Birkhoff sums for interval exchange maps: the Kontsevich-Zorich cocycle (II)

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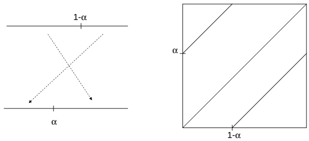

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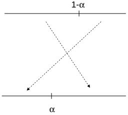

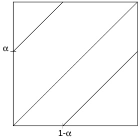

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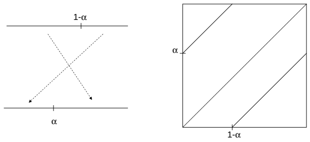

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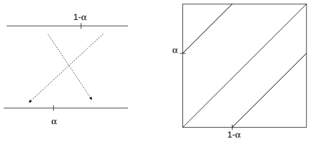

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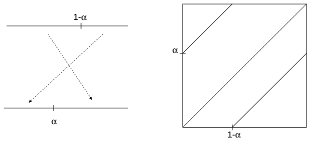

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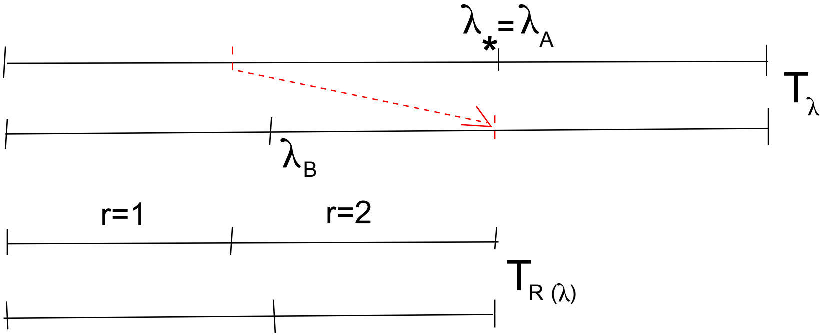
We can thus define a *renormalization map*:
$\mathcal{R} : \mathbb{R}_+^2 - \{\lambda_A = \lambda_B\} \to \mathbb{R}_+^2$

$$\mathcal{R}(\lambda_A, \lambda_B) = \begin{cases} (\lambda_A - \lambda_B, \lambda_B) & \text{if } \lambda_A > \lambda_B, \\ (\lambda_A, \lambda_B - \lambda_A) & \text{if } \lambda_A < \lambda_B. \end{cases}$$

The case $\lambda_A > \lambda_B$



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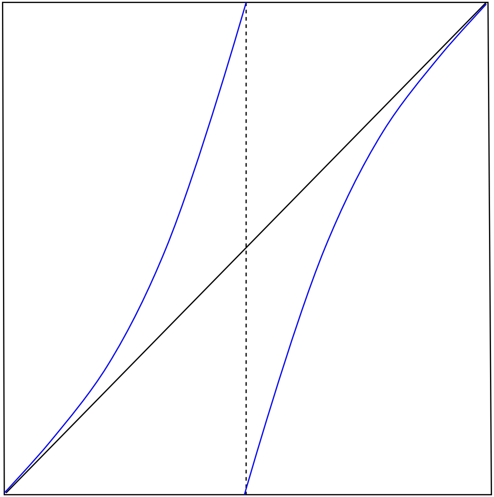
$$\mathcal{R}(\alpha) = \begin{cases} \frac{\alpha}{1-\alpha} & \text{if } \alpha \in (0, \frac{1}{2}), \\ \frac{2\alpha-1}{\alpha} & \text{if } \alpha \in (\frac{1}{2}, 1). \end{cases}$$

Graph of $\mathcal{R}$



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- The map $\mathcal{R}$ commutes with the involution $\iota(x) = 1 - x$.

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$$G(x) = \{\frac{1}{x}\}, \forall x \in (0, 1).$$

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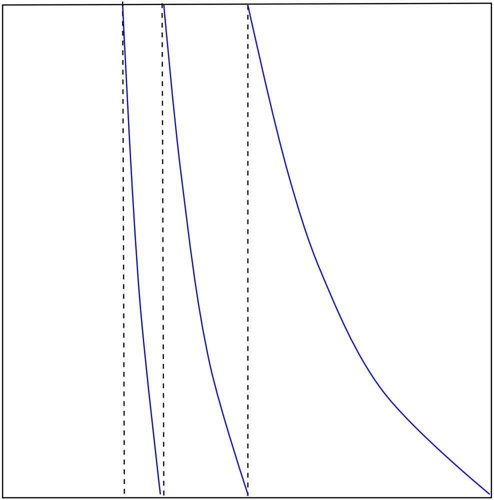

The continued fraction algorithm

For an irrational number $x \in (0,1)$ and $n \geq 0$, we set $x_n := G^n(x)$ and $a_{n+1} = \lfloor \frac{1}{x_n} \rfloor$, so that $x_n = \frac{1}{a_{n+1}+x_{n+1}}$.

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The rational numbers $(\frac{p_n}{q_n})$ are called the *convergents* of x.

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It has full Lebesgue measure in $\mathbb{R}$ (or $\mathbb{T}$).

Birkhoff sums of smooth functions for diophantine rotations

Theorem: Let α be a diophantine number in $\mathbb{T}$ and let $\varphi \in C^\infty(\mathbb{T})$. There exists a smooth function $\psi \in C^\infty(\mathbb{T})$ such that

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Corollary: The Birkhoff sums of φ w.r.t. R_α satisfy

$$S_n\varphi(x) = n \int_{\mathbb{T}} \varphi(t)\, dt + O(1) \ ,$$

uniformly in $x \in \mathbb{T}$.

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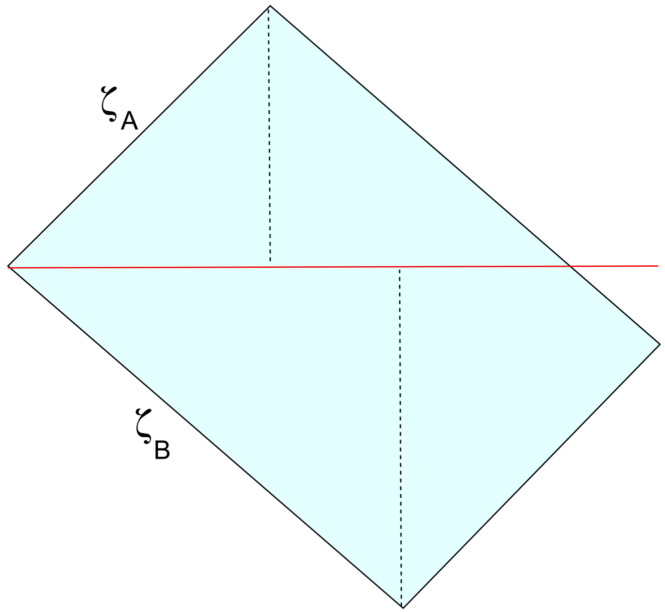
The return map on the horizontal segment $(0, \lambda_A + \lambda_B)$ of the flow on $\mathbb{T}_\zeta$ generated by the vertical vectorfield $\frac{\partial}{\partial y}$ is the transformation T_λ.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The proof is left as an exercise.

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The quotient $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ is then identified with the unit tangent bundle of the modular surface $\mathbb{H}/SL(2,\mathbb{Z})$.

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The quotient $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$ is then identified with the unit tangent bundle of the modular surface $\mathbb{H}/SL(2, \mathbb{Z})$. It is also identified with the space of lattices of covolume 1.

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The action by left multiplication of the diagonal one-parameter subgroup $g^t := \mathrm{diag}(e^t, e^{-t})$ on $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$ is the geodesic flow on the modular surface, endowed with its metrics of constant negative curvature inherited from $\mathbb{H}$.

Relation with the continued fraction algorithm

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One recognizes a fast homogeneous version of the renormalization algorithm!