Birkhoff sums of i.e.t.'s: KZ cocycle (3rd lecture)

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Table of contents

Definition of i.e.t.

Definition

Let $I \subset \mathbb{R}$ be a bounded open interval. $T : D_T \to D_{T^{-1}}$ is an *interval exchange transformation* (i.e.t. for short) if

- $D_T, D_{T^{-1}} \subset I$,
- $\#(I - D_T) = \#(I - D_{T^{-1}}) = d - 1 < \infty$,
- T is injective, and
- the restriction of T to any connected component of D_T is a translation onto some connected component of $D_{T^{-1}}$.

Some concrete examples

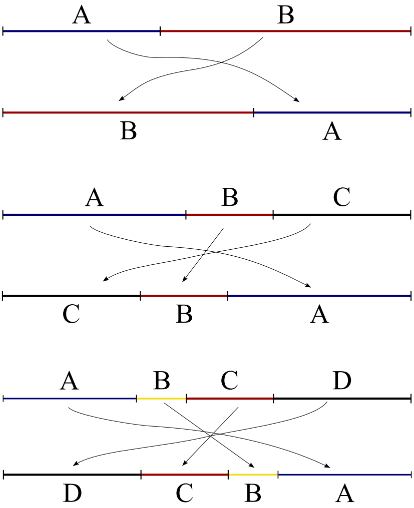

Figure: 3 examples of i.e.t.'s

Combinatorial data

A *combinatorial marking* for an i.e.t. T is an alphabet $\mathcal{A}$ with $\#\mathcal{A} = d$ and bijections $\pi_t, \pi_b : \mathcal{A} \to \{1, \ldots, d\}$ s.t. $\forall\, \alpha \in \mathcal{A}$ the image under T of the connected comp. of D_T at position $\pi_t(\alpha)$ (from left to right) is the c.c. of $D_{T^{-1}}$ at position $\pi_b(\alpha)$.

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We'll denote a comb. marking also as $\begin{pmatrix} \pi_t^{-1}(1) & \dots & \pi_t^{-1}(d) \\ \pi_b^{-1}(1) & \dots & \pi_b^{-1}(d) \end{pmatrix}$.

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Two comb. markings $(\mathcal{A}, \pi_t, \pi_b)$ and $(\mathcal{A}', \pi'_t, \pi'_b)$ are *equivalent* when $\exists$ bijection $i : \mathcal{A} \to \mathcal{A}'$ s.t. $\pi'_t = \pi_t \circ i$, $\pi'_b = \pi_b \circ i$.

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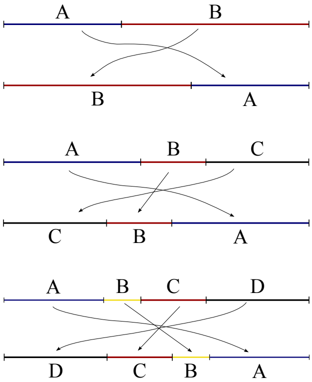
Clearly, an i.e.t. T determines a equiv. class of comb. markings called the *combinatorial data* of T.

Combinatorial markings for 3 i.e.t.'s

The 3 i.e.t.'s below



have combinatorial markings

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}, \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}, \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$$

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But, we'll *not* do so because this breaks the natural *symmetry* between T and T^{-1}, i.e., the past and future, or top and bottom lines, as $\pi_b \circ \pi_t^{-1}$ implicitly "prefers" to number the top line as $1, \ldots, d$.

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Also, by simplifying it here, we pay a price later when doing computations with i.e.t.'s because every time we have to present the comb. data, we must firstly *renumber* the top line as $1, \dots, d$ before calculating the bottom line.

Irreducible combinatorial data

A combinatorial marking $(\mathcal{A}, \pi_t, \pi_b)$ is *irreducible* when

$$\pi_t^{-1}(\{1, \ldots, k\}) \neq \pi_b^{-1}(\{1, \ldots, k\})$$

for each $1 \leq k < d$.

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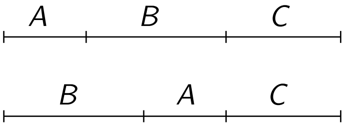
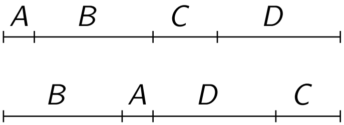
From the dynamical point of view, it suffices to consider i.e.t.'s T with irred. comb. data: indeed, an i.e.t. T with reducible data, say $\pi_t^{-1}(\{1, \ldots, k\}) = \pi_b^{-1}(\{1, \ldots, k\})$, is the juxtaposition of two i.e.t.'s, one with k intervals, and another with $d - k$ intervals.

Some i.e.t.'s admitting reducible combinatorial data





Length data

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For each $\alpha \in \mathcal{A}$, let $I_\alpha^{t,b}$ be the c.c. of $D_{T^{\pm 1}}$ in position $\pi_{t,b}(\alpha)$, and let $\lambda_\alpha = |I_\alpha^{t,b}|$ be the (common) length of $I_\alpha^{t,b}$.

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For each $\alpha \in \mathcal{A}$, let δ_α be the real number such that $I_\alpha^b = I_\alpha^t + \delta_\alpha$. The *column* vector $\delta = (\delta_\alpha)_{\alpha \in \mathcal{A}}$ is the *translation vector* of T.

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A combinatorial data $(\mathcal{A}, \pi_t, \pi_b)$ and a length data $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$ determine an unique i.e.t. on $I = (0, |\lambda|)$ where $|\lambda| = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha$.

Length data, translation vector and the matrix Ω

The length vector $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}}$ and the translation vector $\delta = (\delta_\alpha)_{\alpha \in \mathcal{A}}$ are related by the following formula

$$\delta_\alpha = \sum_{\pi_b(\beta) < \pi_b(\alpha)} \lambda_\beta - \sum_{\pi_t(\beta) < \pi_t(\alpha)} \lambda_\beta = \sum_\beta \Omega_{\alpha\beta} \lambda_\beta$$

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where $\Omega = (\Omega_{\alpha\beta})$ is the *antisymmetric* matrix

$$\Omega_{\alpha\beta} = \begin{cases} +1, & \text{if } \pi_b(\beta) < \pi_b(\alpha) \text{ and } \pi_t(\beta) > \pi_t(\alpha) \\ -1, & \text{if } \pi_b(\beta) > \pi_b(\alpha) \text{ and } \pi_t(\beta) < \pi_t(\alpha) \\ 0, & \text{otherwise} \end{cases}$$

Suspension data

Definition

Let $(\mathcal{A}, \pi_t, \pi_b)$ be a combinatorial data of an i.e.t. $T : D_T \to D_{T^{-1}}$. We say that $\tau \in \mathbb{R}^{\mathcal{A}}$ is a *suspension data* (or suspension vector) if

$$\sum_{\pi_t(\alpha)\leq k} \tau_\alpha > 0 \quad \text{and} \quad \sum_{\pi_b(\alpha)\leq k} \tau_\alpha < 0$$

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Example (Canonical suspension data)

$\tau_\alpha^{can} := \pi_b(\alpha) - \pi_t(\alpha)$ is a susp. vector iff the comb. data $(\mathcal{A}, \pi_t, \pi_b)$ is irred.. Also, the set of susp. vectors is empty when the comb. data is reducible.

Masur's suspension construction I

Let T an i.e.t. with irred. comb. data $(\mathcal{A}, \pi_t, \pi_b)$ and length vector $\lambda = (\lambda_\alpha)_{\alpha\in\mathcal{A}}$, and let $\tau = (\tau_\alpha)_{\alpha\in\mathcal{A}}$ be a susp. vector. Set $\zeta_\alpha = \lambda_\alpha + i\tau_\alpha \in \mathbb{C} \simeq \mathbb{R}^2$.

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We suspend T using τ as follows: starting from $0 \in \mathbb{C} \simeq \mathbb{R}^2$, we construct a "top", "bottom" polygonal line by connecting the points $0, \zeta_{\pi_{t,b}^{-1}(1)}, \zeta_{\pi_{t,b}^{-1}(1)} + \zeta_{\pi_{t,b}^{-1}(2)}, \dots, \zeta_{\pi_{t,b}^{-1}(1)} + \cdots + \zeta_{\pi_{t,b}^{-1}(d)}$.

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Remark

These polygonal lines have the same endpoints (namely $\sum\limits_{\alpha\in\mathcal{A}} \zeta_\alpha$).

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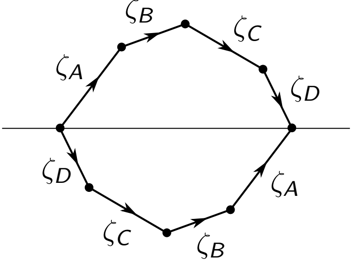
Remark

Since τ is a suspension vector, the intermediary points of top, bottom polygonal line stays in the upper, lower half-plane.

Masur's suspension construction II

If these lines *don't* intersect except at endpoints, one has a (compact, orient.) surface M by gluing ζ_α-sides of the top and bottom lines by an adequate *translation* by $\theta_\alpha \in \mathbb{C} = \mathbb{R}^2$:



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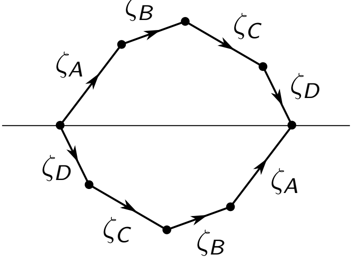

Exercise

The θ_α's above satisfy $\theta = \delta - ih$ where θ is the column vector of coord. θ_α, h is the column vector $h = -\Omega^t \tau$, and δ is the (column) transl. vector $\delta = \Omega^t \lambda$.

Masur's suspension construction III

The top and bottom lines don't intersect if $\sum_\alpha \tau_\alpha = 0$ (hence for $\tau = \tau^{can}$) or $\lambda_{\pi_t^{-1}(d)} = \lambda_{\pi_b^{-1}(d)}$, but they may intersect in *general*.

Masur's suspension construction III

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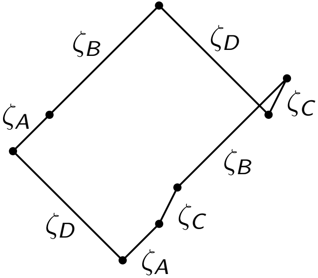
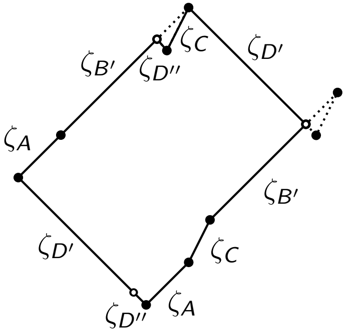
For instance, this happens for: comb. data $\begin{pmatrix} A & B & D & C \\ D & A & C & B \end{pmatrix}$, $\zeta_A = 1+i$, $\zeta_B = 3+3i$, $\zeta_C = \varepsilon + i$, $\zeta_D = 3-3i$, and $0 < \varepsilon < 1$:



Masur's suspension construction IV

A way of getting around this difficulty is by stopping before the self-intersection and gluing it at an appropriate place:



Veech's zippered rectangles construction I

Another way of overcoming this problem is via *Veech's zippered rectangles construction*. One considers

- $R_\alpha^{t,b} = I_\alpha^{t,b} \times [0, h_\alpha]$;
- $S_i^{t,b} = \{u_i^{t,b}\} \times [0, \sum\limits_{\pi_{t,b}(\alpha)\leq i} \tau_\alpha)$;
- $C_0 = (u_0, 0),\ C_d = (u_d, \sum\limits_\alpha \tau_\alpha),\ C_i^{t,b} = C_0 + \sum\limits_{\pi_{t,b}(\alpha)\leq i} \zeta_\alpha$;
- S^* is the vertical segment joining $(u_d, 0)$ and C_d,

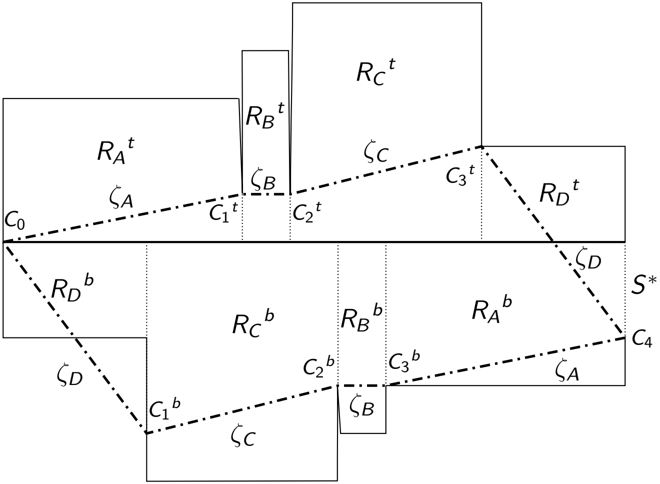
and one builds a surface M by gluing these elements with the following rules.

Veech's zippered rectangles construction II

- R^t_α and $R^b_\alpha = R^b_\alpha + \theta_\alpha$;
- $C^t_{\pi_t(\alpha)}$ and $C^b_{\pi_b(\alpha)} = C^t_{\pi_t(\alpha)} + \theta_\alpha$;
- if $\sum\limits_\alpha \tau_\alpha > 0$, we glue the top part of $S^t_{\pi_t \pi_b^{-1}(d)}$ with S^* by $\theta_{\pi_b^{-1}(d)}$, and if $\sum\limits_\alpha \tau_\alpha < 0$, we glue S^* with the bottom part of $S^t_{\pi_b \pi_t^{-1}(d)}$ with S^* by $\theta_{\pi_t^{-1}(d)}$

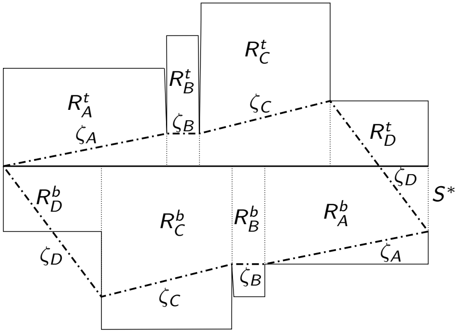
The following picture shows in a nutshell the main features of Veech's construction:

Veech's zippered rectangles construction III



A word on the nomenclature "suspension"

The name "suspension" for the transl. surf. M coming from an i.e.t. T and a susp. vector τ is motivated by the fact that T is naturally the first return map of the vertical vector field Y on an adequate transversal:



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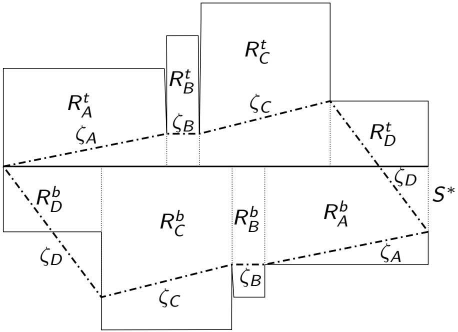

In other words, it is natural to study the dynamics of the i.e.t. T and the translation (vertical) flow Y *together*.

Translation surfaces I

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The surface M comes with:

- a special finite set $\Sigma = \{A_1, \dots, A_s\}$ corresponding to the intermediary points on the polygonal lines, and
- a (maximal) atlas ξ on $M - \Sigma$ such that the changes of charts are *translations* of $\mathbb{C} \simeq \mathbb{R}^2$.
- moreover, for each $1 \leq i \leq s$, one can find a nbd. $V_i \subset M$ of A_i, a nbd. $W_i \subset \mathbb{R}^2$ of 0 and $\pi : (V_i, A_i) \to (W_i, 0)$ a ramified covering of finite degree $\kappa_i \geq 1$ such that every injective restriction of π is a chart of ξ.

Translation surfaces II

The data of a compact orientable topological surface M of genus $g \geq 1$, a finite set $\Sigma = \{A_1, \ldots, A_s\} \subset M$, a list $\kappa = (\kappa_1, \ldots, \kappa_s)$ of *ramification indices* and a (max.) atlas ξ with the properties described in the last slide is a *translation surface structure*.

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Hence, we just saw that susp.'s of i.e.t.'s are transl. surfaces.

Translation surfaces III

Any translation surface (M, Σ, κ, ξ) comes equipped with

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- an *area form* on M obtained by pull-back of $dx \wedge dy$ on $\mathbb{R}^2$;
- a canonical choice of *vertical vector field* Y on $M - \Sigma$ obtained by pull-back of $\partial/\partial y$ on $\mathbb{R}^2$;

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- a *flat metric* on $M - \Sigma$ obtained by pull-back of the Euclidean metric $dx^2 + dy^2$ on $\mathbb{R}^2$;
- an *area form* on M obtained by pull-back of $dx \wedge dy$ on $\mathbb{R}^2$;
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because the corresponding objects are transl.-inv. on $\mathbb{C} \simeq \mathbb{R}^2$.

Translation surfaces IV

In fact, a transl. surf. is *completely* determined by a non-trivial Ab. diff. ω on a Riemann surf. M: indeed, by letting Σ be the finite set of zeroes of ω and by taking local primitives $z \mapsto \int_p^z \omega$ of ω near a point $p \in M - \Sigma$, we get a translation surface structure.

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Moreover, the ramification indices κ_i relate to the orders of zeroes d_i via $\kappa_i = d_i + 1$.

Translation surfaces V

So far, transl. surf.s came from susp.s constructions, but they naturally appear in other contexts, e.g. in the study of *billiards in rational polygons* (a subject itself motivated by *Boltzmann gases*).

However, due to the usual space-time limitations, we'll not discuss this here: instead we refer to surveys written by Zorich, and Masur and Tabachnikov.

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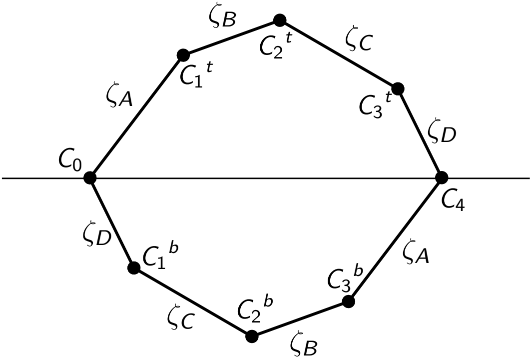
Now, let's come back to transl. surf. obtained by susp. of i.e.t.'s.

Ramification indices for suspensions of i.e.t.'s I

In the notation of Veech's construction, let $\mathcal{C} = \{C_i^{t,b} : 0 < i < d\}$. It has cardinality $2d - 2$. Turning around anticlockwise in the suspension M correspond to the successor map $\sigma : \mathcal{C} \to \mathcal{C}$:

- $\sigma(C_i^t) = C^b_{\pi_b \pi_t^{-1}(i)}$ except for $\pi_b \pi_t^{-1}(i) = 1$ when $\sigma(C_i^t) = C^b_{\pi_b \pi_t^{-1}(1)-1}$;
- $\sigma(C_j^b) = C^t_{\pi_t \pi_b^{-1}(j)}$ except for $\pi_t \pi_b^{-1}(j) = d$ when $\sigma(C_i^b) = C^t_{\pi_t \pi_b^{-1}(d)}$.

Ramification indices for suspensions of i.e.t.'s Ia



Ramifications indices for suspensions of i.e.t.'s II

Since σ exchange C_i^t and C_j^b, its cycles have *even* length.

Furthermore, the marked points Σ of the translation structure are in bijection with the cycles of σ. In particular,

- the number $s = \#\Sigma$ of marked pts is the nb. of cycles of σ,

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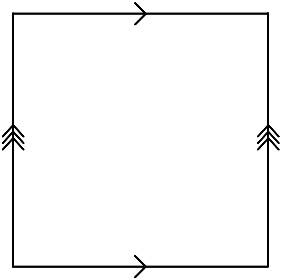
Hence, the genus g of the transl. surf. M relates to the size d of the alphabet $\mathcal{A}$ of the i.e.t. and the number s of marked points by:

$$d = 2g + s - 1$$

Homology of suspensions of i.e.t.'s I

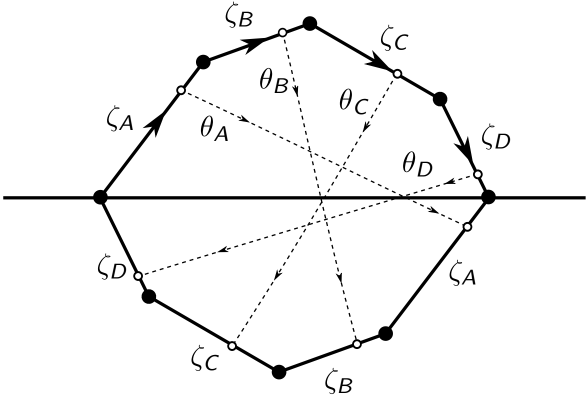
We denote by $H_1(M,\mathbb{Z})$, $H_1(M-\Sigma,\mathbb{Z})$, $H_1(M,\Sigma,\mathbb{Z})$ the *homology* groups of M (for instance, by Hurewicz theorem we can think of $H_1(M,\mathbb{Z})$ as the *Abelianization* of the *fundamental* group $\pi_1(M)$).

For example, in genus 1:



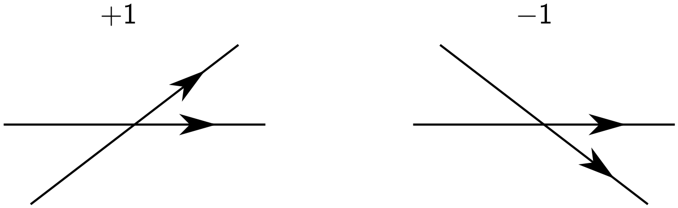
Homology of suspensions of i.e.t.'s Ia

These groups are related by a *short exact sequence* $H_1(M - \Sigma, \mathbb{Z}) \to H_1(M, \mathbb{Z}) \to H_1(M, \Sigma, \mathbb{Z})$, and we have a natural basis $[\theta_\alpha]$ of $H_1(M - \Sigma, \mathbb{Z})$ and $[\zeta_\alpha]$ of $H_1(M, \Sigma, \mathbb{Z})$ when M is a suspension of an i.e.t.:



Homology of suspensions of i.e.t.'s II

The natural intersection form $\langle ., . \rangle$



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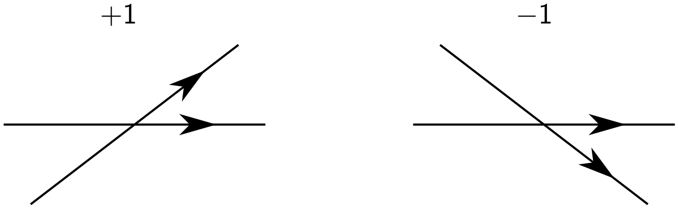

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By direct inspection, one sees that

$$\langle [\theta_\alpha], [\zeta_\beta] \rangle = \delta_{\alpha\beta},$$

i.e., the basis $([\theta_\alpha])_{\alpha \in \mathcal{A}}$ and $([\zeta_\beta])_{\beta \in \mathcal{A}}$ are dual of each other.

Homology of suspensions of i.e.t.'s III

Consider now $[\theta_\alpha] \in H_1(M, \mathbb{Z})$. Then, $\langle [\theta_\alpha], [\theta_\beta] \rangle = \Omega_{\beta\alpha}$ because the image $\overline{[\theta_\alpha]}$ of $[\theta_\alpha]$ in $H_1(M, \Sigma, \mathbb{Z})$ is

$$\overline{[\theta_\alpha]} = \sum_\beta \Omega_{\alpha\beta} [\zeta_\beta]$$

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In particular, we have that the antisymmetric matrix Ω has rank

$$\mathrm{rank}(\Omega) = 2g = \dim_{\mathbb{Z}} H_1(M, \mathbb{Z}).$$

Cohomology of suspensions of i.e.t.'s I

We have a short exact sequence of *cohomology* groups

$$H^1(M, \Sigma, \mathbb{Z}/\mathbb{R}/\mathbb{C}) \to H^1(M, \mathbb{Z}/\mathbb{R}/\mathbb{C}) \to H^1(M - \Sigma, \mathbb{Z}/\mathbb{R}/\mathbb{C})$$

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The Abelian differential ω associated to the translation surface M gives rise to $[\omega] \in H^1(M, \Sigma, \mathbb{C})$ and

$$\langle [\omega], [\zeta_\alpha] \rangle = \zeta_\alpha, \quad \langle \overline{[\omega]}, [\theta_\alpha] \rangle = \theta_\alpha$$

where $\overline{[\omega]}$ is the image of $[\omega]$ in $H^1(M - \Sigma, \mathbb{C})$.

Cohomology of suspensions of i.e.t.'s II

In this way, we can interpret the vectors λ and τ as elements of $H^1(M, \Sigma, \mathbb{R})$, $\zeta = \lambda + i\tau$ as an element of $H^1(M, \Sigma, \mathbb{C})$, δ, h as elements of $H^1(M - \Sigma, \mathbb{R})$ and $\theta = \delta - ih$ as an element of $H^1(M - \Sigma, \mathbb{C})$.

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Finally, the area of the translation surface M is

$$A = \sum_\alpha \lambda_\alpha h_\alpha = \tau \Omega^t \lambda$$