Birkhoff sums of i.e.t.'s: KZ cocycle (10th lecture)

Carlos Matheus and Jean-Christophe Yoccoz

CNRS (Paris 13) and Collège de France

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Quick review from yesterday

Yesterday, we reduced Avila-Viana simplicity criterium for *pinching* and *twisting* cocycle A over countable shifts to show that:

Proposition 3

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where

- $\underline{\ell}(x, n) = (x_{-n}, \dots, x_{-1})$ is the terminal word of length n of $x = (\dots, x_{-n}, \dots, x_{-1}) \in \Sigma_-$;
- $\nu = r_* \widehat{m}$,
- $\widehat{m}$ is a u-state, and
- $r : \widehat{\Sigma} \times G(k) \to G(k)$ is the can. proj. to the Grass. $G(k)$.

Pinching and twisting

Recall that a cocycle A is:

- *pinching* if $\exists\,\underline{\ell}^*$ word s.t. $P := A^{\underline{\ell}^*}$ is a matrix whose eigenv. are all real and of distinct modulus, i.e., P is a *pinching matrix*;

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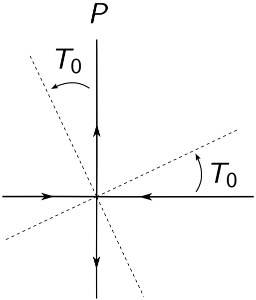
- *twisting* if $\forall\, 1 \leq k \leq d/2$, $\exists\, \underline{\ell}(k)$ s.t. $T_0 := A^{\underline{\ell}(k)}$ is *twisting wrt pinching matrix* P, i.e.,

$$T_0(F) \cap F' = \{0\}$$

for every P-inv. $F \in G(k)$ and $F' \in G(d-k)$.

Pinching and twisting



Why $\nu_n(x)$ accumulate Dirac masses? I

We start the proof of Prop. 3 by noticing that *pinching* and *twisting* for A implies a uniform twisting:

(ut) $\exists\ \underline{\ell}_1, \dots, \underline{\ell}_m \in \Omega$ and $\delta > 0$ s.t. $\forall\ F' \in G(d-k)\ \exists\, i$ with

$$A^{\underline{\ell}_i}(F_+(P)) \cap F' = \{0\} \quad \text{and} \quad \angle(A^{\underline{\ell}_i}(F_+(P)), F') \geq \delta$$

Here, $F_+(P)$ is the subsp. of kth largest eigenvalues of P.

Remark

Here, the compactness of $G(k)$ (and Leb. cov. lemma) were used.

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Notation

Let $T_i := A^{\underline{\ell}_i}$ be the matrices appearing in item (ut).

Why $\nu_n(x)$ accumulate Dirac masses? II

The key lemma for Prop. 3 is:

Lemma 1

Let $\varepsilon > 0$ and ρ prob. meas. on G(k). $\exists\ n_0 = n_0(\rho, \varepsilon)$ and, for each $\underline{\widetilde{\ell}} \in \Omega$, there exists $i = i(\underline{\widetilde{\ell}}) \in \{1, \dots, m\}$ s.t., for $n \geq n_0$, we have

$$A_*^{\underline{\ell}}(\rho)(B) = (A^{\underline{\widetilde{\ell}}} T_i P^n T_0 P^n)_*(\rho)(B) > 1 - \varepsilon$$

where

$$\underline{\ell} := (\underline{\ell}^*)^n \underline{\ell}(k) (\underline{\ell}^*)^n \underline{\ell}_i \underline{\widetilde{\ell}}$$

(here $(\underline{\ell}^*)^n := \underbrace{\underline{\ell}^* \dots \underline{\ell}^*}_{n}$) and B is the ball of radius $\varepsilon > 0$ centered at $\xi_{\underline{\ell}}$.

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- $A^{\underline{\ell}}$ concentrates the most of the mass of any prob. meas. ρ on G(k) on a tiny ball B.

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by applying $A^{\widetilde{\underline{\ell}}}$ one sees that the mass of ρ starts to concentrate near $F_+(A^{\widetilde{\underline{\ell}}})$ ...

However, if ρ charges a lot a neighb. of $F_-(A^{\widetilde{\underline{\ell}}})$, we run in trouble and this is the enemy we should fight by choosing a convenient "start word" $s(\widetilde{\underline{\ell}})$...

Why $\nu_n(x)$ accumulate Dirac masses? V

Formally, the proof of Lemma 1 goes as follows. Apply P^n to ρ. In this way, $\rho' := P^n_* \rho$ concent. its mass near a P-inv. pt. of $G(k)$.

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- in 1st appl. of P^n, we get near some P-inv. k-plane, but
- after adjusting with the "twisting" T_0, in 2nd appl. of P^n, we get near the very specific k-plane $F_+(P)$ (assoc. to the k largest eigenv. of P).

Why $\nu_n(x)$ accumulate Dirac masses? VI

Continuing the argument, we take $F' = F_-(A^{\widetilde{\underline{\ell}}})$ (our enemy) and, by item (ut) above, we select $i = i(\widetilde{\underline{\ell}})$ s.t.

$$\angle(T_i(F_+(P)), F') \geq \delta$$

That is, by applying T_i, we get prob. meas. $\rho'''' = A^{\underline{\ell}_i}\rho'''$ concent. near a pt unif. transv. to the enemy $F' = F_-(A^{\widetilde{\underline{\ell}}})$.

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Thus, if $\widetilde{\underline{\ell}}$ is long word, we have that prob. meas. $\rho^{(v)} = A^{\widetilde{\underline{\ell}}}\rho''''$ is concentrated nearby $\xi_{\underline{\ell}}$, where

$$\underline{\ell} = (\underline{\ell}^*)^n \underline{\ell}(k)(\underline{\ell}^*)^n \underline{\ell}_i \widetilde{\underline{\ell}} := s(\widetilde{\underline{\ell}})\widetilde{\underline{\ell}}.$$

This completes the proof of Lemma 1.

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However, by inspecting the statements, one sees that such a direct application of Lemma 1 is not possible as it deals only with words of the form

$$s(\widetilde{\underline{\ell}})\widetilde{\underline{\ell}}.$$

But, by arguments similar to the proof of ergodicity of Bernoulli shifts, one can show that for μ_--a.e. $x \in \Sigma_-$

$$\underline{\ell}(x, N) \text{ has the form } s(\widetilde{\underline{\ell}})\widetilde{\underline{\ell}}$$

for infinitely many ("times") N.

At this point, it is safe to use Lemma 1 to conclude Prop. 3!

What lies beyond Masur-Veech measures? I

After studying the Lyap. exp. of KZ cocycle wrt Masur-Veech measures μ_{MV} and its applications to the dynamics of almost every i.e.t., one can ask

What about other measures?

Indeed, besides the intrinsic interest, this question is motivated e.g. by the fact that *rational billiards* are not detected by μ_{MV} (and so Lyap. exp. of μ_{MV} don't help in this situation).

What lies beyond Masur-Veech measures? II

On the other hand, it is not a good idea to try to attack all Teichmüller flow g_t erg. inv. prob. at once.

Indeed, since g_t is a non-unif. hyp. flow, it has plenty of inv. meas. (e.g., the ones supported on periodic orbits) and it might be tricky to check whether the Lyap. exp. of KZ cocycle are simple wrt such meas. (cf. 1st exercise in the slides of yesterday's lecture).

What lies beyond Masur-Veech measures? III

Fortunately, g_t is part of a $SL(2,\mathbb{R})$-action and, since the celebrated works of M. Ratner, one has the feeling that $SL(2,\mathbb{R})$-orbits tend to behave better than g_t-orbits.

Remark (for experts)

It was recently announced by A. Eskin and M. Mirzakhani that "the $SL(2,\mathbb{R})$-action on moduli spaces of translation surfaces behave as "predicted" by Ratner's theory"...

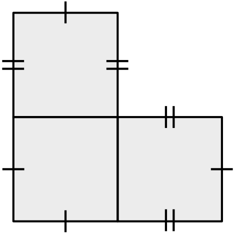
What lies beyond Masur-Veech measures? IV

So, from now on, μ is a $SL(2,\mathbb{R})$-inv. g_t-ergodic prob. on a (c.c. of a) stratum of the moduli space of transl. surf.

What lies beyond Masur-Veech measures? IV

So, from now on, μ is a $SL(2,\mathbb{R})$-inv. g_t-ergodic prob. on a (c.c. of a) stratum of the moduli space of transl. surf.

Such μ's are not very difficult to find: for instance, $SL(2,\mathbb{R})$-orbits of square-tiled surfaces (i.e., transl. surfaces obtained as finite ram. cov. of $\mathbb{T}^2$ branched only at $0 \in \mathbb{T}^2$ – see figure below) are closed and they support an unique $SL(2,\mathbb{R})$-inv. prob.; moreover, the class of square-tiled surfaces is dense in moduli spaces.



EKZ formula I

In a recent work, A. Eskin, M. Kontsevich and A. Zorich completed the proof of a formula (announced 15 years ago...) for the sum

$$\lambda_1^\mu + \cdots + \lambda_g^\mu = 1 + \lambda_2^\mu + \cdots + \lambda_g^\mu$$

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It is beyond scope to present a proof of EKZ formula here: the current version of EKZ paper has 106 pp. and it uses sophisticated tools from Alg. Geom. such as Deligne-Mumford compactification, Grothendieck-Hirzebruch-Riemann-Roch theorem, etc...

In some sense, EKZ paper is the hardest integration by parts "exercise" ever...

EKZ formula II

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To get started, here is a 1st order approximation to EKZ formula:

Theorem (EKZ formula)

Let μ be a $SL(2,\mathbb{R})$-inv. g_t-ergodic prob. on a stratum $\mathcal{M}^{(1)}(k_1,\dots,k_s)$. Then,

$$\lambda_1^\mu + \cdots + \lambda_g^\mu = \frac{1}{12}\sum_{j=1}^{s}\frac{k_j(k_j+2)}{(k_j+1)} + c(\mu)$$

where $c(\mu) > 0$ is a *geometric* quantity associated to μ called *Siegel-Veech constant*.

A technical remark

In fact, there is a subtlety about the assumptions on μ: in EKZ paper, besides $SL(2,\mathbb{R})$-inv. (and g_t-erg.), they require μ to be

- "algebraic", i.e., the support of μ is an affine orbifold (in period coord.) and μ is the "Lebesgue" measure of this affine orbifold, and
- "regular" (a technical assumption for a certain integration by parts argument to work).

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Nevertheless, as we told, Eskin-Mirzakhani showed that any $SL(2,\mathbb{R})$-inv. μ fit Ratner's theory predictions and, in particular, they're alg. Also, all known examples of $SL(2,\mathbb{R})$-inv. prob. are "regular" (in EKZ sense).

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So, for today, let's pretend that "μ is $SL(2,\mathbb{R})$-inv." means

"μ is $SL(2,\mathbb{R})$-inv., g_t-erg., algebraic and regular".

EKZ formula III

Even without knowing what are Siegel-Veech constants, we already can extract consequences of EKZ formula:

Corollary

In genus $g \geq 7$, any $SL(2,\mathbb{R})$-inv. μ satisfies

$$\lambda_2^\mu > 0 \text{ (and actually } \lambda^\mu_{[(g-1)g/(6g-3)]} > 0).$$

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Proof

Since $\lambda_1^\mu = 1$, it suffices to check that the r.h.s. of EKZ formula is > 1 to get $\lambda_2^\mu > 0$, and this follows from

$$\frac{1}{12}\sum \frac{k_j(k_j+2)}{(k_j+1)} + c(\mu) > \frac{1}{12}\sum k_j = \frac{2g-2}{12} \geq 1 \quad (\text{as } g \geq 7)$$

Siegel-Veech constants I

Given S a transl. surf. and γ a *closed regular* geodesic, we can form a maximal cylinder C by collecting all closed regular geodesics of S parallel to γ.

Siegel-Veech constants I

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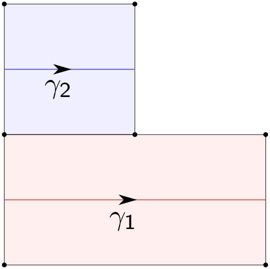
For instance, we see below a transl. surf. with two max. cylinders C_1 and C_2 (assoc. to γ_1 and γ_2) in the horizontal direction.



Siegel-Veech constants II

The width $w(C)$ of a cylinder is the length of its *waist* curve, its height $h(C)$ is the distance across it, and its modulus $\mathrm{mod}(C)$ is $h(C)/w(C)$.

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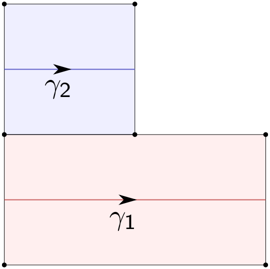
In the figure of previous slide (recalled below), $w(C_1) = 2$, $h(C_1) = 1$ and $\mathrm{mod}(C_1) = 1/2$.



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Of course $N_{area}(S, L)$ depends a lot on S and L, but W. Veech and Ya. Vorobets proved that, for any $SL(2,\mathbb{R})$-inv. μ, the quantity

$$c(\mu) := \frac{\pi}{3L^2} \int N_{area}(S, L) d\mu(S)$$

independs on L.

Siegel-Veech constants IV

As you can guess, $c(\mu)$ is the Siegel-Veech constant: in some sense, $c(\mu)$ measures how transl. surf. $S \in \mathrm{supp}(\mu)$ are filled by max. cyl. of bounded length in average.

Siegel-Veech constants IV

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In particular, $c(\mu)$ is a quantity related to the geometry of transl. surf. in the support of μ.

Siegel-Veech constants V

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Siegel-Veech constants V

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Instead, we present only two contexts where Siegel-Veech constants for $SL(2,\mathbb{R})$-inv. μ are known:

- for any μ in genus 2;
- for μ supported on $SL(2,\mathbb{R})$-orbits of *square-tiled surfaces*.

SV constants and individual values of exponents

In genus 2 there are two strata $\mathcal{M}(2)$ and $\mathcal{M}(1,1)$. It was shown by EKZ that

$$c(\mu) = \begin{cases} 10/9 & \text{if } \operatorname{supp}(\mu) \subset \mathcal{M}(2) \\ 15/12 & \text{if } \operatorname{supp}(\mu) \subset \mathcal{M}(1,1) \end{cases}$$

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By EKZ formula, this means that we know all Lyap. exp. in genus 2:

$$1 + \lambda_2^\mu = \lambda_1^\mu + \lambda_2^\mu = \begin{cases} 4/3 & \text{if } \mathrm{supp}(\mu) \subset \mathcal{M}(2) \\ 3/2 & \text{if } \mathrm{supp}(\mu) \subset \mathcal{M}(1,1) \end{cases}$$

that is

$$\lambda_2^\mu = \begin{cases} 1/3 & \text{if } \mathrm{supp}(\mu) \subset \mathcal{M}(2) \\ 1/2 & \text{if } \mathrm{supp}(\mu) \subset \mathcal{M}(1,1) \end{cases}$$

SV constants of square-tiled surfaces and sums of exp.

It was shown by EKZ that SV const. of μ associated to $SL(2,\mathbb{R})$-orbits of a square-tiled surface M_0 is given by

$$c(\mu) = \frac{1}{\#SL(2,\mathbb{Z})\cdot M_0}\sum_{M_i\in SL(2,\mathbb{Z})\cdot M_0}\sum_{M_i=\cup C_{ij}}\frac{h(C_{ij})}{w(C_{ij})}$$

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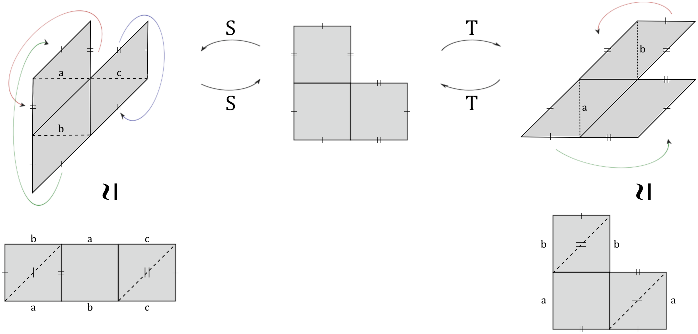
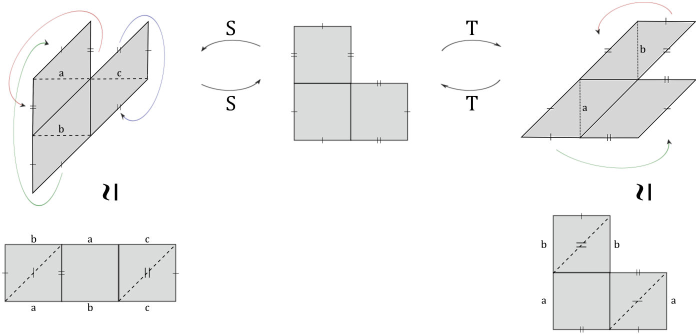
In particular, for a square-tiled surface $M_0\in\mathcal{M}^{(1)}(k_1,\dots,k_s)$, the sum $\Lambda(\mu)=\lambda_1^\mu+\dots+\lambda_g^\mu$ of exponents is

$$\frac{1}{12}\sum_{j=1}^{s}\frac{k_j(k_j+2)}{(k_j+1)}+\frac{1}{\#SL(2,\mathbb{Z})\cdot M_0}\sum_{M_i\in SL(2,\mathbb{Z})\cdot M_0}\sum_{M_i=\cup C_{ij}}\frac{h(C_{ij})}{w(C_{ij})}$$

Example I: a genus 2 square-tiled surface in $\mathcal{M}(2)$



Example I: a genus 2 square-tiled surface in $\mathcal{M}(2)$

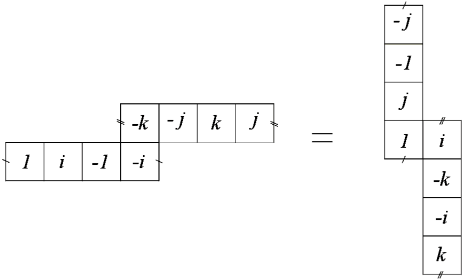


From this picture, we infer that the sum of Lyap. exp. is

$$\Lambda(\mu) = \frac{1}{12}\frac{2\cdot 4}{3} + \frac{1}{3}\left\{\left(\frac{1}{3}\right) + \left(\frac{1}{1}+\frac{1}{2}\right) + \left(\frac{1}{1}+\frac{1}{2}\right)\right\} = \frac{4}{3},$$

a fact that we already knew (from the genus 2 discussion above).

Example II: a genus 3 square-tiled surface



This genus 3 square-tiled surface $M_{EW} \in \mathcal{M}(1,1,1,1)$ satisfies $SL(2,\mathbb{Z}) \cdot M_{EW} = M_{EW}$, so that the sum of Lyap. exp. is

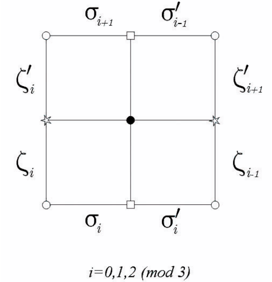
$$\Lambda(\mu) = \frac{1}{12} 4 \cdot \frac{1 \cdot 3}{2} + \frac{1}{1}\left(\frac{1}{4} + \frac{1}{4}\right) = 1,$$

i.e., $\lambda_2^\mu = \lambda_3^\mu = 0$! So, there is no Avila-Viana thm for M_{EW} at all!

Example IIa: a genus 3 square-tiled surface

The square-tiled surface from previous slide was discovered by G. Forni and it is called *Eierlegende Wollmilchsau*:

Example III: a genus 4 square-tiled surface



This genus 4 square-tiled surface $M_{EW} \in \mathcal{M}(2,2,2)$ satisfies $SL(2,\mathbb{Z}) \cdot M_O = M_O$, so that the sum of Lyap. exp. is

$$\Lambda(\mu) = \frac{1}{12} 3 \cdot \frac{2 \cdot 4}{3} + \frac{1}{1}\left(\frac{1}{6} + \frac{1}{6}\right) = 1,$$

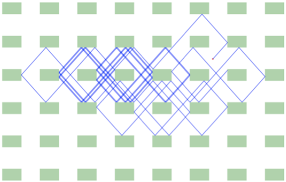
i.e., $\lambda_2^\mu = \lambda_3^\mu = \lambda_4^\mu = 0!$

Example IIIa: a genus 4 square-tiled surface

The square-tiled surface from previous slide was discovered by G. Forni and a coauthor, and it is called *Ornithorynque*:

Why trying to get explicit values for these exponents?

The knowledge KZ cocycle has an exp. $2/3 > 1/2$ in a certain example recently allowed V. Delecroix, P. Hubert and S. Lelièvre to confirm a prediction of the physicists J. Hardy and J. Weber that the Ehrenfest wind-tree model:



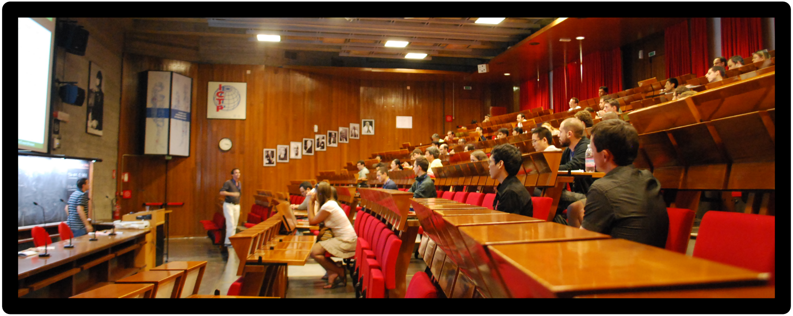
has *abnormal diffusion rate* for typical choices of parameters, i.e.,

$$\limsup_{t\to\infty} \frac{\log d(x, \phi_\theta^t(x))}{\log t} = 2/3 > 1/2$$

for a.e. $0 < a, b < 1$ (sizes of the rectangles), θ and x.

Acknowledgment

Besides the organizers, I'm thankful to all young brave warriors



who "survived" this intense minicourse, and I hope that by now you are convinced that, after working hard to setup appropriate renormalization dynamics, one is frequently payed off with beautiful theorems!