Renormalization for Lorenz maps

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Lorenz dynamics

The geometric Lorenz flow (R. F. Williams, 79): a model capturing the qualitative behaviour of the Lorenz flow with extra conditions. Specifically, the return map

- map preserves a one-dimensional foliation in a section transversal to the flow;
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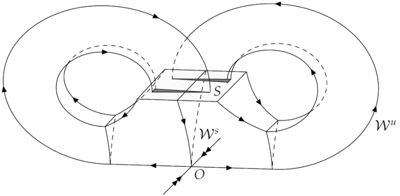

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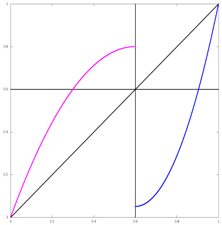

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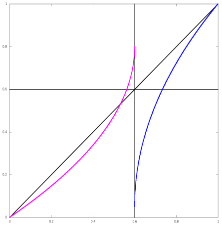

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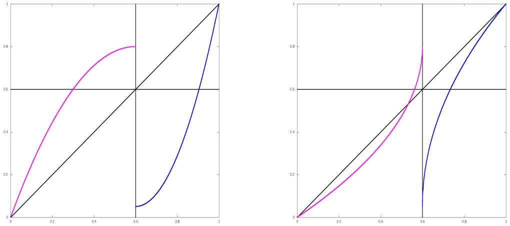

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• There is an open set of vector fields in $\mathbb{R}^3$, that generate a geometric Lorenz flow with a "smooth" Lorenz map of $\rho < 1$ (J. Guckenheimer and R. F. Williams, '79). "Similarly", for $\rho \geq 1$.

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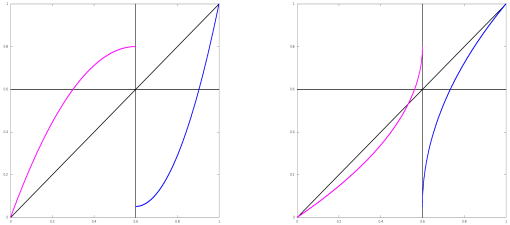

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• As with the unimodal maps, Lorenz maps with $\rho > 1$ have a richer dynamics that combines contraction with expansion.

Let $u \in [0,1]$, $v \in [0,1]$, $c \in (0,1)$ and $\rho > 0$. The standard Lorenz family $(u,v,c) \mapsto Q(x)$ is the family of maps $Q : [0,1] \setminus \{c\} \mapsto [0,1]$ with a single critical point at which the map is discontinuous:

$$Q(x) = \begin{cases} u\left(1 - \left(\frac{c-x}{c}\right)^{\rho}\right), & x \in [0,c), \\ 1 + v\left(-1 + \left(\frac{x-c}{1-c}\right)^{\rho}\right), & x \in (c,1], \end{cases}$$

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More generally,

A C^k-Lorenz map $f : [0,1] \setminus \{c\} \mapsto [0,1]$ is defined as

$$f(x) = \begin{cases} f_0(x) \equiv \phi(Q(x)), & x \in [0,c), \\ f_1(x) \equiv \psi(Q(x)), & x \in (c,1], \end{cases}$$

where ϕ and ψ are C^k orientation preserving diffeomorphisms of $[0,1]$ (this space will be denoted by $\mathcal{D}^k$).

The set of C^k-Lorenz maps will be denoted $\mathcal{L}^k$

Kneading sequences

• For any $x \in [0,1] \setminus \{c\}$ such that $f^n(x) \neq c$ for all $n \in \mathbb{N}$, define the itinerary $\omega(x) \in \{0,1\}^{\mathbb{N}}$ of x as the sequence $\{\omega^0(x), \omega^1(x), \ldots\}$, such that

$$\omega^i = \begin{cases} 0, & f^i(x) < c, \\ 1, & f^i(x) > c. \end{cases}$$

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• The kneading invariant $K(f)$ of f is the pair $(K^-(f), K^+(f)) = (\omega(c^-), \omega(c^+))$.

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• Can it happen that the limiting dynamics concentrates on a subset of the domain, i.e. is there an attractor for dynamics?

Renormalization

A Lorenz map f is renormalizable if there exist p and q, $0 < p < c < q < 1$, s. t. the first return map (f^n, f^m), $n > 1, m > 1$, of $C = [p, q]$ is affinely conjugate to a nontrivial Lorenz map. Choose C such that it is maximal. The rescaled first return map of such $C \setminus \{c\}$ is called the renormalization of f and denoted $\mathcal{R}[f]$.

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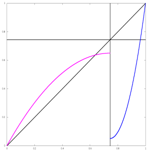
- A Lorenz map f with $c_1^+ < c < c_1^-$ is called nontrivial, otherwise f has a globally attracting fixed point.



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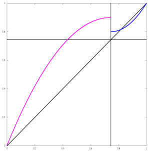

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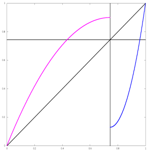

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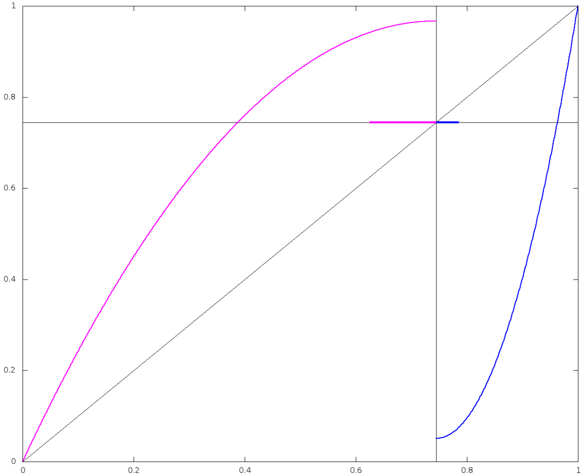
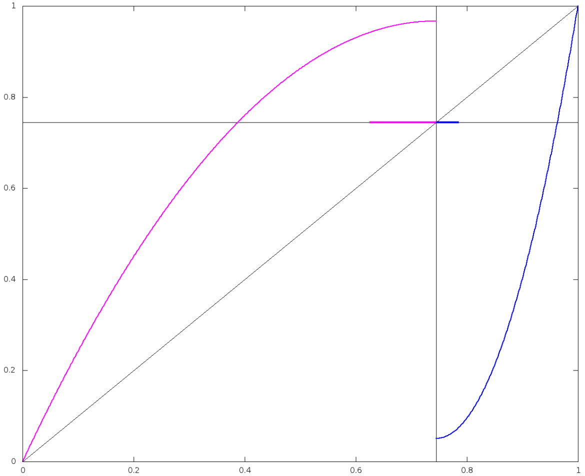


A Lorenz map of renormalization type $(01, 1000)$.



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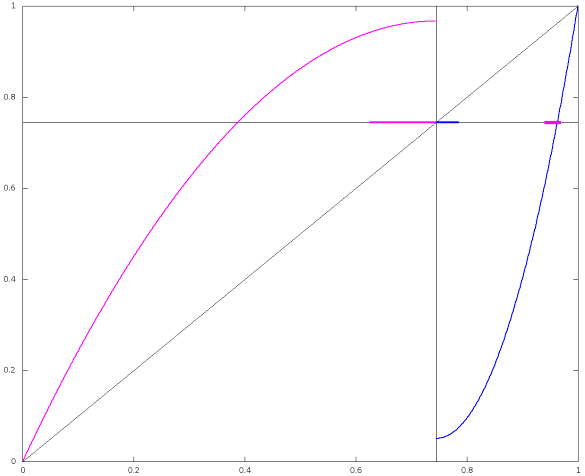

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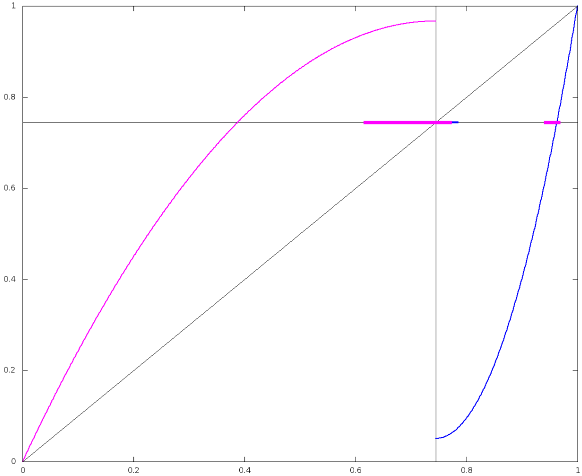

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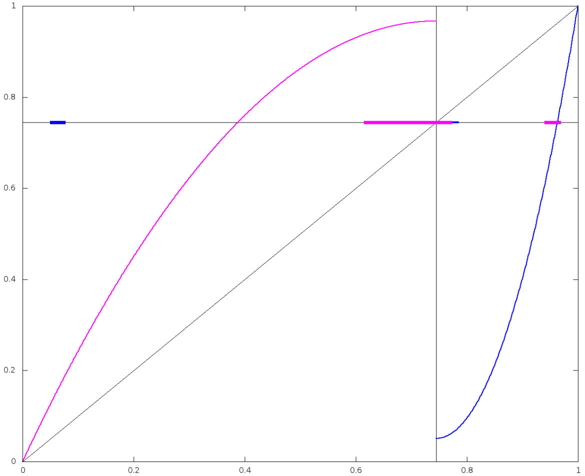

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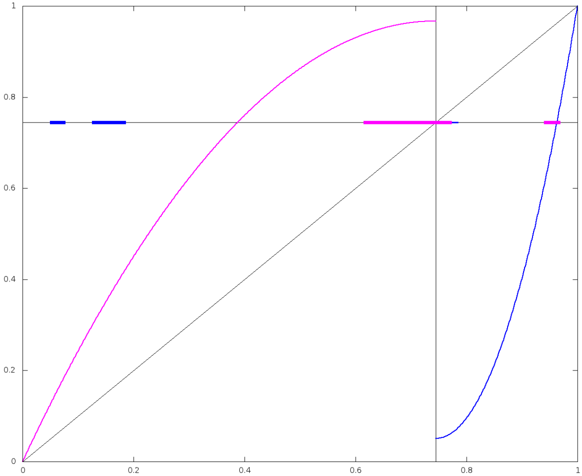

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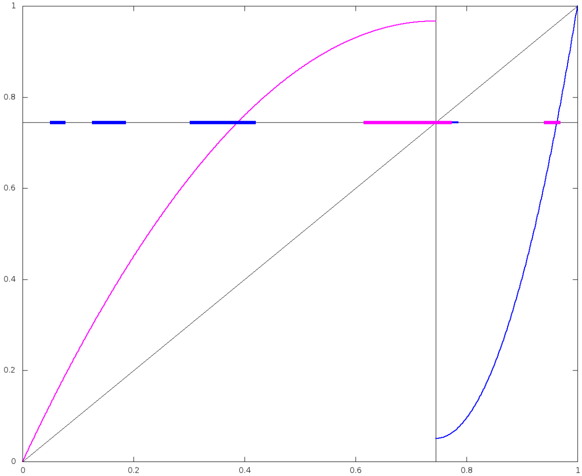

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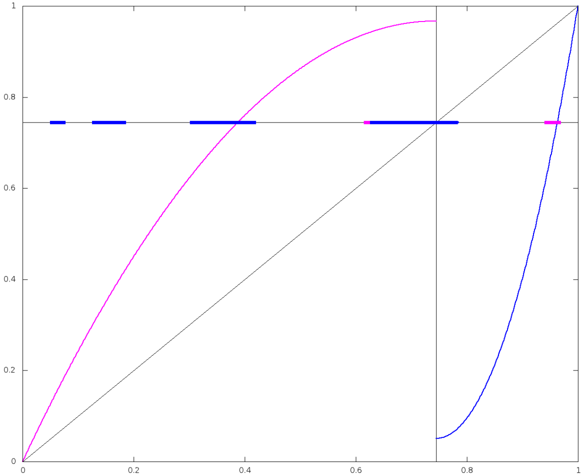

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• Denote $L = [p, c)$, $R = (c, q]$, the first return map will be denoted $\mathcal{P}[f]$ - the prerenormalization. If f is renormalizable, then $\exists$ minimal $n > 1, m > 1$ such that

$$\mathcal{P}[f](x) = \begin{cases} f^{n+1}(x), & x \in L, \\ f^{m+1}(x), & x \in R, \end{cases}$$

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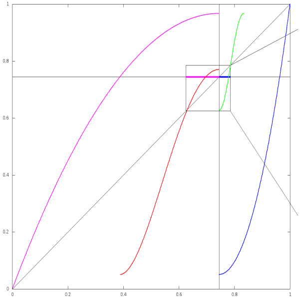

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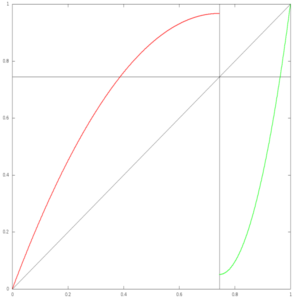

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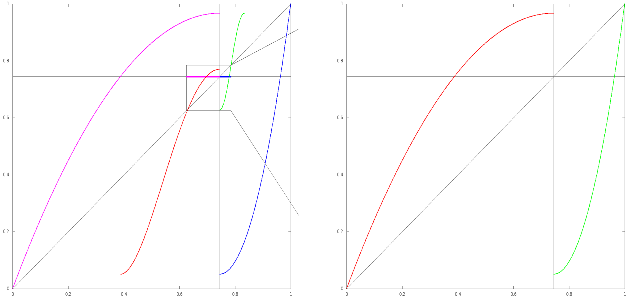

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• $$\mathcal{R}[f] = A^{-1} \circ \mathcal{P}[f] \circ A,$$

where A is the affine orientation preserving rescaling of $[0, 1]$ onto C.

• The intervals $f^i(L)$, $1 \le i \le n$, and $f^i(R)$, $1 \le i \le m$, are pairwise disjoint, and disjoint from C. Associate a finite sequence of 0 and 1 to each of these two sequences of intervals:

$$\omega^- = \{K_0^-, \ldots, K_n^-\},\ \omega^+ = \{K_0^+, \ldots, K_m^+\},$$

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$$\omega = (0\overbrace{1\ldots1}^{n}, 1\overbrace{0\ldots0}^{m}) \tag{1}$$

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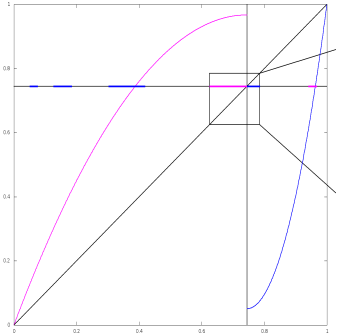
• If $\mathcal{R}^i[f] \subset \mathcal{L}_{\omega_i}$ and $\bar{\omega} = (\omega_0, \omega_1, ..., \omega_n)$, with n finite or infinite, then $f \in \mathcal{L}_{\bar{\omega}}$.

• Suppose we can construct a compact $\mathcal{K} \subset \mathcal{L}^k$ which is renormalization invariant: $R[\mathcal{K}] \subset \mathcal{K}$. Then consider a $f \in \mathcal{K} \cap \mathcal{L}^S_{\bar{\omega}}$, $\bar{\omega} = \{\omega_0, \omega_1, \omega_2, \dots\} \in \mathcal{M}^{\mathbb{N}}$.



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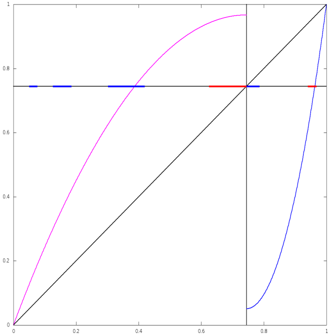

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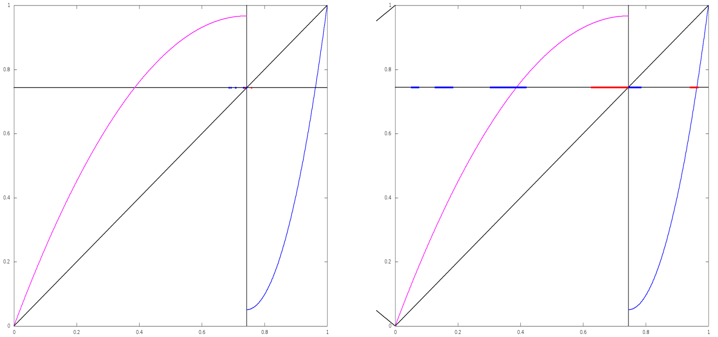

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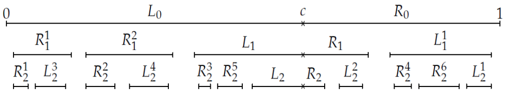

Generations of a Cantor set for an infinitely renormalizable map of type $(01, 100)$.

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Intervals in Λ_n are intervals of generation n. Components of $\Lambda_{n-1} \setminus \Lambda_n$ are gaps of generation n.

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Cantor attractor

Suppose that $\mathcal{K}$ is a compact non-empty subset of $\mathcal{L}^k$, $k \geq 3$, such that $R[\mathcal{K} \cap \mathcal{L}^S_{\mathcal{M}}] \subset \mathcal{K}$.
Let $f \in \mathcal{L}^S_{\mathcal{M}} \cap \mathcal{K}$ be infinitely renormalizable, and let Λ be the closure of the orbits of the critical values. Then,

1) *Λ is a Cantor set of Lebesgue measure 0 and a Hausdorff dimension strictly inside $(0,1)$;*
2) *Λ is uniquely ergodic (if bounded combinatorics);*
3) *the complement of the basin of attraction of Λ in $[0,1]$ has zero Lebesgue measure.*

Compactness

- A Lorenz map can be identified with a quintuple (u, v, c, ϕ, ψ):

$$\mathcal{L}^k \approx [0,1]^2 \times (0,1) \times \mathcal{D}^k \times \mathcal{D}^k.$$

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Given $\pi > 0$, $\epsilon > 0$, *set*

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Given $\pi > 0$ *and* $\epsilon > 0$, *the set* $\mathcal{K}_\epsilon^\pi$ *is relatively compact in* $\mathcal{L}^0$.

A-priori bounds

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1) Denote the affine transformation that takes $[0,1]$ onto I as ξ_I.
The quintuple for the renormalized map is

$$\tilde{u} = \frac{|Q(L)|}{|U|}, \quad \tilde{v} = \frac{|Q(R)|}{|V|}, \quad \tilde{c} = \frac{|L|}{|C|},$$

$$\tilde{\phi} = \xi^{-1}_{\bar{\phi}(U)} \circ \bar{\phi} \circ \xi_U, \quad \tilde{\psi} = \xi^{-1}_{\bar{\psi}(V)} \circ \bar{\psi} \circ \xi_V, \quad \bar{\phi} = f_1^n \circ \phi, \quad \bar{\psi} = f_0^m \circ \psi,$$

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2) Demonstrate that for any $x, y \in [0,1]$,

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Use the Koebe principle!

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3) Estimate $\tilde{c} = \frac{|L|}{|C|}$, and show that there exists $\epsilon > 0$ such that $\tilde{c} \in [\epsilon, 1-\epsilon]$ whenever c is.

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