Central shadowing and partial hyperbolicity

Sergey Tikhomirov[1]

[1]Free University of Berlin

Trieste, 2012

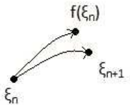
- S. Kryzhevich, S. Tikhomirov, Partial hyperbolicity and central shadowing, // http://arxiv.org/abs/1112.4272.

Outline

- Shadowing for partially hyperbolic systems.
  - Central shadowing.
  - Main steps of the proof.
- Open questions.
  - Shadowing for flows.

Shadowing property

- $f : M \to M$, $f \in C^1$, $M \in C^\infty$, dist.
- $\{y_n\}$ is d-pseudotrajectory, if $\text{dist}(y_{n+1}, f(y_n)) < d$

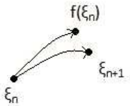


- Standard Shadowing (StSh)
  $\forall \varepsilon > 0$ $\exists d > 0$ such that $\forall$ d-pseudotrajectory $\{y_n\}$ there exists exact trajectory $\{x_n\}$ such that

$$\text{dist}(x_n, y_n) < \varepsilon.$$

Shadowing property

- $f : M \to M$, $f \in C^1$, $M \in C^\infty$, dist.
- $\{y_n\}$ is d-pseudotrajectory, if $\mathrm{dist}(y_{n+1}, f(y_n)) < d$



- Standard Shadowing (StSh)
  $\forall \varepsilon > 0\ \exists d > 0$ such that $\forall$ d-pseudotrajectory $\{y_n\}$ there exists exact trajectory $\{x_n\}$ such that

$$\mathrm{dist}(x_n, y_n) < \varepsilon.$$

- Applications:
  - Numerical simulations
  - Stability theory

Hyperbolicity and shadowing

- **Shadowing lemma (Anosov):** Let Λ be hyperbolic set. Then f has shadowing property in some neighborhood $U(\Lambda)$.

Hyperbolicity and shadowing

- **Shadowing lemma (Anosov):** Let Λ be hyperbolic set. Then f has shadowing property in some neighborhood $U(\Lambda)$.
- SS – set of structurally stable diffeomorphisms.
  SS = Axiom A + Strong Transversality Condition.
- SS $\subsetneq$ StSh

Hyperbolicity and shadowing

- **Shadowing lemma (Anosov):** Let Λ be hyperbolic set. Then f has shadowing property in some neighborhood $U(\Lambda)$.
- SS – set of structurally stable diffeomorphisms. SS = Axiom A + Strong Transversality Condition.
- SS $\subsetneq$ StSh
- Lipschitz shadowing property (LipSh) ($\varepsilon = Ld$) $\exists L, d_0 > 0$ such that $\forall d < d_0$ and d-pseudotrajectory $\{y_n\}$ there exists an exact trajectory $\{x_n\}$ such that

$$\mathrm{dist}(x_n, y_n) < Ld.$$

Partial hyperbolicity

Partial hyperbolicity

We say that f is partially hyperbolic if there exists an invariant splitting $TM = E^s \oplus E^c \oplus E^u$ such that

- Splitting is dominated.
- E^s, E^u are hyperbolic

Dynamical coherence

We say that f is dynamically coherent if there exists unique foliations W^{cs}, W^{cu} tangent to $E^{cs} = E^s \oplus E^c$, $E^{cu} = E^c \oplus E^u$.

Foliation $W^c = W^{cs} \cap W^{cu}$ is tangent to E^c.
Note: Usually foliation tangent to $E^s + E^u$ do not exists.

Partial hyperbolicity

Partial hyperbolicity

We say that f is partially hyperbolic if there exists an invariant splitting $TM = E^s \oplus E^c \oplus E^u$ such that

- Splitting is dominated.
- E^s, E^u are hyperbolic

Dynamical coherence

We say that f is dynamically coherent if there exists unique foliations W^{cs}, W^{cu} tangent to $E^{cs} = E^s \oplus E^c$, $E^{cu} = E^c \oplus E^u$.

Foliation $W^c = W^{cs} \cap W^{cu}$ is tangent to E^c.
Note: Usually foliation tangent to $E^s + E^u$ do not exists.

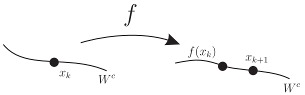
Question: Which kind of shadowing we have for partially hyperbolic diffeomorphisms?

Central Shadowing

Central pseudotrajectory

We say that $\{x_n\}$ is a d central pseudotrajectory, if $\text{dist}(x_{n+1}, f(x_n)) < d$ and

$$x_{n+1} \in W^c_{loc}(f(x_n)).$$



Central Shadowing

$\forall \varepsilon > 0 \; \exists d > 0$ such that $\forall$ d-pseudotrajectory $\{y_n\}$ there exists an ε central pseudotrajectory $\{x_n\}$ such that

$$\text{dist}(x_n, y_n) < \varepsilon.$$

- Lipschitz Central Shadowing: $\varepsilon = Ld$.

Main Theorem

Theorem (Kryzhevich, Tikhomirov, 2011)

Partially hyperbolic, dynamically coherent diffeomorphism satisfy Lipschitz Central Shadowing.

Plaque expansivity

f – partially hyperbolic, dynamically coherent.

Plaque expansivity

We say that f is plaque expansive if there exists $d > 0$ such that for any d central pseudotrajectories $y_n^{(1)}$ and $y_n^{(2)}$ satisfying

$$\mathrm{dist}(y_n^{(1)}, y_n^{(2)}) < d$$

holds inclusion $y_n^{(1)} \in W^c(y_n^{(2)})$.

Conjecture (Hirsh-Pugh-Shub)

All partially hyperbolic dynamically coherent diffeomorphisms are plaque expansive.

Plaque expansivity

f – partially hyperbolic, dynamically coherent.

Plaque expansivity

We say that f is plaque expansive if there exists $d > 0$ such that for any d central pseudotrajectories $y_n^{(1)}$ and $y_n^{(2)}$ satisfying

$$\mathrm{dist}(y_n^{(1)}, y_n^{(2)}) < d$$

holds inclusion $y_n^{(1)} \in W^c(y_n^{(2)})$.

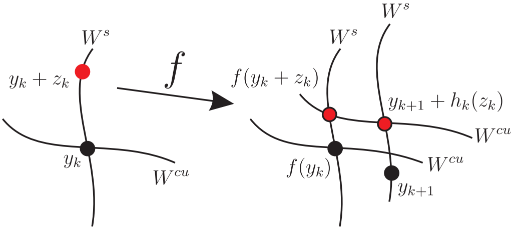
Conjecture (Hirsh-Pugh-Shub)

All partially hyperbolic dynamically coherent diffeomorphisms are plaque expansive.

Note: Central shadowing perfectly matches with plaque expansivity.
Difficulty: $E^{s,c,u}(x)$ depends on x only Holder, but not Lipschitz.

Proof

- Consider splitting $E^s \oplus E^{cu}$.
- Let $\{y_k\}$ be a d-pseudotrajectory.
- Consider vector $z_k \in E^s(y_k)$.
- Define $h_k(z_k)$ as on the picture:



- If $|z_k| < Ld$ and $h_k(z_k) = z_{k+1}$ then $x_k = y_k + z_k$ is central pseudotrajectory.

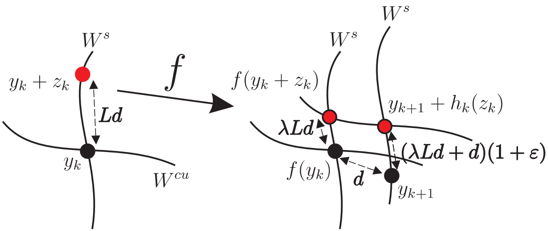
Proof

Presentation assumptions:

- Contraction in one step; $E^s \perp E^{cu}$.

Lemma

There exists $L > 0$ such that if $|z_k| < Ld$ then $|h_k(z_k)| < Ld$



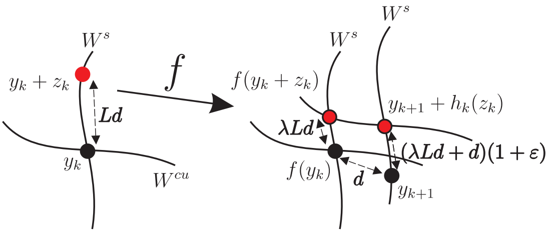
Proof

Presentation assumptions:

- Contraction in one step; $E^s \perp E^{cu}$.

Lemma

There exists $L > 0$ such that if $|z_k| < Ld$ then $|h_k(z_k)| < Ld$



Let $\mathbf{X} = \{|z_k| < Ld\}$. $H(\{z_k\}) = \{h_k(z_k)\}$.
Tikhonov-Shauder fixed point theorem gives us fixed point, which gives us a shadowing central pseudotrajectory.

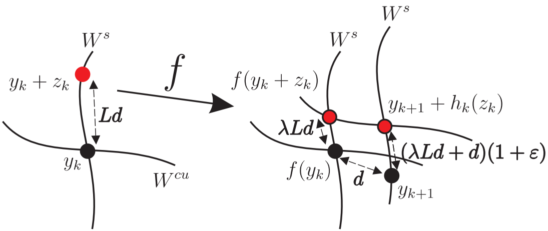
Proof

Presentation assumptions:

- Contraction in one step; $E^s \perp E^{cu}$.

Lemma

There exists $L > 0$ such that if $|z_k| < Ld$ then $|h_k(z_k)| < Ld$

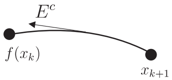


Let $\mathbf{X} = \{|z_k| < Ld\}$. $H(\{z_k\}) = \{h_k(z_k)\}$.
Tikhonov-Shauder fixed point theorem gives us fixed point, which gives us a shadowing central pseudotrajectory.
Remark: We do not have uniqueness of fixed point.

Is Dynamical coherence important?

Central-curve pseudotrajectory

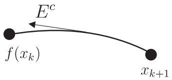
We say that $\{x_n\}$ is a d central pseudotrajectory, if there exists a curve $\gamma(t)$ of length $< d$ tangent to E^c, with $\gamma(0) = x_{n+1}, \gamma(1) = f(x_n)$.



Is Dynamical coherence important?

Central-curve pseudotrajectory

We say that $\{x_n\}$ is a d central pseudotrajectory, if there exists a curve $\gamma(t)$ of length $< d$ tangent to E^c, with $\gamma(0) = x_{n+1}, \gamma(1) = f(x_n)$.



Central-curve Shadowing

$\forall \varepsilon > 0\ \exists d > 0$ such that $\forall$ d-pseudotrajectory $\{y_n\}$ there exists an ε central-curve pseudotrajectory $\{x_n\}$ such that $\text{dist}(x_n, y_n) < \varepsilon$.

- Lipschitz Central-curve Shadowing: $\varepsilon = Ld$.

Conjecture 1.

Partially hyperbolic diffeomorphism satisfy Lipschitz Central-curve Shadowing.

Hyperbolicity and shadowing

- Consider space of diffeomorphisms endowed with C^1-topology.
- $\mathrm{Int}^1(\mathrm{StSh}) = \mathrm{SS}$ (Sakai, 1994).
- Conjecture: C^1-generically $\mathrm{StSh} = \mathrm{SS}$ (Abdenur, Diaz, 2003)
  - proved for tame.
- $\mathrm{LipSh} = \mathrm{SS}$, (Pilyugin, Tikhomirov, 2010).

Open Questions

Context:

- $f \in C^1$.
- $W^c(x)$ – foliation, $f(W^c(x)) = W^c(f(x))$.

Open Questions

Context:

- $f \in C^1$.
- $W^c(x)$ – foliation, $f(W^c(x)) = W^c(f(x))$.

Conjecture 2.

If f is transitive and has Lipschitz central shadowing with respect to W^c, then f is partially hyperbolic with $E^c(x) \subset T_x W^c(x)$.

Conjecture 3.

If f has Lipschitz central shadowing and plaque expansivity with respect to W^c, then f is partially hyperbolic with $E^c(x) \subset T_x W^c(x)$.

Analog of LipSh = SS.

Open Questions

Context:

- $f \in C^1$, U_f – C^1-neighborhood of f.
- For any $g \in U_f$ there exists foliation $W_g^c(x)$ such that, $g(W_g^c(x)) = W_g^c(g(x))$.
- W_g^c depends continuously on g.

Open Questions

Context:

- $f \in C^1$, U_f – C^1-neighborhood of f.
- For any $g \in U_f$ there exists foliation $W_g^c(x)$ such that, $g(W_g^c(x)) = W_g^c(g(x))$.
- W_g^c depends continuously on g.

Conjecture 4.

If for all $g \in U_f$, g is transitive and has central shadowing with respect to W_g^c, then f is partially hyperbolic with $E^c(x) \subset T_x W^c(x)$.

Conjecture 5.

If all $g \in U_f$, has central shadowing and plaque expansivity with respect to W^c, then f is partially hyperbolic with $E^c(x) \subset T_x W^c(x)$.

Analog of $\mathrm{Int}^1(\mathrm{StSh}) = \mathrm{SS}$.

Shadowing for Vector fields

- Vector field X on M. Flow $\varphi(t, x) : \mathbf{R} \times M \to M$.

Shadowing for Vector fields

- Vector field X on M. Flow $\varphi(t,x) : \mathbf{R} \times M \to M$.
- $f(x) = \varphi(1,x)$. Assume that f is non-singular.

Shadowing for Vector fields

- Vector field X on M. Flow $\varphi(t, x) : \mathbf{R} \times M \to M$.
- $f(x) = \varphi(1, x)$. Assume that f is non-singular.
- Let $< X(x) >$ be the "central" direction of f.
- Consider notion of "central shadowing". It is equivalent to the classical one.

Theorems

- $\mathrm{Int}^1(\mathit{Shadowing} +$ "no singularities"$) \subset$ Structurally stable (Lee, Sakai, 2008) (corresponds to conjectures 4, 5)
  - $\mathrm{Int}^1(\mathit{Shadowing}_{\mathit{Oriented}}) \neq$ Structurally stable.
- Lipschitz Shadowing equivalent to structural stability (Pilyugin, Palmer, T., 2012)(corresponds to conjectures 2, 3)

Thank you very much for your attention!