Pseudo-integrable billiards and arithmetic dynamics

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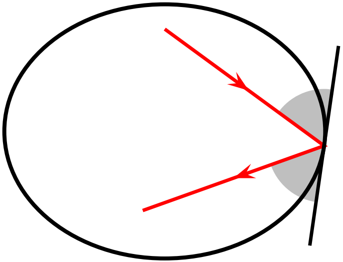
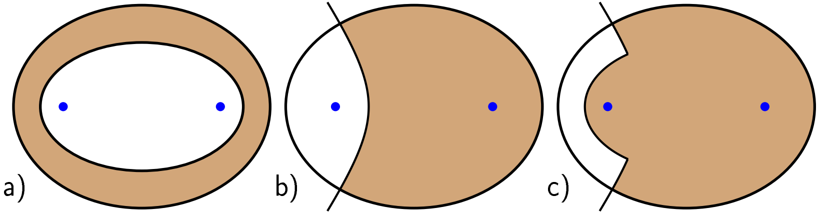
Definition of billiard

Billiard within a given domain is a dynamical system where a particle is moving freely inside the domain, and reflecting absolutely elastically on the boundary.

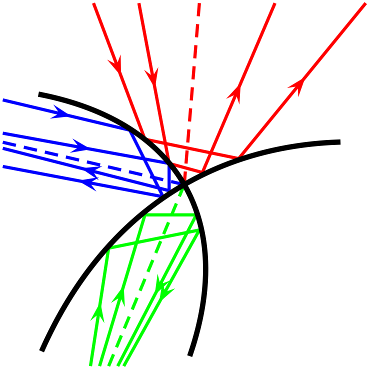
Trajectories are polygonal lines with vertices lying on the domain boundary, with congruent impact and reflection angles at each vertex, while the particle speed remains constant.



Today, we discuss billiards in domains bounded by arcs of several confocal conics.



Reflection in the right angle

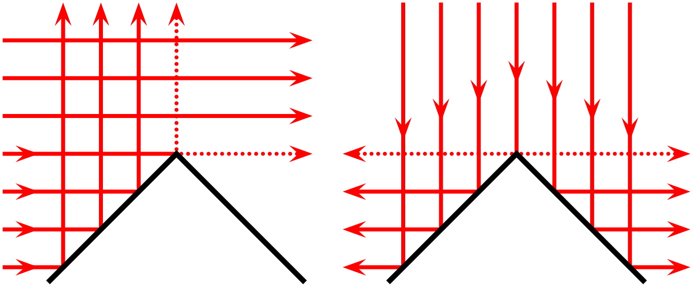


The limit exists

The reflection off the vertex of a right angle is sending the particle back in the opposite direction.

Reflection in angle $> 180°$

The limit does not exist, thus the reflection cannot be defined uniquely.

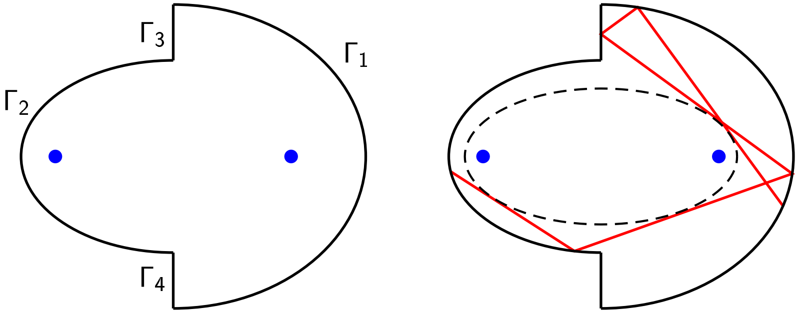


Our aim

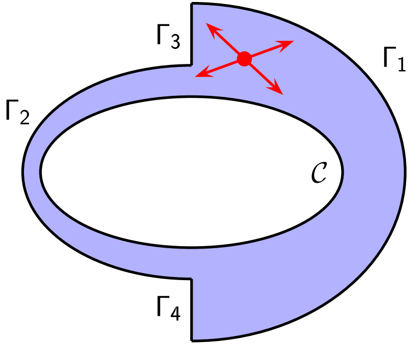
Billiard dynamics in a domain bounded by arcs of a few confocal conics containing reflex angles on the boundary.

Main example

A domain bounded by two confocal ellipses and two segments placed on the smaller axis of theirs. We fix a caustic to be an ellipse completely contained in the domain.



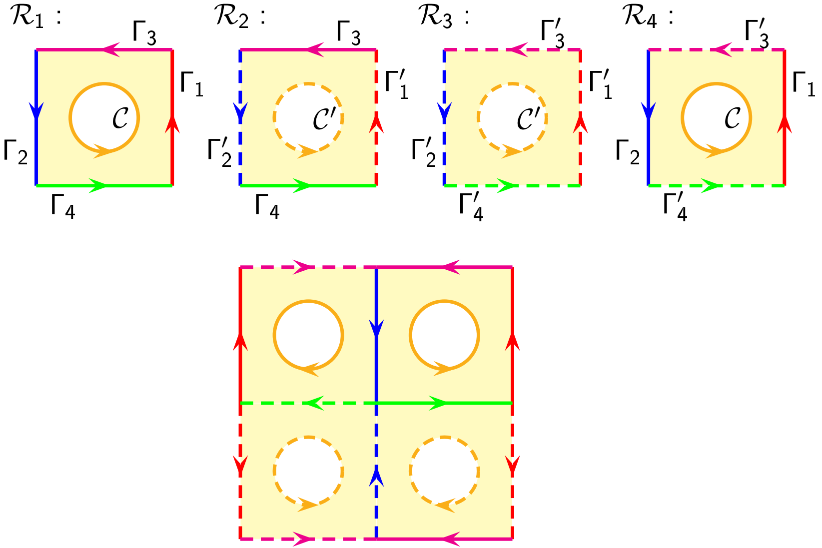
Billiard trajectories are placed in the the ring between the billiard border and the caustic.



Each point of the ring is the projection of four points from the phase space.

The corresponding leaf of the phase space is naturally decomposed into four such rings, which are glued to each other along the border arcs.

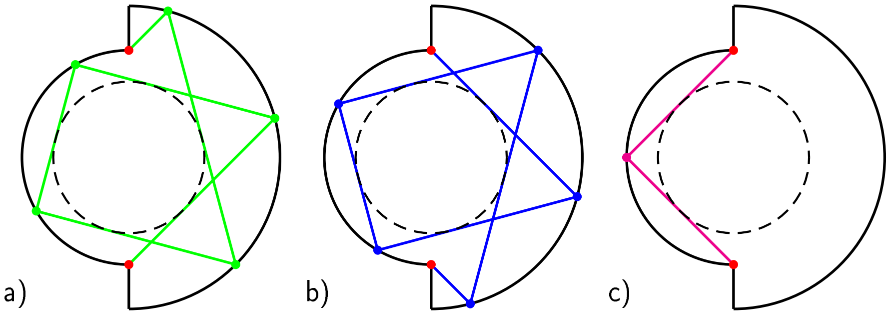
The leaf is an orientable surface of genus 3



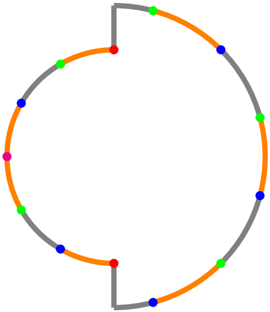
Circles with rotation numbers $\frac{1}{3}$ and $\frac{1}{4}$

The domain with two concentric half-circles of radii $2R$, $R\sqrt{2}$ and the corresponding segments; the caustic with radius R.

There exist six trajectories connecting singular points corresponding to concave angles of the boundary.

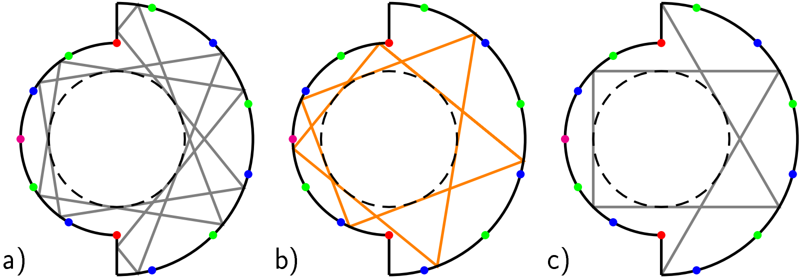


Vertices of the saddle connections divide the billiard border into thirteen parts.



All trajectories are periodic:

- either all bouncing points of a given trajectory are in gray parts – the billiard particle hits twice each gray part. It is 12-periodic: four bounces on the smaller circle, six bounces on the bigger one, and one on each of the segments on the y-axis;
- or all bouncing points are in orange parts – the particle will hit each part once; it is 7-periodic: it hits three times the bigger circle and four times the smaller one.

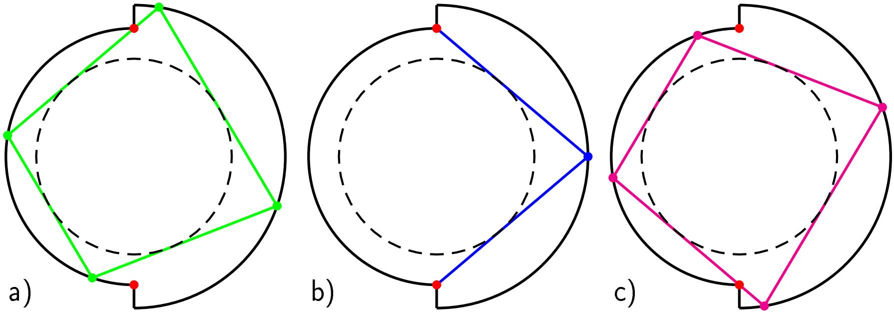


The corresponding level set in the phase space is divided by the saddle-connections into three parts:

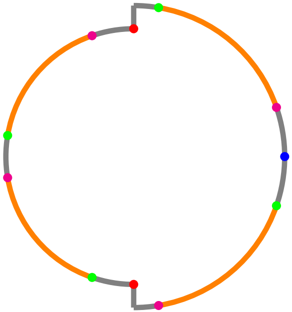
- the part containing all 12-periodic trajectories: this part is bounded by four saddle-connections;
- two parts containing all 7-periodic trajectories winding about the caustic in the clockwise and counterclockwise direction: these parts are bounded by saddle-connections winding in the same direction.

Circles with rotation numbers $\frac{5-\sqrt{5}}{10}$ and $\frac{\sqrt{5}}{10}$

There exist six saddle-connections.

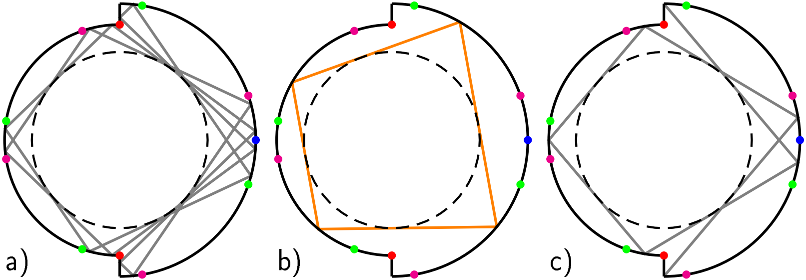


Vertices of the saddle connections divide the billiard border into eleven parts.



All trajectories are periodic:

- either all bouncing points of a given trajectory are in gray parts – in this case the billiard particle hits twice each gray part; the trajectory is 14-periodic. A trajectory bounces six times on each of the circles and once on each of the segments;
- or all bouncing points are in orange parts – the particle will hit each part once; it is 4-periodic. A trajectory reflects twice on each of the circular arcs.


a) b) c)

The level set in the phase space is divided by the saddle-connections into three parts:

- the part containing all 14-periodic trajectories: this part is bounded by four saddle-connections.
- two parts containing all 4-periodic trajectories winding about the caustic in the clockwise and counterclockwise direction: these parts are bounded by saddle-connections winding in the same direction.

Local Poncelet porisms

For the billiard within one ellipse the famous Poncelet porism holds:

(A) if there is a periodic billiard trajectory with one initial point of the boundary, then there are infinitely many such periodic trajectories with the same period, sharing the same caustic;

(B) even more is true, if there is one periodic trajectory, then all trajectories sharing the same caustic are periodic with the same period.

For domains bounded with more than one confocal conic (A) is still generally true. However, (B) is not true any more.
The Poncelet porism is true locally, but not globally.

Maier's theorem (1943) vs. Liouville-Arnol'd's theorem

The theory of measured foliations in pseudo-billiard context leads to a version of the Poncelet theorem.

Theorem [V. Dragović, M.R. (2012)]

There exist saturated subsets $D_1, \dots, D_N$ of the boundary Γ, with the following properties:

- $D_1, \dots, D_N$ are pairwise disjoint;
- each D_k is a finite union of d_k open subarcs of Γ: $D_k = \bigcup_{j=1}^{d_k} \ell_j^k$;
- closure of $D_1 \cup \dots \cup D_N$ is Γ,

such that they satisfy:

- if one billiard trajectory with bouncing points within D_k is periodic, then all such trajectories are periodic with period n_k. Moreover, n_k is a multiple of d_k and every such a trajectory bounces the same number $\dfrac{n_k}{d_k}$ of times off each arc ℓ_j^k;
- if billiard trajectories having vertices in D_k are non-periodic, then the bouncing points of each trajectory are dense in D_k.

In either case, the boundary of D_k consists of bouncing points of saddle connections.

Necessary conditions for periodicity

The confocal family

$\mathcal{C}_\lambda \ : \ \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} = 1$, $a > b > 0$

Cayley-type condition, obtained by application of [V. Dragović, M.R, J.Phys.A (2004)]

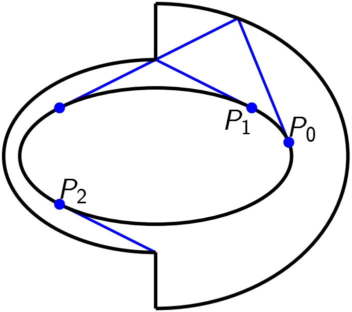
Consider domain $\mathcal{D}$ bounded by half-ellipses $\mathcal{C}_{\beta_1}$ and $\mathcal{C}_{\beta_2}$ and two corresponding segments belonging to y-axis. A necessary condition for the existence of a billiard trajectory within $\mathcal{D}$ with $\mathcal{C}_{\alpha_0}$ as a caustic which becomes closed after n_1 reflections off $\mathcal{C}_{\beta_1}$ and n_2 reflections off $\mathcal{C}_{\beta_2}$ is:

$$n_1\mathcal{A}(P_{\beta_1}) + n_2\mathcal{A}(P_{\beta_2}) = (n_1 + n_2)\mathcal{A}(P_{\alpha_0})$$

$\mathcal{A}$ – the Abel-Jacobi map of the ellitic curve:
$s^2 = (a-t)(b-t)(\alpha_0 - t)$.
P_δ is $(\delta, \sqrt{(a-\delta)(b-\delta)(\alpha_0-\delta)})$ on Γ.

Interval exchange transformation

The part of the boundary where the billiard particle is going to hit depends on the direction of motion and the touching point with the caustic.



Interval exchange transformation

To see the billiard dynamics as an interval exchange transformation, we make the following identification:

$$(X, +) \sim p(X), \quad (X, -) \sim q(X).$$

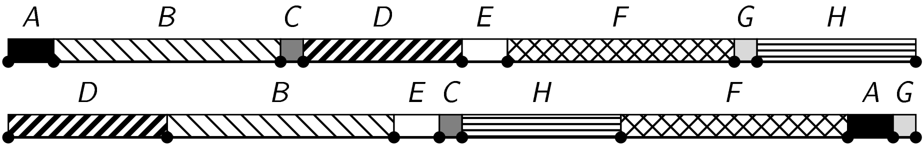
In other words:

- we identify the joint point X of a given trajectory with the caustic with $p(X) \in [0, 1)$ if the particle is moving in the counterclockwise direction on the corresponding segment;
- for the motion in the clockwise direction, we identify X with $q(X) \in [-1, 0)$.

$p(X)$ and $q(X)$ are natural parametrizations of the caustic with respect to the invariant measure corresponding to the billiard dynamics.

P_0 above the x-axis

$$\xi \mapsto \begin{cases} \xi + r_1 + \frac{3}{2}, & \xi \in [-1, r_1 - r_2 - 1), \\ \xi + r_2, & \xi \in [r_1 - r_2 - 1, r_1 - r_2 - \frac{1}{2}), \\ \xi + r_1, & \xi \in [r_1 - r_2 - \frac{1}{2}, -r_1), \\ \xi + r_1 - 1, & \xi \in [-r_1, 0), \\ \xi + r_1 - \frac{1}{2}, & \xi \in [0, r_1 - r_2), \\ \xi + r_2, & \xi \in [r_1 - r_2, r_1 - r_2 + \frac{1}{2}), \\ \xi + r_1, & \xi \in [r_1 - r_2 + \frac{1}{2}, 1 - r_1), \\ \xi + r_1 - 1, & \xi \in [1 - r_1, 1). \end{cases}$$



Pencil independence – arithmetic dynamics

The interval exchange transformations depends only on the rotation numbers r_1, r_2.

Theorem [V. Dragović, M.R. (2012)]

The billiard dynamics inside the domain $\mathcal{D}$ with an ellipse as the caustic, does not depend on the confocal family but only on rotation numbers r_1, r_2.

Billiard-like interval exchange transformations

An interval exchange transformation f of $I = [-1, 1)$ is billiard-like if the partition into subintervals satisfies the following:

- for each α, I_α is contained either in $[-1, 0)$ or $[0, 1)$;
- both $[-1, 0)$ and $[0, 1)$ contain at least two intervals of the partition.

Keane condition

We will say that a billiard-like interval exchange transformation f satisfies the modified Keane condition if $f^m(p_\alpha) \neq p_\beta$ for all $m \geq 1$, $\alpha \in \mathcal{A}$, and $\beta \in \mathcal{A}$ such that $p_\beta \notin \{-1, 0\}$.

Proposition [V. Dragović, M.R. (2012)]

If an irreducible billiard-like interval exchange transformation f satisfies the modified Keane condition, then f is minimal.

Cayley-type condition vs. Keane condition

Rotation numbers

$r_1 = \frac{5}{11} + \frac{1}{22\pi}$, $r_2 = \frac{5}{11} - \frac{1}{220\pi}$

Periodicity condition is satisfied

$r_1 + 10r_2 = 5$

Proposition [V. Dragović, M.R. (2012)]

The corresponding transformation satisfies the Keane condition.

In this example, although the Cayley-type conditon for periodicity is satisfied, not only that closed trajectories do not exist, but each trajectory densely fills the ring between the billiard border and the caustic.

The End