Birkhoff sums of i.e.t.'s: KZ cocycle (9th lecture)

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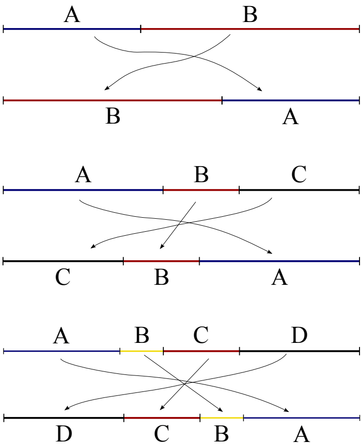
Table of contents

I.e.t.'s as generalization of circle rotations



I.e.t.'s and their combinatorial and length data

I.e.t.'s T are determined by

- a *comb. data* $(\mathcal{A}, \pi_t, \pi_b)$ and
- a *length data* λ_α, $\alpha \in \mathcal{A}$.

Main goal of the course

The idea was to study Birkhoff sums of i.e.t.'s via adequate renorm. dyn. for them: this technique is well-known in Dyn. and it is best represented by Adrien Douady's phrase

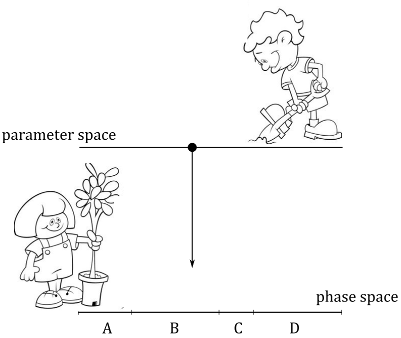
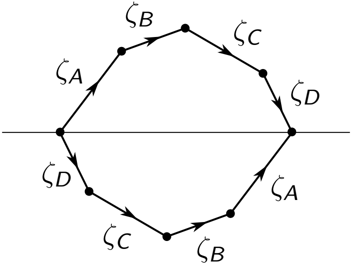


Figure: "to plough in parameter space and harvest in phase space"

Interlude: Suspensions of i.e.t.'s and translation surfaces

Using either Masur's suspension construction or Veech's "zippered rectangles" construction, we saw that i.e.t.'s are first return maps of translation flows on *translation surfaces*:



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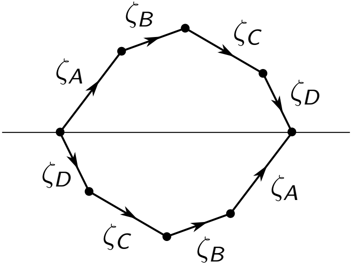
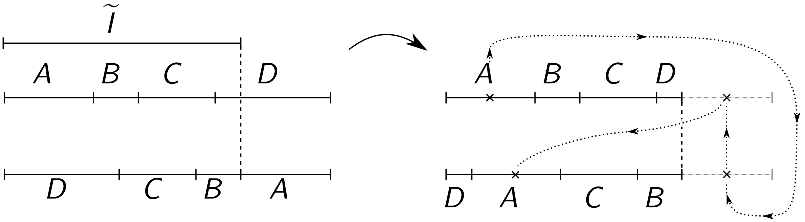


In particular, it is natural to study the dynamics of i.e.t.'s and translation flows *together*!

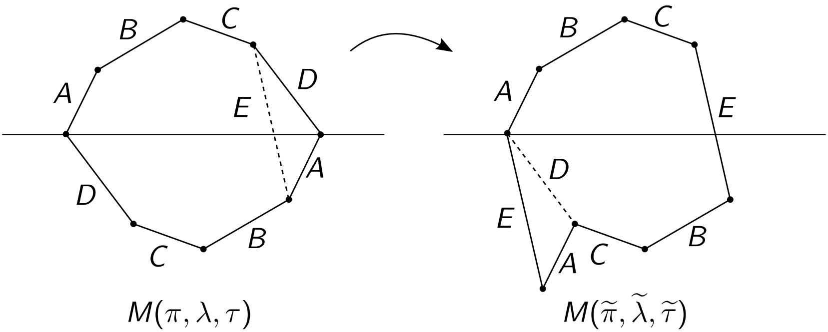
Rauzy-Veech algorithm for i.e.t.'s

The basic step (of *top* type) of the renorm. process for i.e.t.'s is:



Rauzy-Veech algorithm for suspensions

Its counterpart for suspensions of i.e.t.'s is:



Matrix B_γ I

For a basic step γ of RV algorithm for i.e.t.'s, the matrix B_γ is $\mathrm{Id} + E_{\beta\alpha}$ where α is the *winner* and β is the *loser*.

In general, for a path $\gamma = \gamma_1 \dots \gamma_n$ obtained from several steps γ_i of RV algorithm, we define $B_\gamma = B_{\gamma_n} \dots B_{\gamma_1}$.

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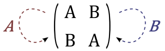
The matrices B_γ are the (discrete time version) of the (extended) Kontsevich-Zorich cocycle.

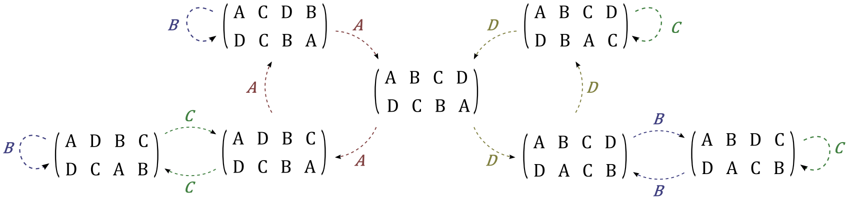
The B_γ's are important because its entries are measuring "how much time $x \in I'^t_\alpha$ spend in I^t_β before returning to I'" for i.e.t.'s and they connect to *special Birkhoff sums*.

Matrix B_γ II

For instance, in the case of an i.e.t. with combinatorial data $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ and $\begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$, the matrices B_γ are computed with the aid of the *Rauzy diagrams* below:



d=2



d=4

Matrix B_γ and Keane's conjecture

Using:

- a *spectral gap* property for B_γ (i.e., its entries are positive as soon as all letters of $\mathcal{A}$ win $2d-3$ times), and
- the fact that the proj. action of the matrices B_γ described conjugacy classes of i.e.t.'s and the cone of inv. meas.,

we saw that i.e.t.'s T with *recurrent* trajectory under RV algorithm are *uniquely ergodic*.

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we saw that i.e.t.'s T with *recurrent* trajectory under RV algorithm are *uniquely ergodic*.

In particular, Keane's conjecture on the unique erg. of almost every i.e.t. would follow once we can justify the recurrence under RV alg. of (a.e.) i.e.t.'s. Keeping this in mind, we started the discussion of Teich. and moduli spaces (as natural spaces for the recurrence to take place).

Teich. and moduli spaces

We considered $Q(M, \Sigma, \kappa)$ and $\mathcal{M}(M, \Sigma, \kappa)$ the Teich. and moduli spaces of transl. surf.: in the first case we identify transl. struct. isomorphic under homeos *isotopic to id.* while in the second case there is no restriction to the isotopy class of the homeos.

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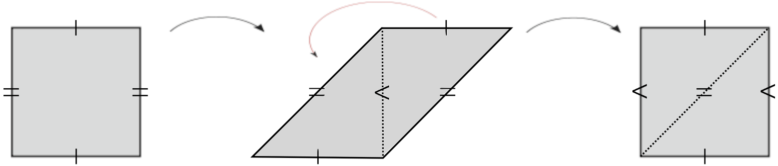
For example, below we illustrate two unit area torii representing the same point in moduli space but not in Teich. space:



"Less concrete but more conceptual"

Paraphrasing J.-C. Yoccoz's lecture, a big advantage of Teich. and moduli spaces is the fact that they allow us to be "less concrete but more conceptual" (an important philosophical step in Dynamics since the works of H. Poincaré).

More precisely, instead of sticking to the (concrete) RV algorithm and KZ cocycle B_γ, we can think (more conceptually) about the Teichmüller flow and the (continuous time version of) KZ cocycle as follows.

Nice structures on Teich. and moduli spaces

Teich. and moduli spaces (of unit area transl. surf.) come equipped with:

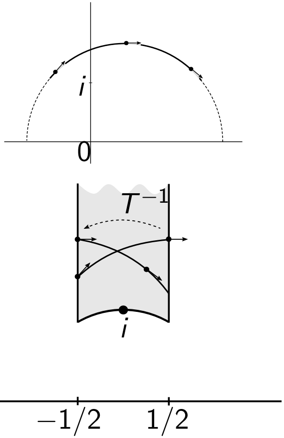
- a $SL(2,\mathbb{R})$-action and hence a Teich. flow $g_t = \mathrm{diag}(e^t, e^{-t})$;
- a complex manifold/orbifold structure;
- a natural "Lebesgue" (Masur-Veech) meas. μ_{MV};

Remark

Masur-Veech measure is finite on $\mathcal{M}^{(1)}(M,\Sigma,\kappa)$, so that moduli spaces are the nice places to get *recurrence*. However, we consider together Teich. and moduli spaces because Teich. spaces work as universal covers of moduli spaces...

Teich. and moduli spaces, and Teich. flow in genus 1

A nice way to appreciate the difference between Teich. and moduli space is to look at the pictures below illustrating the genus 1 case:



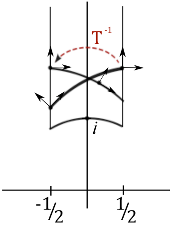
KZ cocycle

In this language, the cont. time version of (restricted) KZ cocycle over Teich. flow g_t was the quotient of the trivial cocycle on $Q^{(1)}(M, \Sigma, \kappa) \times H_1(M, \mathbb{R})$,

$$(p, v) \mapsto (g_t(p), v)),$$

by the mapping class group $\mathrm{Diff}^+(M, \Sigma)/\mathrm{Diff}_0^+(M, \Sigma)$.

In genus 1, a schematic representation of KZ cocycle is:



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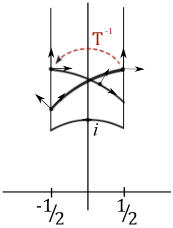
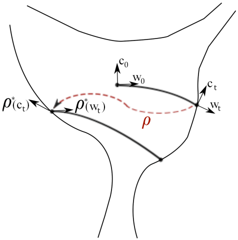


In general, KZ cocycle "is":



Deviations of ergodic averages for i.e.t.'s I

Before yesterday, J.-C. Yoccoz showed that Veech boxes in moduli spaces allow to connect the RV algorithm and the Teich. flow, and the matrices B_γ and KZ cocycle.

Deviations of ergodic averages for i.e.t.'s I

Before yesterday, J.-C. Yoccoz showed that Veech boxes in moduli spaces allow to connect the RV algorithm and the Teich. flow, and the matrices B_γ and KZ cocycle.

Then, yesterday, using this connection, he showed that *Zorich phenomenon* for deviations of Birkhoff sums $S_N T(x)$ of a.e. i.e.t. T

$$\limsup_{N\to\infty} \frac{\log \mathrm{dist}(S_N T(x), D_j(T))}{\log N} = \lambda_j$$

is naturally explained by the (non-negative) Lyapunov exponents $1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_g (\geq 0)$ of KZ cocycle *w.r.t. Masur-Veech measures*.

Deviations of ergodic averages for i.e.t.'s II

Actually, the picture for Zorich phenomenon was only complete under the so-called *Kontsevich-Zorich conjecture* that the Lyapunov exponents λ_i of Masur-Veech measures are simple (i.e., they have multiplicity 1).

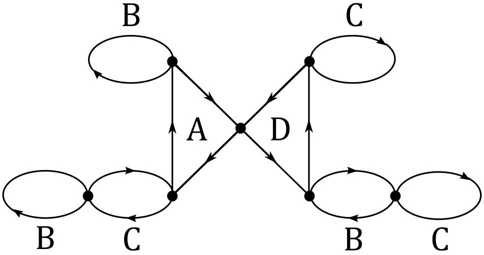
Deviations of ergodic averages for i.e.t.'s II

Actually, the picture for Zorich phenomenon was only complete under the so-called *Kontsevich-Zorich conjecture* that the Lyapunov exponents λ_i of Masur-Veech measures are simple (i.e., they have multiplicity 1).

After an important partial result of G. Forni, we know that this conjecture is true due the works of A. Avila and M. Viana.

Reduction to a countable shift

As J.-C. Yoccoz mentioned yesterday, one of the ideas of Avila and Viana is to think of KZ cocycle (wrt Masur-Veech measures) as a locally constant symplectic cocycle over a countable shift by looking at B_γ's over loops in Rauzy diagrams:



Setting for the simplicity criterium I

Let:

- Λ be a *countable* alphabet, $\Sigma = \Lambda^{\mathbb{N}}$, $\widehat{\Sigma} = \Lambda^{\mathbb{Z}} := \Sigma_- \times \Sigma$;
- $p_+ : \widehat{\Sigma} \to \Sigma$ and $p_- : \widehat{\Sigma} \to \Sigma_-$ are the canonical projections;
- $f : \Sigma \to \Sigma$ unilateral shift, $\widehat{f} : \widehat{\Sigma} \to \widehat{\Sigma}$ bilateral shift;
- $\Omega = \bigcup_{n \in \mathbb{N}} \Lambda^n$ is the set of *words* of Λ;
- given a word $\underline{\ell} \in \Omega$, we form the *cylinders*

$$\Sigma(\underline{\ell}) := \{x \in \Sigma : x \text{ starts by } \underline{\ell}\},$$

$$\Sigma_-(\underline{\ell}) := \{x \in \Sigma : x \text{ ends by } \underline{\ell}\}$$

- for μ a f-inv. prob., $\widehat{\mu}$ is the *unique* $\widehat{f}$-inv. prob. s.t. $p_+^*(\widehat{\mu}) = \mu$ and $\mu_- := p_-^*(\widehat{\mu})$.

Setting for the simplicity criterium II

Let μ a f-inv. prob. with *bounded distortion*, i.e., $\exists\, C(\mu) > 0$ s.t.

$$C(\mu)^{-1}\mu(\Sigma(\underline{\ell}_1))\mu(\Sigma(\underline{\ell}_2)) \leq \mu(\Sigma(\underline{\ell}_1\underline{\ell}_2)) \leq C(\mu)\mu(\Sigma(\underline{\ell}_1))\mu(\Sigma(\underline{\ell}_2))$$

In other words, μ is not very far from being a *Bernoulli measure*.

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Exercise

By mimicking the proof of ergodicity of Bernoulli measure, show that any μ with bounded distortion is *ergodic*.

Setting for the simplicity criterium III

Let $A : \Sigma \to Sp(d, \mathbb{R})$ be a locally constant log-integrable symplectic cocycle, that is,

$$A(\underline{x}) = A_{x_0} \text{ for any } \underline{x} = (x_0, x_1, \dots) \in \Sigma$$

and

$$\int_{\Sigma} \log \|A^{\pm 1}(x)\| d\mu(x) = \sum_{\ell \in \Lambda} \mu(\Sigma(\ell)) \log \|A_{\ell}^{\pm}\| < \infty$$

Setting for the simplicity criterium IV

Definition

A cocycle A as above is *pinching* if there exists a word $\underline{\ell}^* \in \Omega$ s.t. the eigenv. of the matrix $A^{\underline{\ell}^*}$ are real and distinct in modulus.

Setting for the simplicity criterium IV

Definition

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Notation: In the sympl. space $\mathbb{R}^d$, d even, let $G(k)$ be the Grassm. of *isotropic*, resp. *coisotropic*, k-planes of $\mathbb{R}^d$ for $1 \leq k \leq d/2$, resp. $d/2 \leq k \leq d$.

Definition

A pinching cocycle A is *twisting* if $\forall\ 1 \leq k \leq d/2\ \exists\ \underline{\ell}(k) \in \Omega$ word s.t. the matrix $A^{\underline{\ell}(k)}$ is twisting wrt $A^{\underline{\ell}^*}$, i.e.,

$$A^{\underline{\ell}(k)}(F) \cap F' = \{0\}$$

for any $A^{\underline{\ell}^*}$-*inv.* $F \in G(k)$, $F' \in G(d-k)$.

Statement of a version of Avila-Viana simplicity criterion

Theorem (Avila-Viana)

Let A be a cocycle over (f, μ) or $(\widehat{f}, \widehat{\mu})$ as above. If A is pinching and twisting, its Lyapunov spectrum (wrt μ or $\widehat{\mu}$) is simple.

Remark

In fact, this simplicity criterion works for other groups of matrices (e.g. $U(p,q)$), but this is another history...

A comment on the twisting hypothesis I

There are several versions of this criterium e.g. by Avila and Viana themselves!

For instance, in one of their works, they required a stronger ("infinitary") version of twisting property asking that for *every* $F \in G(k)$, $F' \in G(d-k)$, one can find a word $\underline{\ell}$ s.t. $A^{\underline{\ell}}(F) \cap F' = \{0\}$.

However, as it turns out, it suffices to ask the weaker ("finitary") version above of twisting wrt to a pinching matrix.

A comment on the twisting hypothesis II

On the other hand, in the weaker version of twisting, it is important to ask for a word $\ell(k)$ "twisting" *all* A^{ℓ^*}-inv. isotropic subspaces *at once*!

Indeed, simplicity of Lyap. spect. may fail if this is not the case: for example

- $\Sigma = \{0,1\}^{\mathbb{N}}$, $\mu = 1/2 - 1/2$ Bernoulli measure
- $A_0 = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

leads to a cocycle with two vanishing exponents.

A few words on reducing KZ cocycle to the shift case I

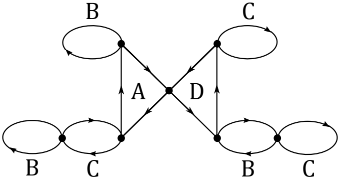
As J.-C. Yoccoz mentioned yesterday, the verification of pinching and twisting in the case of the cocycle over a countable shift derived from KZ cocycle depends on a combinatorial study of Rauzy diagrams and the idea is to proceed by induction.

In particular, as announced yesterday, we'll not enter on this verification here. Instead, we'll propose again the following exercise showing that even the pinching property may be not automatic in general.

A few words on reducing KZ cocycle to the shift case II

Exercise

Compute B_γ of the path $D \to B^2 \to D \to C \to D \to A^3$ on our "preferred" genus 2 Rauzy diagram (see below), calculate its characteristic polynomial and determine the moduli of its eigenvalues. Is this B_γ a pinching matrix?



Key Theorem

The key result towards Avila-Viana simplicity criterion is:

Theorem 1 (Avila-Viana)

For every $1 \leq k \leq d/2$, $\exists$ a map $\Sigma_- \to G(k)$, $x \mapsto \xi(x)$ s.t.

- the map $\widehat{\xi} := \xi \circ p_-$ is *invariant*: $\widehat{A}(x)\widehat{\xi}(x) = \widehat{\xi}(f(x))$;
- for μ_--a.e. $x \in \Sigma_-$, $\frac{\sigma_k(A^{\underline{\ell}(x,n)})}{\sigma_{k+1}(A^{\underline{\ell}(x,n)})} \to \infty$ and $\xi_{\underline{\ell}(x,n)} \to \xi(x)$
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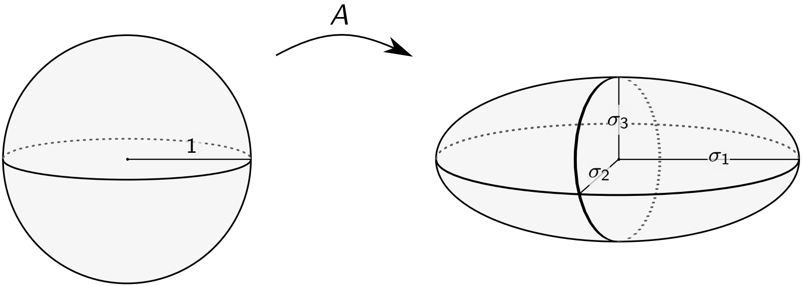
Here,

- $\underline{\ell}(x,n)$ is the terminal word of x of length n;
- $\sigma_1 \geq \cdots \geq \sigma_d$ are singular values of a matrix (see next slide);
- $\xi_{\underline{\ell}}$ is the subspace generated by the k largest semi-axis of the ellipse $A^{\underline{\ell}}(\{\|v\| = 1\})$ (see next slide).

Semi-axes of ellipses and singular values



Reduction of simplicity criterion to Theorem 1

Intuitively, the simplicity of Lyap. spect. of a cocycle A as above follows from Theorem 1 because:

(a) starting with $\xi(x) \in G(k)$ for $x \in \Sigma_-$ and "reversing time", we get analogous objects $\xi_*(y) \in G(d-k)$ for $y \in \Sigma$;

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(c) by the 1st and 2nd items of Thm 1, $\xi(x) \leftrightarrow k$ largest exponents of A, and $\xi_*(y) \leftrightarrow d-k$ smallest exponents of A;

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(c) by the 1st and 2nd items of Thm 1, $\xi(x) \leftrightarrow k$ largest exponents of A, and $\xi_*(y) \leftrightarrow d-k$ smallest exponents of A;

(d) by (c) and (b), one can "separate" the kth exponent of A from the $(k+1)$th exponent of A.

u-states

Definition

A u-state ($u =$ unstable) is a prob. $\widehat{m}$ on $\widehat{\Sigma} \times G(k)$ s.t.

- $q^* \widehat{m} = \widehat{\mu}$, where $q : \widehat{\Sigma} \times G(k) \to \widehat{\Sigma}$ is the can. proj., and
- for some constant $C(\widehat{m}) < \infty$,

$$\frac{\widehat{m}(\Sigma_-(\underline{\ell}_0) \times \Sigma(\underline{\ell}) \times X)}{\mu(\Sigma(\underline{\ell}))} \leq C(\widehat{m}) \frac{\widehat{m}(\Sigma_-(\underline{\ell}_0) \times \Sigma(\underline{\ell}') \times X)}{\mu(\Sigma(\underline{\ell}'))}$$

$\forall\ \underline{\ell}_0,\ \underline{\ell},\ \underline{\ell}' \in \Omega$ and $X \subset G(k)$.

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$\forall\ \underline{\ell}_0, \underline{\ell}, \underline{\ell}' \in \Omega$ and $X \subset G(k)$.

Example

Given ν prob. on $G(k)$, $\widehat{m} = \widehat{\mu} \times \nu$ is a u-state w/ $C(\widehat{m}) = C(\mu)^2$.

Existence of invariant *u*-states

Proposition 1

There are $(\widehat{f}, A)$-inv. u-states.

This result follows from the "usual" Krylov-Bogolyubov arg.: take $\widehat{m}_0$ any u-state, let $\widehat{m}(n) = (\widehat{f}, A)_*^n \widehat{m}_0$ and extract an accum. pt $\widehat{m}$ from Cesaro averages. Of course, $\widehat{m}$ is $(\widehat{f}, A)$-inv.

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Proposition 1

There are $(\widehat{f}, A)$-inv. *u*-states.

This result follows from the "usual" Krylov-Bogolyubov arg.: take $\widehat{m}_0$ any *u*-state, let $\widehat{m}(n) = (\widehat{f}, A)^n_* \widehat{m}_0$ and extract an accum. pt $\widehat{m}$ from Cesaro averages. Of course, $\widehat{m}$ is $(\widehat{f}, A)$-inv.

So, it remains to show that $\widehat{m}$ is a *u*-state. Here, this follows from:

Exercise

Show that $C(\widehat{m}(n)) \leq C(\widehat{m}_0) \cdot C(\mu)^2$ for all n.

What is the current goal?

Our goal is to find the section $x \mapsto \xi(x) \in G(k)$ (cf. Thm 1) via an $(\widehat{f}, A)$-inv. u-state because such meas. should capture the dynamical behavior of $(\widehat{f}, A)$ and hence it should "see" $x \mapsto \xi(x)$.

In order to formalize this, we'll need a *martingale convergence theorem*.

Proposition 2

Let $\widehat{m}$ s.t. $q^*\widehat{m} = \mu$ and let

$$m_n(x)(X) := \frac{\widehat{m}(\Sigma_-(\underline{\ell}(x,n)) \times \Sigma \times X)}{\widehat{m}(\Sigma_- \times \Sigma \times X)}$$

for any $x \in \Sigma_-$ and $X \subset G(k)$. Then, for μ_--a.e. $x \in \Sigma_-$,

$$m_n(x) \to m(x) \text{ prob. on } G(k).$$

Assume now that $\widehat{m}$ is a $(\widehat{f}, A)$-inv. u-state and let's apply the martingale convergence theorem.

We get a family of $\widehat{m}(x)$ of prob. on $G(k)$ and we claim that if we can show that $\widehat{m}(x)$ is a Dirac mass $\widehat{m}(x) = \delta_{\xi(x)}$ for a.e. x, then $x \mapsto \xi(x)$ satisfies Theorem 1.

For ex., $A(x)(\xi \circ p_-(\widehat{x})) = \xi \circ p_-(\widehat{f}(\widehat{x}))$ (1st ["inv."] item of Thm 1) would follow from the $(\widehat{f}, A)$ invariance of $\widehat{m}$, while the other two items depend on the pinching and twisting properties of $(\widehat{f}, A)$.

Why are $\widehat{m}(x)$ Dirac masses? I

By definition,

$$m_n(x)(X) = \frac{\widehat{m}(\Sigma_-(\underline{\ell}(x,n)) \times \Sigma \times X)}{\widehat{m}(\Sigma_-(\underline{\ell}(x,n)) \times \Sigma \times G(k))}$$

Thus, by $(\widehat{f}, A)$-inv. of $\widehat{m}$, we get

$$m_n(x)(X) = \frac{\widehat{m}(\Sigma_- \times \Sigma(\underline{\ell}(x,n)) \times A^{-\underline{\ell}(x,n)}(X))}{\widehat{m}(\Sigma_- \times \Sigma(\underline{\ell}(x,n)) \times G(k))}$$

Since $\widehat{m}$ is a u-state, it follows that

$$m_n(x)(X) \leq C(\widehat{m})^2 \frac{\widehat{m}(\Sigma_- \times \Sigma \times A^{-\underline{\ell}(x,n)}(X))}{\widehat{m}(\Sigma_- \times \Sigma \times G(k))},$$

that is,

$$m_n(x)(X) \leq C(\widehat{m})^2 \widehat{m}(\Sigma_- \times \Sigma \times A^{-\underline{\ell}(x,n)}(X)) := \nu_n(x)(X)$$

Why are $\widehat{m}(x)$ Dirac masses? II

Actually, the previous inequality is "symmetric" (because the def. of *u*-state is "symmetric") and one has

$$C(\widehat{m})^{-2} \leq m_n(x)(X)/\nu_n(x)(X) \leq C(\widehat{m})^2,$$

that is, $m_n(x)(X)$ is *equivalent* to $\nu_n(x)(X)$.

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On the other hand, we can rewrite

$$\nu_n(x)(X) := \widehat{m}(\widehat{\Sigma} \times A^{-\underline{\ell}(x,n)}(X)) = A_*^{\underline{\ell}(x,n)}\nu(X)$$

where $\nu = r_*\widehat{m}$ and $r : \widehat{\Sigma} \times G(k) \to G(k)$ is the can. proj.

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Thus, $m(x) = \lim m_n(x)$ is *equivalent* to any accum. pt of $\nu_n(x)$.

Why are $\widehat{m}(x)$ Dirac masses? III

In view of the previous slide, our task is reduced to show that

Proposition 3

$\nu_n(x) = A_*^{\ell(x,n)}\nu$ accum. some Dirac mass for μ_--a.e. $x \in \Sigma_-$.