Birkhoff sums for interval exchange maps: the Kontsevich-Zorich cocycle (I)

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All orbits of the rotation R_α are periodic of exact period q.

Theorem: Let $\alpha \in \mathbb{T}$ be irrational.

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Theorem: Let $\alpha \in \mathbb{T}$ be irrational. Then, every orbit of the rotation R_α is dense and *equidistributed* in $\mathbb{T}$: for any continuous function $\varphi \in C(\mathbb{T})$, one has

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{0}^{n-1} \varphi(x + n\alpha) = \int_{\mathbb{T}} \varphi(t)\, dt \ ,$$

uniformly in $x \in \mathbb{T}$.

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For $\mu \in \mathcal{M}(X)$, we denote by $f^*\mu$ the *direct image* of μ under f, defined by

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Equivalently, for every Borel set $A \subset X$, we have $\mu(f^{-1}(A)) = \mu(A)$.

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The ergodic f-invariant measures are exactly the extremal points of $\mathcal{M}_f(X)$.

Birkhoff sums

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When f is invertible, one can also define $S_n\varphi$ for $n < 0$ through

$$S_n\varphi = -\sum_{i=n}^{-1} \varphi \circ f^i \,.$$

Birkhoff sums and invariant measures

For $x \in \mathbb{T}$, $n \geq 0$, denote by $\mu_{n,x}$ the barycentric combination of Dirac measures

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Let (x_n) be any sequence in X. Any accumulation point (for the weak topology) of the sequence (μ_{n,x_n}) is a f-invariant measure.

Birkhoff's ergodic theorem

Assume that μ is an **ergodic** f-invariant measure.

Theorem: (Birkhoff) Let φ be a μ-integrable function. For μ-almost all $x \in X$, we have

$$\lim_{n \to +\infty} \frac{1}{n} S_n \varphi(x) = \int \varphi \, d\mu \, .$$

For a proof, see for instance Milnor's notes
www.math.sunysb.edu/ jack/DYNOTES/dn9.pdf

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Proposition: The following are equivalent

1. f is uniquely ergodic;
2. for every continuous function φ, the Birkhoff averages $(\frac{1}{n}S_n\varphi)$ converge uniformly to a constant;
3. for every continuous function φ and any $\varepsilon > 0$, there exists a constant c and a continuous function ψ such that

$$||\varphi - c - \psi \circ f + \psi||_{C^0} \leqslant \varepsilon \,.$$

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Theorem: Irrational rotations are uniquely ergodic.

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For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, consider the constant vectorfield $X_\alpha = \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2}$ on $\mathbb{T}^2$.

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It generates a flow (one-parameter group) $(\Phi_\alpha^t)_{t\in\mathbb{R}}$ given by

$$\Phi_\alpha^t(x_1, x_2) = (x_1 + t\alpha_1, x_2 + t\alpha_2) \mod \mathbb{Z} .$$

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Consider the *return map R* of the flow $(\Phi_\alpha^t)_{t\in\mathbb{R}}$ on the circle $\mathbb{T} \times \{0\}$:

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$$r(x_1) := \min\{t > 0, \Phi_\alpha^t(x_1, 0) \in \mathbb{T} \times \{0\}\}$$

is the return time to $\mathbb{T} \times \{0\}$.

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Definition The linear flow $(\Phi_\alpha^t)_{t \in \mathbb{R}}$ is *rational* if $\frac{\alpha_1}{\alpha_2} \in \mathbb{Q} \cup \{\infty\}$, *irrational* otherwise.

The fundamental dichotomy for linear flows

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$$\lim_{T\to\infty} \int_0^T \varphi \circ \Phi_\alpha^t(x)\, dt = \int_{\mathbb{T}^2} \varphi(s_1, s_2)\, ds_1\, ds_2,$$

uniformly in $x = (x_1, x_2) \in \mathbb{T}^2$.