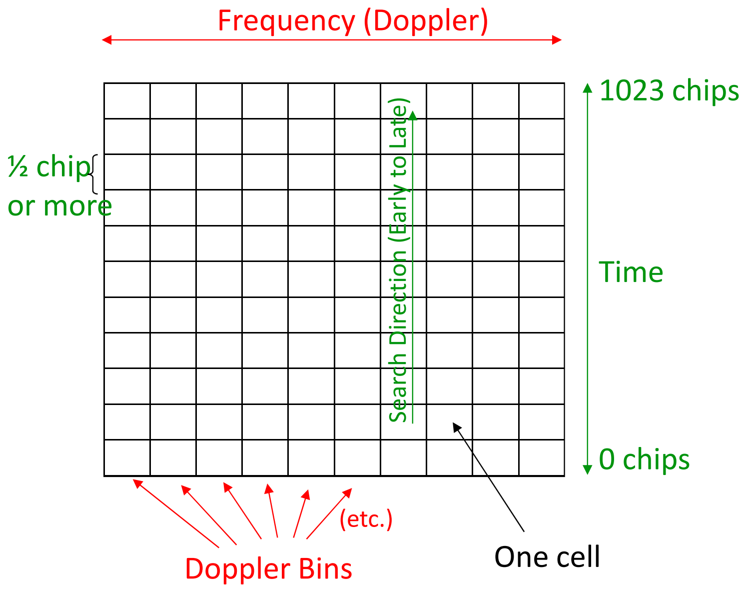
How is Acquisition Accomplished?



A Little More Detail



Acquistion 2-Dimensional Search



- Time is searched early to late to avoid multipath
  - Also easy to implement—just skip a coder clock cycle on occasion

Acquisition Issues

- Tradeoffs to be made

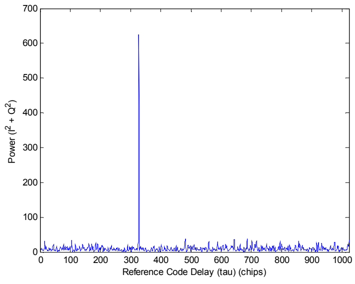
	Signal Not Present	Signal Present
Signal Not Detected	Correct Dismissal	Missed Detection (P_{md})
Signal Detected	False Alarm (P_{fa})	Correct Detection (P_d)

- Variables to control
  - Dwell time – how long you consider a single cell
  - Signal present threshold – power at which you declare a signal present

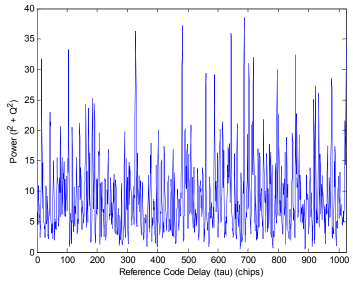
Pretend You are a Receiver…

- Have we acquired a GPS signal?
  - Looking across time samples (single Doppler bin):

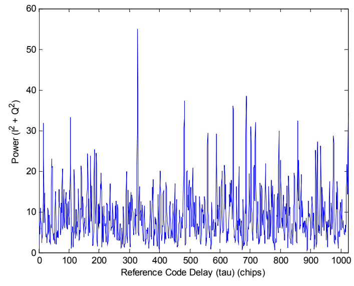
Case 1



Case 2



Case 3

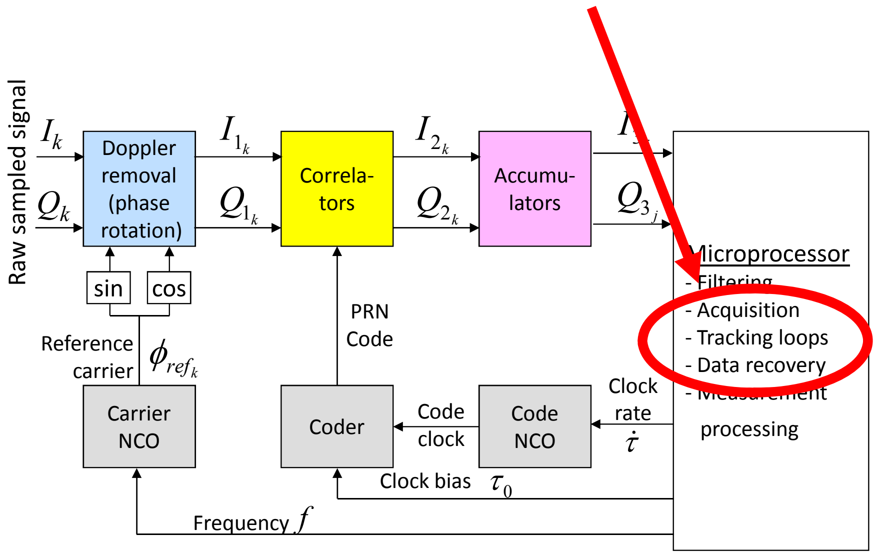


GPS Tracking Loops - Overview

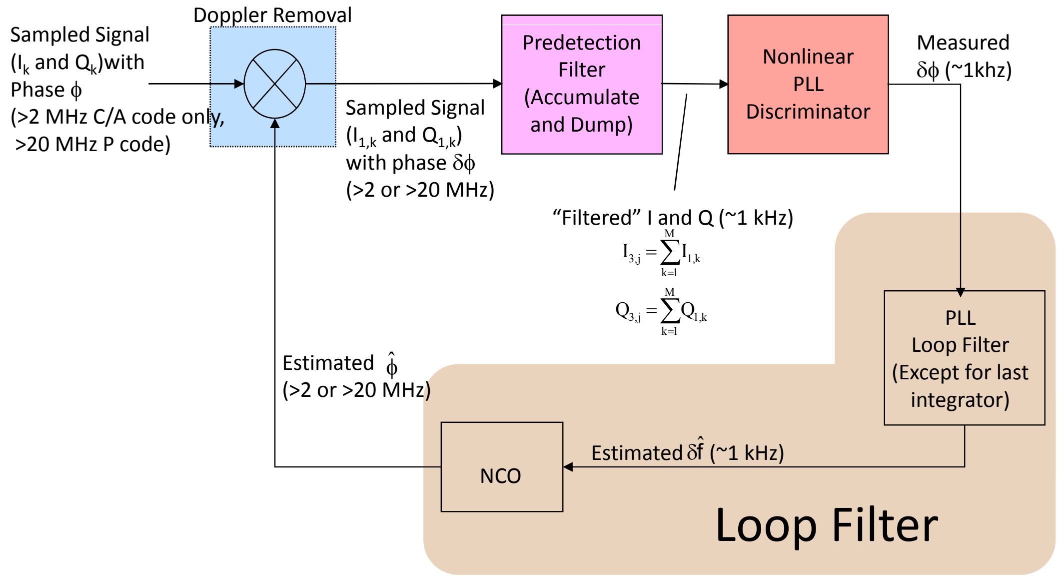
- Overview
- Basic Tracking Loop Design
- Aiding
  - Tracking Doppler and clock drift
  - Carrier-aiding of DLL
  - FLL-assisting PLL
- Tracking loop performance rules of thumb

GPS Tracking Loops – Where Are They?

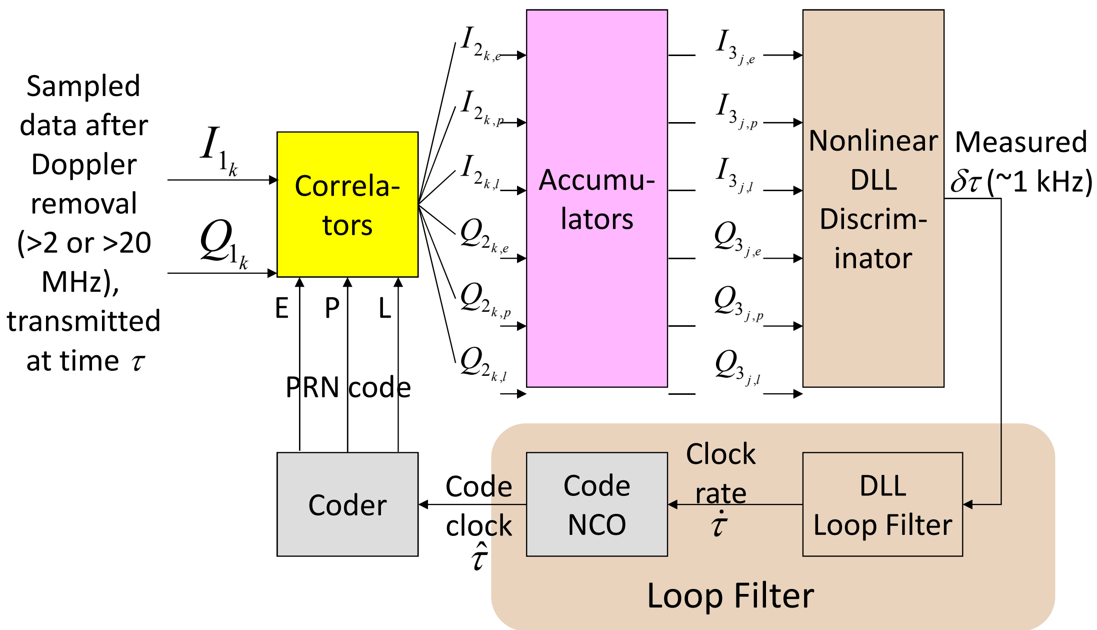
Answer: They are all in software!



PLL Loop Filter

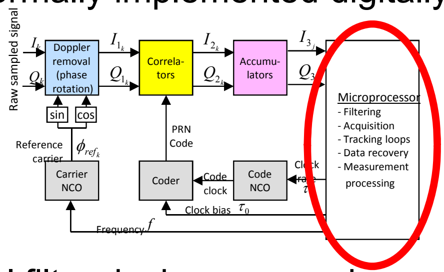


DLL Loop Filter



Loop Filter Design

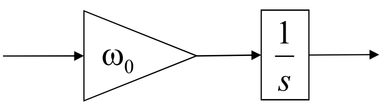
- Loop Filters are normally implemented digitally (within microprocessor)



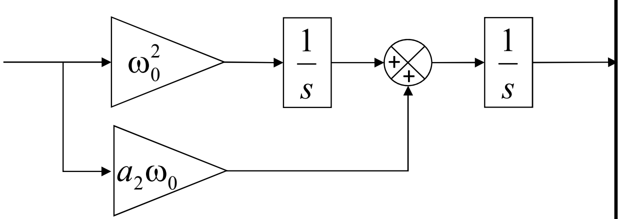
- Can use any digital filter design approach
  - Z-domain methods (root-locus, pole placement, etc.)
  - Analog design → Digital equivalent
    - Easy to understand
    - Approach that we'll discuss
- Goal of loop filter
  - Reduce noise
  - Generate an accurate estimate of the desired signal
  - Generate inputs to coder and NCO

Generic Analog Filter Block Diagrams

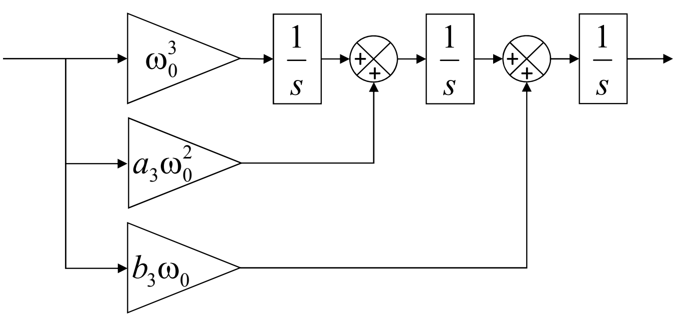
First Order



Second Order



Third Order



Typical Tracking Loop Parameters

- Type of tracking loop chosen depends on
  - Desired tracking performance
  - Desired noise bandwidth (and resulting S/N ratio)
  - Anticipated dynamics of tracking loop

Loop Order	***Noise Bandwidth Bn (rad/sec)***	***Typical Filter Values***	***Steady State Error***	***Characteristics***
First	$\frac{\omega_0}{4}$	ω_0 $B_n = 0.25\omega_0$	$\frac{\dot{R}}{\omega_0}$	Sensitive to velocity stress. Used in aided code loops. Unconditionally stable at all noise bandwidths
Second	$\frac{\omega_0(1+a_2^2)}{4a_2}$	ω_0^2 $a_2 = 1.414$ $B_n = 0.53\omega_0$	$\frac{\ddot{R}}{\omega_0^2}$	Sensitive to acceleration stress. Used in aided and unaided carrier loops. Unconditionally stable at all noise bandwidths.
Third	$\frac{\omega_0(a_3 b_3^2 + a_3^2 - b_3)}{4(a_3 b_3 - 1)}$	ω_0^3 $a_3 = 1.1$ $b_3 = 2.4$ $B_n = 0.7845\omega_0$	$\frac{\dddot{R}}{\omega_0^3}$	Sensitive to jerk stress. Used in all unaided carrier loops. Remains stable at Bn <= 18 Hz

P. Ward, "Satellite Signal Acquisition and Tracking", Chapter 5 of *Understanding GPS: Principles and Applications* (ed. Kaplan), 1996.

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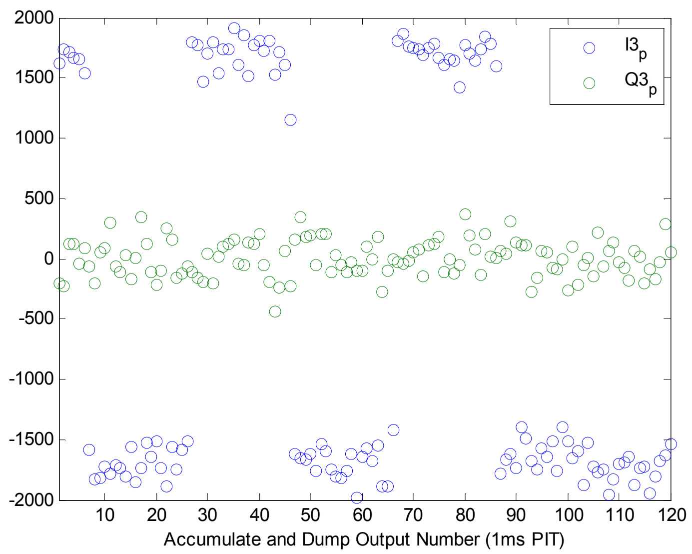
Bit Synchronization

- Need to determine at which 1ms C/A code sequence is the first one in each data bit
- Can be done by watching for 180 deg phase reversals over time
  - Potential for error (especially for weaker signals)
  - Need to calculate it statistically
    - Look for bit changes
    - When found, increment counter for corresponding C/A code sequence by one
    - Only declare bit sync when one of the C/A code sequences stands out from the rest
- Once bit synchronization has occurred, receiver can then start reading (and interpreting) data
  - Will give proper timing information that can be used to resolve the 1ms timing ambiguity

Bit Synchronization Example (High C/N_0 Case)

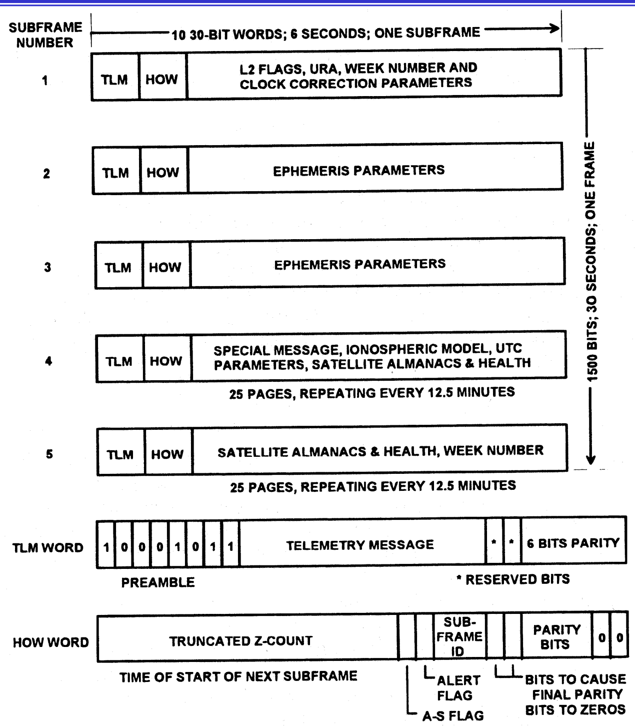


Once we figure out one which integration period the data bit occurs, it will always occur at increments of 20 ms after that

Frame Synchronization

- Once data bit boundaries are known, location in navigation data structure must be determined
  - Sign ambiguity (because we can only determine bit transitions, not absolute sign)
  - How can this be resolved?
- After frame synchronization, we can determine exact time of signal transmission (resolves 1ms ambiguity)
- Alternate method
  - Ambiguity resolution technique

Ephemeris Message Structure



Van Dierendonck, A. J., GPS Receivers, Ch. 8 of Global Positioning System: Theory and Applications, Vol. I, AIAA, 1996.

Example of Data Bits

- Here is an example of 30s worth of GPS data bits (processed using Simulink receiver!)
  - Each row is 1 second
  - Potential preambles are highlighted (both standard and inverted)
  - Where is the beginning of each subframe?

1011101101010011101101011001000110000110000000111
0111010000000100111111111111110000000000000100011101
010000010111100100**01110100**001000**10001011**0000100010
0010000111110110011001010010110101100011000000111
00000100101110010010000100100010010110001101011010
0010111100100100110000100111111100000010001100100000
000101101010010111100000111110111010110001111110011
01001010010111100111000000110110111101000111110100
01**10001011**11111011101011001001001**10001011**0000100010
0010000111110110011001010010010100000000100111111111
10110101001010010101001000000111011101100110001010
10000000000100000000100100011000001100111010111010
011000000011000011100001111100010000010000110110101
10100100101001101110110000000001100011111110010110
111111100011110010000000101000000**10001011**0000100010
00100001111101100110010100011100111111010001111110
10011001011111010101000110110101000010110000001100
101001101110001110111000011000100100000001110110111
11001101001000001111001011001100101011001100001000
11000000111110010100110000011101100110101100001111
00111011011101011100100011101100**10001011**0000100010
00100001111101100110010100010100100111110001001100
00011011111011001010000000111100010001101011111011
01000000101**10001011**11111110100010101111011111001001
101010010101001001110010000001110010110101**01110100**1
10101010111001110010101010110011101101100110100000
01000100111111111111010010111100**10001011**0000100010
00100001111101100110010100001101100111100010001000
1001000000000000**01110100**110000100100101110001110000
11100010010111011111100100110111010101010100110111
1011101101010011101101011001000110000110000000111
011101000000010011111111111111000000000000100101001

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Measurements

- Pseudorange
  - Difference between receiver clock time and time from prompt code generator (code NCO) (multiplied by the speed of light)
    - In our nomenclature: $c\left(t_{rec} - t_{code_{prompt}}\right)$
    - Note that there is a 1ms ambiguity that must be resolved
- Doppler
  - Value taken straight out of carrier NCO (which is keeping track of frequency of carrier)
- Carrier-Phase
  - The "non-carrier" portion of the phase of the signal
  - Will be the integral of the Doppler, but with a specific phase
  - At any point in time the carrier-phase measurement can be calculated by

$$\phi_{meas} = \phi_{ref} - f_{baseband} t_{rec} \qquad (\phi_{ref} \text{ in units of cycles})$$

    - In this equation ϕ_{ref} should continuously be accumulated while the loop is tracking (not wrapped around to zero).
    - Note that there is a 1 cycle ambiguity in the carrier-phase measurement

Overview

What we plan to cover over the next two days

1. GPS Navigation Solutions
2. Differential GPS
3. GNSS Receiver Design
4. Kalman Filtering and Inertial Navigation Systems

Introduction to Kalman Filters
--
Assumptions and Pitfalls

Dr. John F. Raquet
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Air Force Institute of Technology

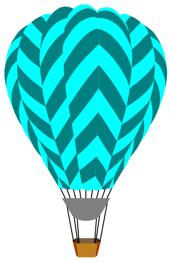
Apr 2012

Kalman Filtering Overview

- Kalman filtering is an estimation approach that can be applied to navigation
  - Many other application areas
- Concepts to be covered
  - Information describing the system
    - State vector
    - Covariance matrix
  - Propagating state and covariance forward in time
  - Using measurements to update the state and covariance
- Assumptions/Limitations

Kalman Filtering:
Information Describing the System (1/2)

- State vector
  - Set of variables that
    - Describe everything you want to know about the system
    - Include all of the information needed to determine how the system changes over time
    - Describe systematic errors in the measurements (anything that's not "noise")
  - Example: Hot air baloon



$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \qquad \begin{aligned} x, y, z &= \text{ENU position of balloon} \\ \dot{x}, \dot{y}, \dot{z} &= \text{ENU velocity of balloon} \end{aligned}$$

  - Does this describe what we want to know?
  - Does this describe how the system changes over time?
  - Would this be a good state vector for a fighter aircraft?
  - Altitude estimation example

Kalman Filtering:
Information Describing the System (2/2)

- Covariance matrix
  - The covariance matrix basically describes how well the state is known
    - If the system only gives a state output, it's not that useful.
    - If it outputs the state and tells how accurate it is, then you have information that you can confidently act upon.
    - Hot air balloon example: the system state tells me that I'm 300 m above the ground descending at a rate of 10 m/sec.
      - Need to know covariance matrix as well.
        - » Case 1: Position accuracy = 10 m 1- σ, velocity accuracy = 1 m/sec 1-σ $\rightarrow$ probably not in danger until ~30 seconds
        - » Case 2: Position accuracy = 400 m 1- σ, velocity accuracy = 15 m/sec 1- σ $\rightarrow$ you could hit the ground any second!
  - How to interpret covariance matrix
    - Diagonal terms are the error variances of the estimated states
    - Off-diagonal terms are cross-covariances, describing the correlations of the errors between the states

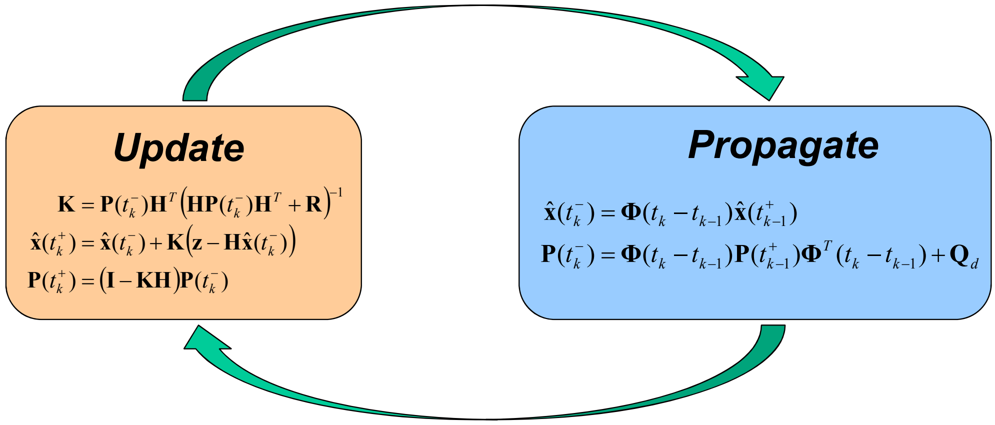
Kalman Filtering: Propagating Covariance and State Forward in Time

- State vector and covariance matrix can be propagated forward in time
  - If you know the current state estimate, you can determine the state estimate at a point in the future
  - If you know the current covariance matrix, you can determine the covariance matrix at a point in the future
  - Information about how the state and covariance changes over time is given in
    - Dynamics matrix $\mathbf{F}$: $\dot{\mathbf{x}} = \mathbf{F}\mathbf{x}$
    - State transition matrix Φ: $\mathbf{x}(t_1) = \mathbf{\Phi}(t_1 - t_0)\mathbf{x}(t_0)$
  - When propagating covariance forward in time, *process noise* is added to account for
    - Unmodeled dynamics
    - Unmodeled system inputs
    - Anything else that decreases the ability to predict the future state using the current state
  - Process noise increases uncertainty (i.e., larger covariance values)

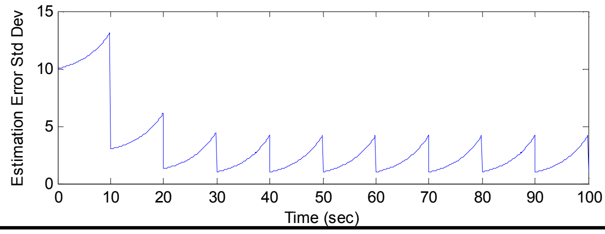
Kalman Filtering: Measurement Updates

- A measurement gives information about the state values
  - Examples: GPS pseudorange (for position or clock bias) or Doppler (for velocity or clock drift)
- Effects of a measurement update
  - State values are adjusted to reflect the measurement
  - Covariance matrix is adjusted to reflect how well the state is known, now that the measurement is available
    - Measurements always decrease uncertainty (i.e., smaller covariance values)
- Measurement noise
  - Description of how precise the measurement is
  - The effect of measurement on state and covariance determined by tradeoff between
    - Measurement noise (how good the measurement is)
    - Covariance matrix (how well the state is known at this point)
- Relationship between measurement and states given by $\mathbf{H}$ matrix (same as least-squares)

The Kalman Filter Iteration



Example of Estimation Error Over Time



Measurement Model

Linear Measurement Model

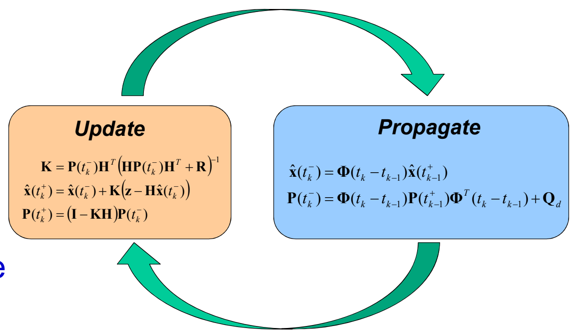
$$\mathbf{z} = \mathbf{H}\mathbf{x} + \mathbf{v}$$

measurement

sensitivity matrix

state

meas noise



Measurement noise is described by the measurement noise covariance matrix:

$$\mathbf{R} = E[\mathbf{v}\mathbf{v}^T] = \begin{bmatrix} E[v_1^2] & E[v_1 v_2] & \cdots & E[v_1 v_n] \\ E[v_1 v_2] & E[v_2^2] & & \\ \vdots & & \ddots & \\ E[v_1 v_n] & & & E[v_n^2] \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & & \\ \vdots & & \ddots & \\ \sigma_{1n} & & & \sigma_n^2 \end{bmatrix}$$

- Key assumptions (in semi-nontechnical language)
  - Measurement errors $\mathbf{v}$ are Gaussian (follow a “bell curve”)
  - Measurement errors $\mathbf{v}$ are “white” (completely random from measurement to measurement)
  - Measurement model is linear

What if Measurements are Non-Linear

- Example of non-linear measurements: a range (distance) measurement (such as with GPS)
- Can use non-linear measurement model

Nonlinear

$$\mathbf{z} = \mathbf{h}(\mathbf{x}) + \mathbf{v}$$

Linear

$$\mathbf{z} = \mathbf{H}\mathbf{x} + \mathbf{v}$$

- Kalman filter is then modified to become an "Extended Kalman Filter" (EKF)
  - Requires linearization about the estimated solution
  - Because of this, an EKF is not, technically speaking, truly optimal like the KF
  - In many cases it would be "nearly optimal"—depends on the nature of the linearization

Dynamics Model

Discrete-Time Dynamics Model

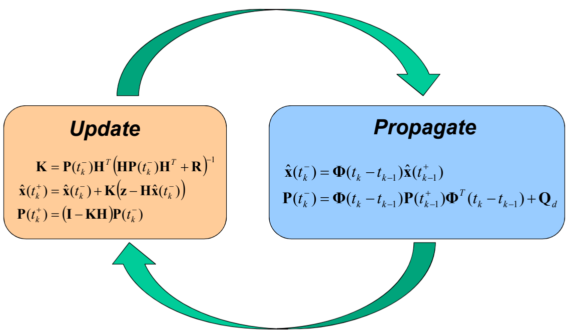
$$\mathbf{x}(t_k^-) = \mathbf{\Phi}\mathbf{x}(t_{k-1}^+) + \mathbf{w}_d$$

state after propagation

dynamics matrix

state before propagation

process noise



Process noise is described by the measurement noise covariance matrix:

$$\mathbf{Q}_d = E[\mathbf{w}_d \mathbf{w}_d^T] = \begin{bmatrix} E[w_1^2] & E[w_1 w_2] & \cdots & E[w_1 w_n] \\ E[w_1 w_2] & E[w_2^2] & & \\ \vdots & & \ddots & \\ E[w_1 w_n] & & & E[w_n^2] \end{bmatrix}$$

- Key assumptions (in semi-nontechnical language)
  - Process noise $\mathbf{w}_d$ is Gaussian (follow a "bell curve")
  - Process noise $\mathbf{w}_d$ is "white" (completely random from epoch to epoch)
  - Dynamics model is perfectly known

Initialization and "Time Constant" of a KF

- Things needed in order to initialize a filter
  - Initial state estimate
  - Initial covariance matrix
  - Measurement model(s)
  - Propagation model(s)
- Time constant (not meant in a precise, technical way)
  - Defines how long a measurement will affect the filter
  - In theory, every measurement will affect the filter for the rest of time
  - In practice, this may not be the case so much
    - Example: Case in which there is high propagation noise—old measurements are significantly "de-weighted" relative to new measurements
  - Warning: Even in a case where a filter has a "short" time constant (i.e., measurements lose impact fairly quickly), a large measurement error (blunder) can have a devastating impact

One Final Thought…

Life is a Kalman Filter

(really, it is!)

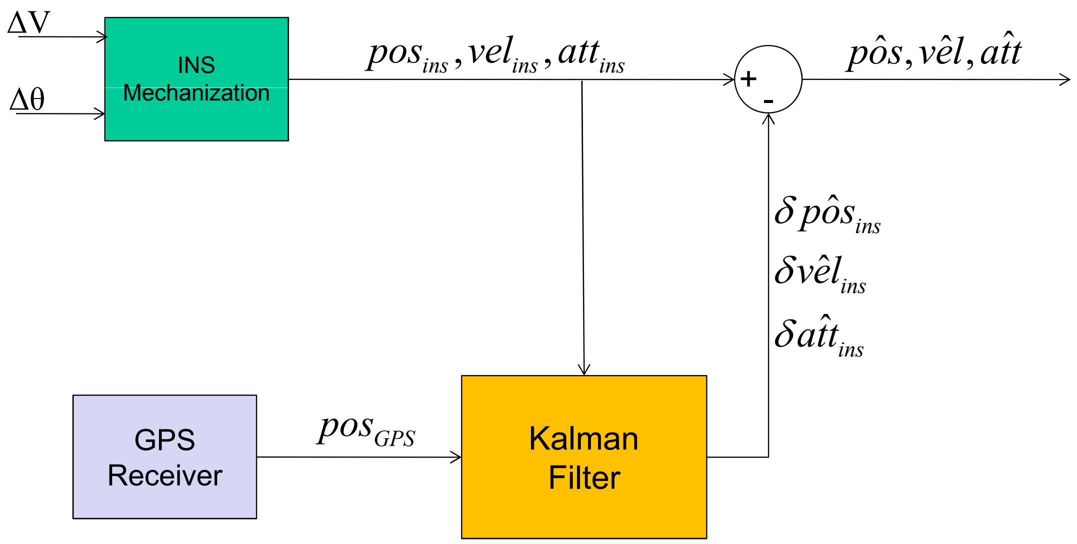
Inertial Navigation Systems

- Sensors
  - Accelerometers
    - Measure specific force f = a + g
  - Gyroscopes
    - Measure rotation about an inertial frame
  - Altitude aiding (required!)
    - Normally a barometric altimeter, but can be other things
- Mechanization equations
  - Attitude computation
  - Resolution of accelerometers into desired frame
  - Subtraction of gravity
  - Double integration
  - Accounting for rotation as vehicle moves around Earth
  - Schuler oscillation

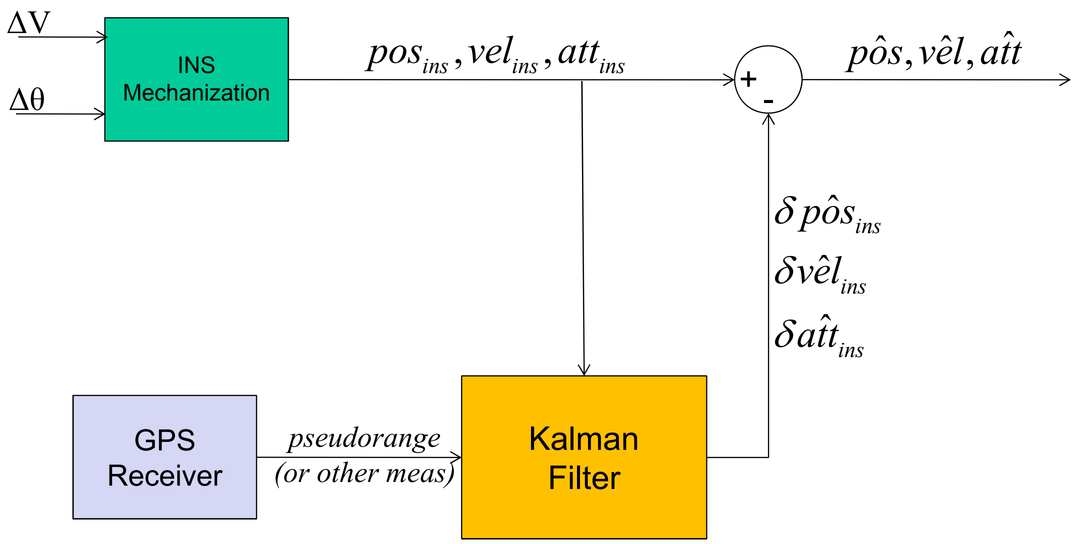
Error Characteristics of Inertial Systems

- Very good high-frequency characteristics
- Long-term drift (poor low-frequency characteristics)
- Categorization of inertial systems
  - Navigation-grade
  - Tactical-grade
  - Commerical-grade
- All inertial systems have errors that grow unbounded unless aided by another sensor
- What would be the ideal sensor?
  - Good low-frequency characteristics (little long term drift)
  - Doesn't necessarily need to have good high-frequency characteristics
- Good candidate: GPS!

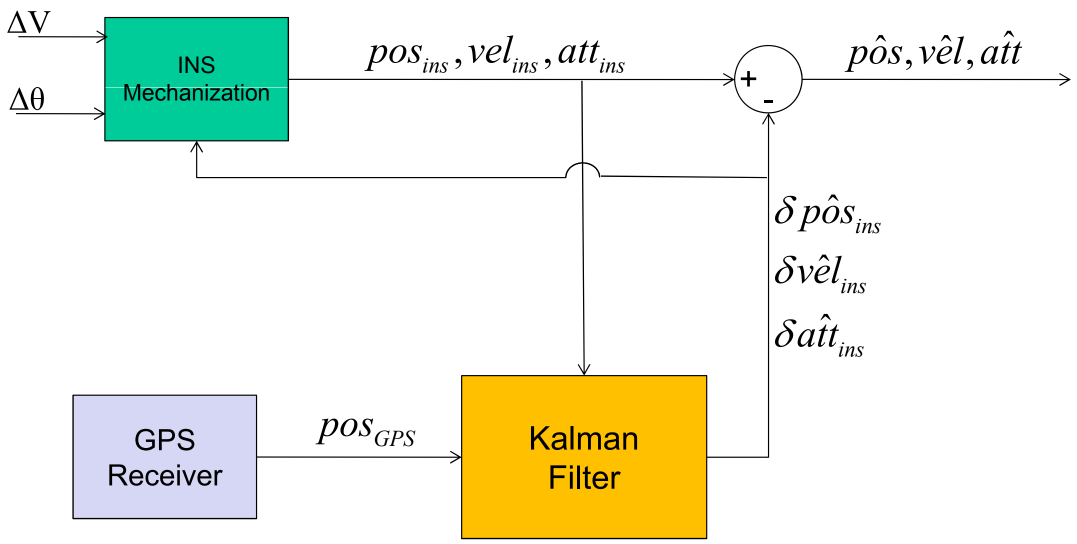
Loosely Coupled, Feed-Forward INS/GPS Integration Approach



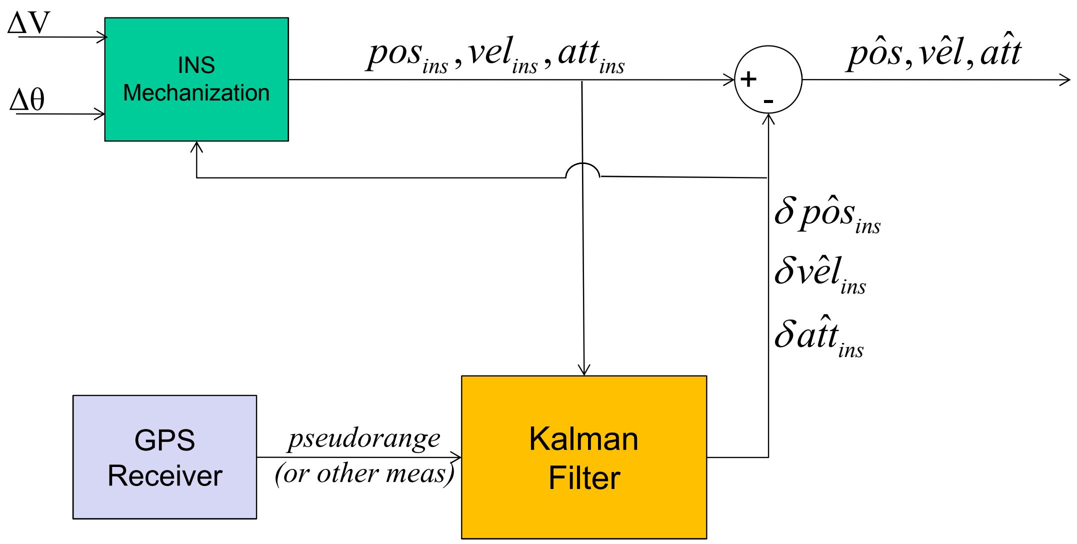
Tightly Coupled, Feed-Forward INS/GPS Integration Approach



Loosely Coupled, Feedback INS/GPS Integration Approach



Tightly Coupled, Feedback INS/GPS Integration Approach



Review

<u>What we plan to cover over the next two days</u>

1. GPS Navigation Solutions
2. Differential GPS
3. GNSS Receiver Design
4. Kalman Filtering and Inertial Navigation Systems