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Measures with historic behavior

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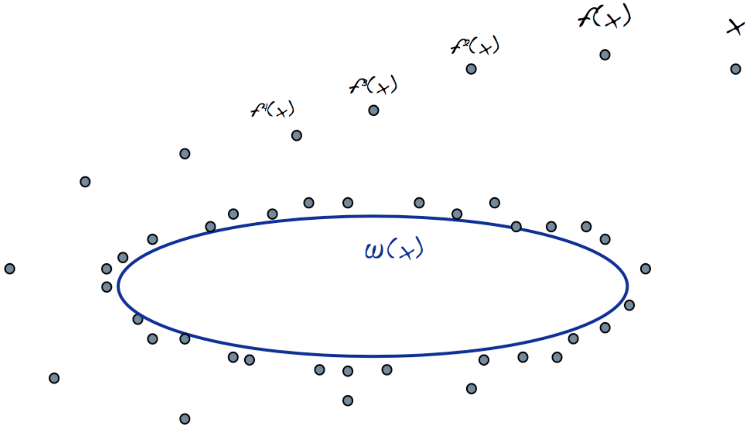
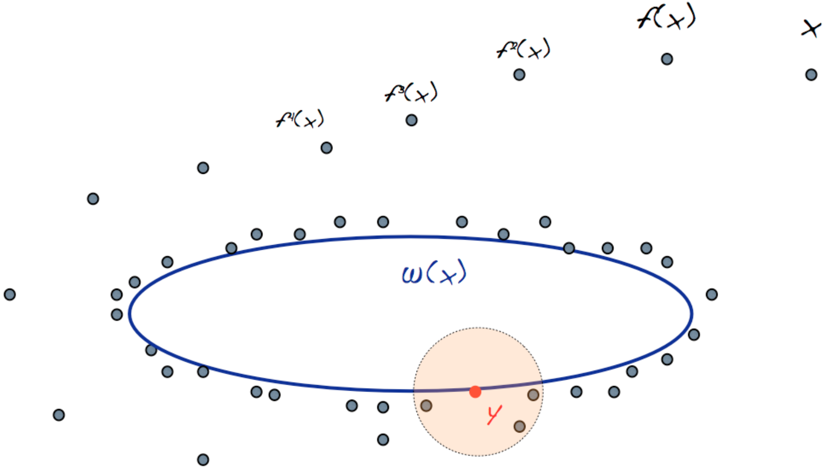
Let $\mathbb{X}$ be a compact metrical space and $f : \mathbb{X} \to \mathbb{X}$ a continuous map.

If you want to understand the asymptotic behavior of the positive orbit $\mathcal{O}^+(x)$ of x, you have to focus on $\omega(x)$ (or a neighborhood of it). At least from the topological point of view.



Nevertheless, the time that $\mathcal{O}^+(x)$ spend near to a point $y \in \omega(x)$, in general, is not homogeneous, it changes with y.



Thus, we can try to understand the frequency of visit of a orbit in a set A. That is,

$$\lim_k \frac{1}{k} \#\{0 \leq j < k\,;\, f^j(x) \in A\} \ \left(= \lim_k \frac{1}{k} \sum_{j=1}^{k-1} \delta_{f^j(x)}(A) \right).$$

Warning!
In general, the limit above does not exists!
(even if $\lim_k f^k(x)$ exists)

Example 1

Let $n_1 = 1,\ n_2 = 11,\ n_3 = 111,\ \cdots,\ n_j = \sum_{i=0}^{j} 10^i$.

Given a point $x \notin \mathrm{Per}(f)$, let

$$A = \{f(x), \cdots, f^{11}(x)\} \cup \{f^{111}(x), \cdots, f^{1111}(x)\} \cup$$
$$\cup \{f^{11111}(x), \cdots, f^{111111}(x)\} \cup \cdots$$

As

$$\frac{1}{n_k} \sum_{j=1}^{n_k - 1} \delta_{f^j(x)}(A) = \begin{cases} \geq \frac{9}{10} & \text{for } k \text{ even} \\ \leq \frac{1}{10} & \text{for } k \text{ odd} \end{cases},$$

it follows that $\lim \frac{1}{n} \sum_{j=1}^{n-1} \delta_{f^j(x)}(A)$ does not exist, even if $\lim_k f^k(x)$ exists!

A way to try to bypass this problem is to consider the convergence of $\lim \frac{1}{n}\sum_{j=1}^{n-1}\delta_{f^j(x)}$ in the weak topology. In this direction we have the Birkhoff theorem

Theorem (Birkhoff)

If μ is an invariant finite measure then $\frac{1}{n}\sum_{j=1}^{n-1}\delta_{f^j(x)}$ converges in the weak topology for μ-almost every x.

That is,

for μ-almost every x

$$\frac{1}{n}\sum_{j=1}^{n-1}\varphi(f^j(x)) \text{ converges } \forall\,\varphi \in C^0(\mathbb{X}).$$

Definition (Orbits with Historic Behavior)

We say that a point x (or its orbit $\mathcal{O}^+(x)$) has historic behavior if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n-1}\varphi(f^j(x))$$

does not exist for some $\varphi \in C^0(\mathbb{X})$.

That is,

$\frac{1}{n}\sum_{j=1}^{n-1}\delta_{f^j(x)}$ does not converges in the weak topology.

This terminology was introduced by Ruelle in 2001.

Ruelle, D.. **Historic behaviour in smooth dynamical systems**. *Global Analysis of Dynamical Systems ed H W Broer et al. 2001(Bristol: Institute of Physics Publishing).*

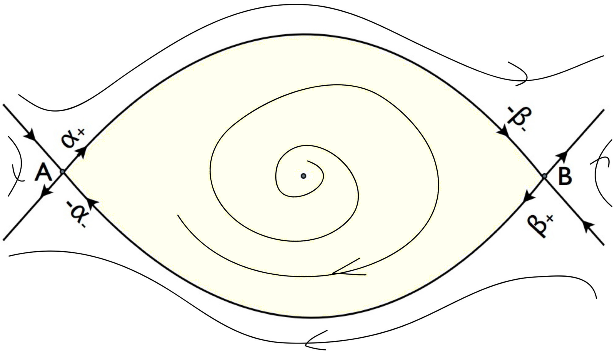
Reference to “Orbits with Historic Behavior”

Takens, F.. *Orbits with historic behaviour, or non-existence of averages*. Nonlinearity 21 (2008), T33–T36.

Example 2 (This example was credited to Bowen by Takens).

Theorem (Bowen?)

Let f be the time 1 of the flow below. If $\alpha_-\beta_- > \alpha_+\beta_+$ *then* $\frac{1}{n}\sum_{j=1}^{n-1}\delta_{f^j(x)}$ *does not converges in the weak topology for every x inside the "Bowen's eye", with the exception of the source in the center.*



Reference to “Bowen’s eye”

Takens, F. Heteroclinic attractors: time averages and moduli of topological stability Bol. Soc. Bras. Mat., 1994, 25, 107-120.

Orbits with historic behavior in hyperbolic dynamics

Let $\sum_2^+ = \{0,1\}^{\mathbb{N}}$ and $\sigma : \sum_2^+ \to \sum_2^+$ be the shift, i.e.,

$$\sigma(x_1 x_2 x_3 \cdots) = (x_2 x_3 x_4 \cdots).$$

Let $\varphi \in C^0(\sum_2^+)$ be given by $\varphi(x_1 x_2 x_3 \cdots) = x_1$ and $\alpha \in \sum_2^+$ by

$$\alpha = 0^{n_1} 1^{n_2} 0^{n_3} 1^{n_4} 0^{n_5} \cdots ,$$

where $0^n = \underbrace{000\cdots0}_{n \text{ times}}$, $1^n = \underbrace{1\cdots1}_{n \text{ times}}$ and $n_1 = 1,\ n_2 = 11,\ n_3 = 111,$ $n_4 = 1111,\ \cdots$ (as defined before).

Orbits with historic behavior in hyperbolic dynamics

Again, a simple calculation shows that

$$\frac{1}{n_k}\sum_{j=1}^{n_k-1}\varphi(\sigma^j\alpha) = \begin{cases} \geq \frac{9}{10} & \text{for } k \text{ even} \\ \leq \frac{1}{10} & \text{for } k \text{ odd} \end{cases},$$

that is, α has historic behavior.

Moreover,

As $\mathcal{O}^-(\alpha) = \bigcup_{j\geq 0}\sigma^{-j}(\{\alpha\})$ is a dense subset of $\sum_2^+$ then

the set of points with historic behavior is dense.

Orbits with historic behavior in hyperbolic dynamics

Let W_n be the set of points $x \in \sum_2^+$ s.t.

$$\frac{1}{m}\sum_{j=1}^{m-1}\varphi(\sigma^j x) > 8/10$$

and

$$\frac{1}{\ell}\sum_{j=1}^{\ell-1}\varphi(\sigma^j x) < 2/10$$

for some m and $\ell \geq n$.

By continuity of φ and σ, it follows that W_n is open and, as $W_n \supset \mathcal{O}^-(\alpha)$, W_n is an open and dense set, $\forall\, n$.

Orbits with historic behavior in hyperbolic dynamics

By construction, if $x \in \bigcap W_n$ then x has a historic behavior!
That is,

the set of points with historic behavior is residual!

Furthermore, as set of points x with $\overline{\mathcal{O}^+(x)} = \sum_2^+$ is also residual, we get the following result.

Generically

the points of $\sum_2^+$ have dense orbits and historic behavior!

Orbits with historic behavior in hyperbolic dynamics

Using Markov partitions and the result above, one easily shows that

Generically

the points of non trivial hyperbolic set have historic behavior!

Measures with historic behavior

Definition

We say that a measure μ has historic behavior if it gives positive measure to the set of points with historic behavior.

Note that Lebesgue has historic behavior to the "Bowen's eye".

Lebesgue with historic behavior for quadratic maps

The logistic (or quadratic) family.

Let $f_t : [0,1] \to [0,1]$ be given by $f_t(x) = 4t\,x(1-x)$, with $t \in [0,1]$.

Theorem (Hofbauer and Keller)

There are uncountably many parameter values t such that the Lebesgue has historic behavior with respect to f_t.

Remark: If f_t has a periodic attractor or if f_t is a ∞-renormalizable map then Lebesgue does not have historic behavior.

Lebesgue with historic behavior for quadratic maps

Reference

Hofbauer, F. and Keller, G.. *Quadratic maps without asymptotic measure*. Commun. Math. Phys 1990, 127, 319-337.

ALTERNATIVE APPROACH

Let $\mathbb{X}$ be a set and $2^{\mathbb{X}}$ be the power set of $\mathbb{X}$, that is, the set of all subsets of $\mathbb{X}$. A measure in the sense of Caratheodory is a function $\mu : 2^{\mathbb{X}} \to [0, +\infty]$ such that

1. $\mu(\emptyset) = 0$;
2. $A \subset B \Rightarrow \mu(A) \leq \mu(B)$;
3. $\mu(\bigcup_{j=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty} \mu(A_j)$.

Result (Caratheodory)

There exists $M(\mu) \subset 2^{\mathbb{X}}$ called the *μ-measurable subsets of $\mathbb{X}$* such that $M(\mu)$ is a "standard" σ-algebra and $\mu|_{M(\mu)}$ is a "standard" σ-additive measure.

A method to construct a "good" Caratheodory measure

Let $\mathbb{X}$ be a compact metrical space let and *dist* be its distance.

Pre-measures

Let $\mathcal{A}$ be a collection of open subset of $\mathbb{X}$ such that $\emptyset \in \mathcal{A}$ (we can assume that $\mathcal{A}$ is a basis of $\mathbb{X}$ topology). A function $\tau : \mathcal{A} \to [0, +\infty]$ is called a pre-measure if

$$\tau(\emptyset) = 0.$$

A method to construct a "good" Caratheodory measure

Given $Y \subset 2^{\mathbb{X}}$, first define

$$\nu_r(Y) := \inf_{\mathcal{I} \in \mathcal{A}(r,Y)} \sum_{I \in \mathcal{I}} \tau(I)$$

and $\mathcal{A}(r, Y)$ is the set of all countable covers $\mathcal{I} = \{I_i\}$ of Y by elements of $\mathcal{A}$ with diameter$(I_i) \leq r \ \forall \, i$.

The Caratheodory (metric) measure constructed from the

pre-measured τ is given by

$$\nu(Y) := \sup_{r>0} \nu_r(Y) = \lim_{r \searrow 0} \nu_r(Y).$$

A method to construct a "good" Caratheodory measure

Theorem (Reference: C.A. Rogers, *Hausdorff measures* , Cambridge Univ. Press 2 edition, 1998.)

The Borel sets of $\mathbb{X}$ are ν-measurable sets and

$$\mu := \nu|_{\text{Borel sets}}$$

is a σ-additive (and regular) measure.

Constructing an invariant measure associated to a given orbit.

Given $Y \subset \mathbb{X}$ let

$$\tau_x(Y) = \limsup_{n \to +\infty} \frac{1}{n}\{0 \leq j < n\,;\, f^j(x) \in Y\};$$

Let $\mathcal{A}$ be a collection of open sets with $\emptyset \in \mathcal{A}$. Assume $\mathcal{A}$ generates the topology of $\mathbb{X}$.

Constructing an invariant measure associated to a given orbit.

τ has the following properties.

1. $\tau_x(\emptyset) = 0$;
2. $\tau_x(\mathbb{X}) = 1$;
3. $\tau_x(Y_1) \leq \tau_y(Y_2)$ whenever $Y_1 \subset Y_2$;
4. $\tau_x(\bigcup_{i=1}^n Y_i) \leq \bigcup_{i=1}^n \tau_x(Y_i)$ for all finite collection $Y_1, Y_2, Y_3, \ldots, Y_n$ of subsets of $\mathbb{X}$;
5. $\tau_x(f^{-1}(Y)) = \tau_x(Y)$ for all $Y \subset \mathbb{X}$.

Remark

Taking, for instance, $Y_j = \{f^j(x)\}$ and assuming $\#\mathcal{O}_f^+(x) = \infty$, we get

$$1 = \tau_x\left(\bigcup_{i=1}^{\infty} Y_i\right) > \bigcup_{i=1}^{\infty} \tau_x(Y_i) = 0.$$

Constructing an invariant measure associated to a given orbit.

Let

$$\eta_x := \nu\big|_{\text{Borel sets of } \mathbb{X}},$$

where ν is the Caratheodory measure associated to τ_x.

Constructing an invariant measure associated to a given orbit.

Theorem (Araujo; __)

The following sentences are equivalent.

1. η_x is a probability.
2. $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f^j(x) = \int \varphi d\eta_x$ for all continuous function φ.
3. $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f^j(x)$ exists for all continuous function φ.

Theorem (Araujo; __)

If f is continuous then η_x is a f-invariant measure $\forall\, x$.

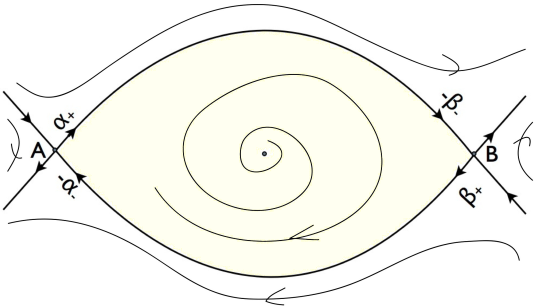
Furthermore, $\operatorname{supp} \eta_x = \omega^*(\delta_x)$, where $\omega^*(\delta_x)$ is the set of all weak accumulation points of the sequence $\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)}$.

Coming back to “Bowen’s eye”

Result

With the exception of the source in the center, for every x inside the “Bowen’s eye” we have

$$\eta_x = \frac{\beta_-}{\beta_- + \alpha_+}\delta_A + \frac{\alpha_-}{\alpha_- + \beta_+}\delta_B$$



Coming back to quadratic maps

Theorem (Blokh and Lyubich)

The Lebesgue measure is ergodic (but not invariant) for every S-unimodal map (in particular, quadratic maps) without periodic attractor.

Lemma

If μ is ergodic (not necessarily invariant) then $\eta_x = \eta_y$ for μ-almost every $x, y \in \mathbb{X}$.

Result

For each $t \in [0,1]$ there is an f_t-invariant measure μ_t such that $\eta_x = \mu_t$ for Lebesgue almost every x.

Ergodic decomposition

Theorem (Ergodic decomposition)

Let $\mathbb{X}$ be a perfect compact metric space and let $f : \mathbb{X} \to \mathbb{X}$ be a measurable map. If μ is an f-invariant finite measure then, for μ almost every $x \in \mathbb{X}$, η_x is a f-invariant ergodic probability. Furthermore,

$$\mu = \int_{x \in \mathbb{X}} \eta_x d\mu.$$

That is,

$$\int \varphi \, d\mu = \int_{x \in \mathbb{X}} \left(\int \varphi \, d\eta_x \right) d\mu$$

for every $\varphi \in L^1(\mu)$ (in particular, $\mu(V) = \int_{x \in \mathbb{X}} \eta_x(V) \, d\mu$ for every Borel set $V \subset \mathbb{X}$).

Open questions

Question 1

$\eta_x(\mathbb{X}) > 1$ implies that η_x is not ergodic?

If the answer of this question is yes, we can use this to give an alternative proof of Birkhoff Ergodic Theorem.

Question 2

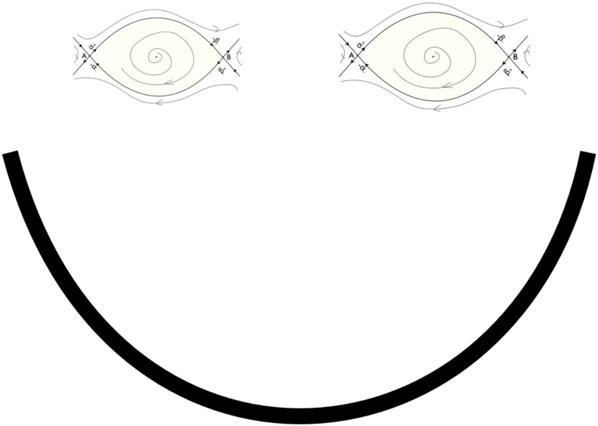
For the parameters values t of Hofbauer and Keller's theorem, what we can say about μ_t. It is finite? Ergodic?

Question 3

One can say something about the decay of correlation of the Lebesgue measure inside the "Bowen's eye".

Question 4 (Ruelle)

There are persistent classes of smooth dynamical systems such that the set of initial states which give rise to orbits with historic behaviour has positive Lebesgue measure?



Thanks!