Birkhoff sums of i.e.t.'s: KZ cocycle (4th lecture)

Carlos Matheus and Jean-Christophe Yoccoz

CNRS (Paris 13) and Collège de France

May 24, 2012

Table of contents

I.e.t.'s, translation flows and renormalization I

We saw (3rd lecture) that suspension constructions bring together the dynamical study of i.e.t.'s and translation flows.

I.e.t.'s, translation flows and renormalization I

We saw (3rd lecture) that suspension constructions bring together the dynamical study of i.e.t.'s and translation flows.

In this course we'll see that i.e.t.'s and translation flows can be efficiently investigated via a *renormalization scheme* sharing some features of the continued fraction algorithm for circle rotations / i.e.t.'s of 2 intervals (cf. first two lectures).

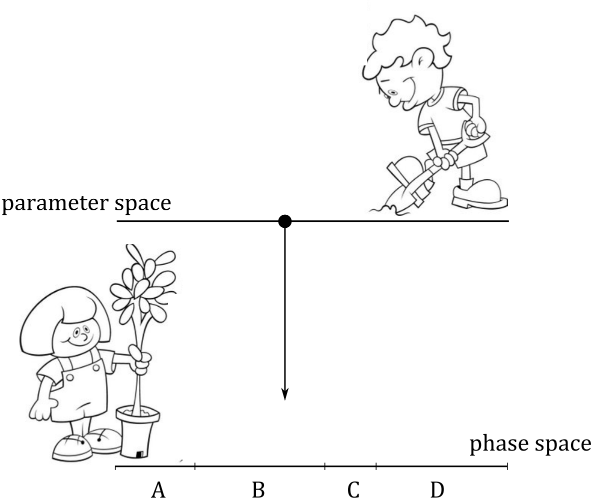
I.e.t.'s, translation flows and renormalization I

We saw (3rd lecture) that suspension constructions bring together the dynamical study of i.e.t.'s and translation flows.

In this course we'll see that i.e.t.'s and translation flows can be efficiently investigated via a *renormalization scheme* sharing some features of the continued fraction algorithm for circle rotations / i.e.t.'s of 2 intervals (cf. first two lectures).

The general philosophy of renormalization dynamics (schemes) is best described by the words of Adrien Douady: "to plough in parameter space, and harvest in phase space".

Renormalization in a nutshell



I.e.t.'s, translation flows and renormalization II

In our context:

- the phase space is the interval I (for i.e.t.'s) or the translation surface M (for translation flows),

I.e.t.'s, translation flows and renormalization II

In our context:

- the phase space is the interval I (for i.e.t.'s) or the translation surface M (for translation flows),
- the parameter space will be an adequate space describing i.e.t's or translations flows (roughly speaking, for i.e.t.'s it is the space of combinatorial and length data), and

I.e.t.'s, translation flows and renormalization II

In our context:

- the phase space is the interval I (for i.e.t.'s) or the translation surface M (for translation flows),
- the parameter space will be an adequate space describing i.e.t's or translations flows (roughly speaking, for i.e.t.'s it is the space of combinatorial and length data), and
- the renormalization dynamics will be a dynamical system in parameter space: in particular, for i.e.t.'s, it will convert a pair (combinatorial, length) data into another pair (combinatorial', length') data.

I.e.t.'s, translation flows and renormalization III

The *magical* fact is that the dynamical (e.g., recurrence, non-uniform hyperbolicity) properties of the renormalization on *parameter space* usually allow to deduce dynamical (e.g., unique ergodicity, deviation spectrum of ergodic averages) properties of generic (typical) i.e.t.'s and translation flows.

I.e.t.'s, translation flows and renormalization III

The *magical* fact is that the dynamical (e.g., recurrence, non-uniform hyperbolicity) properties of the renormalization on *parameter space* usually allow to deduce dynamical (e.g., unique ergodicity, deviation spectrum of ergodic averages) properties of generic (typical) i.e.t.'s and translation flows.

For a survey on the marvelous properties of renorm. dyn. in *several* contexts, see Artur Avila's article on ICM'2010 proceedings.

I.e.t.'s, translation flows and renormalization III

The *magical* fact is that the dynamical (e.g., recurrence, non-uniform hyperbolicity) properties of the renormalization on *parameter space* usually allow to deduce dynamical (e.g., unique ergodicity, deviation spectrum of ergodic averages) properties of generic (typical) i.e.t.'s and translation flows.

For a survey on the marvelous properties of renorm. dyn. in *several* contexts, see Artur Avila's article on ICM'2010 proceedings.

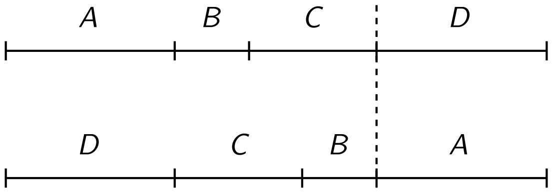
Unfortunately, it takes some time to setup the definitions and notations to describe renorm. dyn. for i.e.t.'s and translation flows. In particular, our plan will be as follows.

Connections for i.e.t.'s

As it turns out, before defining the Rauzy-Veech (renormalization) algorithm, we need the following concept:

Definition

A *connection* of an i.e.t. $T : D_T \to D_{T^{-1}}$ is a triple (m, u^t, u^b) where $u^t \in I - D_T$ is a singularity of T, $u^b \in I - D_{T^{-1}}$ is a singularity of T^{-1} and $T^m(u^b) = u^t$.

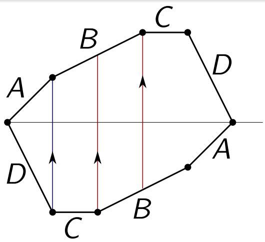


Connections for translation flows

Of course, we have an analogous concept for translation flows:

Definition

A *connection* of the vert. flow ϕ^t of a transl. surf. (M, Σ, κ) is a trajectory starting and ending at Σ.

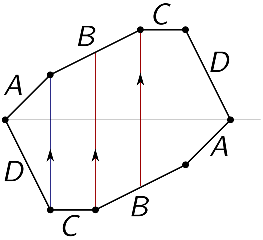


Connections for translation flows

Of course, we have an analogous concept for translation flows:

Definition

A *connection* of the vert. flow ϕ^t of a transl. surf. (M, Σ, κ) is a trajectory starting and ending at Σ.



Remark

If T has a connection, then ϕ^t on a susp. has a connection as well.

Keane's theorem

Theorem (M. Keane)

T i.e.t. w/o connections $\implies$ T is *minimal*: all half-orbit is dense.

Keane's theorem

Theorem (M. Keane)

T i.e.t. w/o connections $\implies$ T is *minimal*: all half-orbit is dense.

The previous result has a version for translation flows:

"Keane's theorem for translation flows"

ϕ^t transl. flow w/o conn. $\implies$ ϕ^t minimal: all half-orbit is dense.

Keane's theorem

Theorem (M. Keane)

T i.e.t. w/o connections $\implies$ T is *minimal*: all half-orbit is dense.

The previous result has a version for translation flows:

"Keane's theorem for translation flows"

ϕ^t transl. flow w/o conn. $\implies$ ϕ^t minimal: all half-orbit is dense.

Remark

Keane's thm for i.e.t.'s follows from Keane's thm for transl. flows (by rmk from the previous slide).

References for connections versus minimality

The proof of the results in the previous slide are not particularly difficult, but we'll not do it here as the discussion of Rauzy-Veech algorithm is our top priority.

Instead, we refer the curious reader to J.-C. Yoccoz's surveys

- "Continued Fraction Algorithms for i.e.t.'s: an introduction" (for a proof of Keane's thm), and
- "Interval exchange maps and translation surfaces" (for a proof of Keane's thm for transl. flows).

References for connections versus minimality

The proof of the results in the previous slide are not particularly difficult, but we'll not do it here as the discussion of Rauzy-Veech algorithm is our top priority.

Instead, we refer the curious reader to J.-C. Yoccoz's surveys

- "Continued Fraction Algorithms for i.e.t.'s: an introduction" (for a proof of Keane's thm), and
- "Interval exchange maps and translation surfaces" (for a proof of Keane's thm for transl. flows).

As a matter of fact, the proof of Keane's theorem for transl. flows in 2nd survey provide *sufficient* conditions for the representability of transl. surf. as suspensions of i.e.t.'s.

Representability of a translation surface as a suspension

Theorem

Let (M, Σ, κ) a transl. surf. with vert. flow ϕ^t w/o connections. Then, inside *any* horizontal *separatrix* S_∞ (i.e., a horiz. segment starting at Σ), we can find an open bounded segment $S \subset S_\infty$ s. t. (M, Σ, κ) is isomorphic to a susp. of the i.e.t. induced on S by ϕ^t.

Sufficient criterion for no connections for i.e.t.'s

Keane's thm for i.e.t.'s motivates the question: when does an i.e.t. has no connections? The result below is an elementary answer.

Theorem

Assume that the coordinates λ_α of the length data $\lambda = (\lambda_\alpha)$ of an i.e.t. T are *rationally* independent. Then, T has no connection.

Sufficient criterion for no connections for i.e.t.'s

Keane's thm for i.e.t.'s motivates the question: when does an i.e.t. has no connections? The result below is an elementary answer.

Theorem

Assume that the coordinates λ_α of the length data $\lambda = (\lambda_\alpha)$ of an i.e.t. T are *rationally* independent. Then, T has no connection.

Remark

This thm is quite natural: since sing. of the top and bottom lines u^t and u^b are sums of certain $\lambda'_\alpha s$ and the coord. of the transl. vector δ of T (from which we can compute T^m) are linear comb. of $\lambda'_\alpha s$, a connection $T^m(u^b) = u^t$ would provide a (non-trivial) rational relation between $\lambda'_\alpha s$.

"Rational independence" is not necessary

Note that in general there is no converse to the previous statement:

Exercise

Show that:

- for $d = \#\mathcal{A} = 2$, an i.e.t. T is minimal $\iff$ T has no connection $\iff$ the coordinates of the length data are rationally independent;
- for $d = \#\mathcal{A} \geq 3$, there are minimal i.e.t.'s T with connections, and i.e.t.'s T with "rationally dependent" length data but no connections.

Review of notations

Let $T : D_T \to D_{T^{-1}}$ be an i.e.t. of an interval $I = (u_0, u_d)$ with comb. data $(\mathcal{A}, \pi_t, \pi_b)$, $d = \#\mathcal{A}$, and length data $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}}$.

Review of notations

Let $T : D_T \to D_{T^{-1}}$ be an i.e.t. of an interval $I = (u_0, u_d)$ with comb. data $(\mathcal{A}, \pi_t, \pi_b)$, $d = \#\mathcal{A}$, and length data $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}}$.

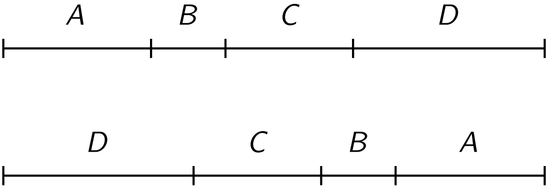
As always, $\{u_1^{t,b} < \cdots < u_{d-1}^{t,b}\} = I - D_{T^{\pm 1}}$ are the sing. of $T^{\pm 1}$.

Review of notations

Let $T : D_T \to D_{T^{-1}}$ be an i.e.t. of an interval $I = (u_0, u_d)$ with comb. data $(\mathcal{A}, \pi_t, \pi_b)$, $d = \#\mathcal{A}$, and length data $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}}$.

As always, $\{u_1^{t,b} < \cdots < u_{d-1}^{t,b}\} = I - D_{T^{\pm 1}}$ are the sing. of $T^{\pm 1}$.

For concreteness, in the sequel our "reference" i.e.t. T_{ref} will be:



Basic step of Rauzy-Veech algorithm I

Suppose T has *no* conn.. In this case, denoting by $\alpha_{t,b} = \pi_{t,b}^{-1}(d)$, the last int. $I_{\alpha_t}^t$ and $I_{\alpha_b}^b$ of the top and bottom lines have *distinct* lengths $\lambda_{\alpha_t} \neq \lambda_{\alpha_b}$ as $u_{d-1}^t \neq u_{d-1}^b$.

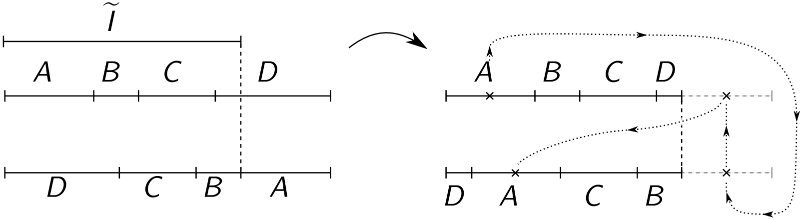
Basic step of Rauzy-Veech algorithm I

Suppose T has *no* conn.. In this case, denoting by $\alpha_{t,b} = \pi_{t,b}^{-1}(d)$, the last int. $I^t_{\alpha_t}$ and $I^b_{\alpha_b}$ of the top and bottom lines have *distinct* lengths $\lambda_{\alpha_t} \neq \lambda_{\alpha_b}$ as $u^t_{d-1} \neq u^b_{d-1}$.

We define $\widetilde{I} := (u_0, \max\{u^t_{d-1}, u^b_{d-1}\})$ and $\widetilde{T}$ the *return map* of T to $\widetilde{I}$, i.e., $\widetilde{T}(x) = T^{r(x)}(x)$, where $r(x) := \min\{r \geq 1 : T^r(x) \in \widetilde{I}\}$.

Basic step of Rauzy-Veech algorithm Ia

For the reference i.e.t. T_{ref}, we have



In particular, we see that $\widetilde{T}_{ref}$ is also an i.e.t. on $\widetilde{I}$ with combinatorial data $\begin{pmatrix} A & B & C & D \\ D & A & B & C \end{pmatrix}$.

Basic step of Rauzy-Veech algorithm II

In general, $\widetilde{T}$ is also an i.e.t. explicitly computable in terms of T depending whether we are in the

- *top case*: $u^b_{d-1} > u^t_{d-1}$, i.e., $\lambda_{\alpha_t} > \lambda_{\alpha_b}$
- or *bottom case*: $u^b_{d-1} < u^t_{d-1}$, i.e., $\lambda_{\alpha_t} < \lambda_{\alpha_b}$.

For simplicity, we present the formulas for $\widetilde{T}$ only in top case, leaving the deduction of (symmetric) formulas in the bottom case as an exercise.

Remark

Symm. formulas in the bottom case *exist* because our comb. data is (π_t, π_b) but *not* the permutation $\pi_b \circ \pi_t^{-1}$! This justifies our comment (yesterday) that (π_t, π_b) is "better" than $\pi_b \circ \pi_t^{-1}$.

Basic step of Rauzy-Veech algorithm III

In *top case* ($\lambda_{\alpha_t} > \lambda_{\alpha_b}$), $\widetilde{T}(x) = \begin{cases} T(x), & x \notin I^t_{\alpha_b} \\ T^2(x), & x \in I^t_{\alpha_b} \end{cases}$, and we use the same alphabet $\mathcal{A}$ to label the intervals $\widetilde{I}^{t,b}_\alpha$ exchanged by $\widetilde{T}$:

$$\widetilde{I}^t_\alpha = \begin{cases} I^t_\alpha, & \text{if } \alpha \neq \alpha_t \\ I^t_{\alpha_t} \cap \widetilde{I}, & \text{if } \alpha = \alpha_t \end{cases}, \quad \widetilde{I}^b_\alpha = \begin{cases} I^b_\alpha, & \text{if } \alpha \neq \alpha_b, \alpha_t \\ T(I^b_{\alpha_b}), & \text{if } \alpha = \alpha_b \\ I^b_{\alpha_t} - T(I^b_{\alpha_b}), & \text{if } \alpha = \alpha_t \end{cases}$$

Also, the new length data is $\widetilde{\lambda}_\alpha = \lambda_\alpha$ if $\alpha \neq \alpha_t$, and $\widetilde{\lambda}_{\alpha_t} = \lambda_{\alpha_t} - \lambda_{\alpha_b}$, and the new combinatorial data is $\widetilde{\pi}_t = \pi_t$ and

$$\widetilde{\pi}_b(\alpha) = \begin{cases} \pi_b(\alpha), & \text{if } \pi_b(\alpha) \leq \pi_b(\alpha_t) \\ \pi_b(\alpha_t) + 1, & \text{if } \alpha = \alpha_b \\ \pi_b(\alpha) + 1, & \text{if } \pi_b(\alpha_t) < \pi_b(\alpha) < d \end{cases}$$

Basic step of Rauzy-Veech algorithm IV

The map $(T, \pi_t, \pi_b) \mapsto (\widetilde{T}, \widetilde{\pi}_t, \widetilde{\pi}_b)$ is the basic step of the *Rauzy-Veech algorithm*. The following exercises contain some basic features of RV algorithm.

Exercise

Show that:

- the combinatorial data $(\widetilde{\pi}_t, \widetilde{\pi}_b)$ is irreducible;
- if T has no connections, $\widetilde{T}$ has no connections (and RV algorithm can be iterated *indefinitely*);
- the return map of T to an interval $\widetilde{I} \subset I' \subset I$ is an i.e.t. of $d+1$ intervals.
- for $d = 2$, the RV algorithm is the Euclidean division $(\lambda_A, \lambda_B) \mapsto \begin{cases} (\lambda_A - \lambda_B, \lambda_B), & \text{if } \lambda_A > \lambda_B \\ (\lambda_A, \lambda_B - \lambda_A), & \text{if } \lambda_A < \lambda_B \end{cases}$

Rauzy-Veech algorithm detects connections

Concerning the 2nd item of the previous exercise, it is possible to prove that the converse is true: the RV algorithm stops *if and only if* the i.e.t. has connections.

Rauzy-Veech algorithm detects connections

Concerning the 2nd item of the previous exercise, it is possible to prove that the converse is true: the RV algorithm stops *if and only if* the i.e.t. has connections.

In other words, if the i.e.t. T has some connection, the RV algorithm will see it some day!

Rauzy diagrams

The combinatorics $\widetilde{\pi} = (\widetilde{\pi}_t, \widetilde{\pi}_b)$ depends only on $\pi = (\pi_t, \pi_b)$ and the *type* (top or bottom) of the step of RV algorithm, that is, $\widetilde{\pi} = R_t(\pi)$ or $\widetilde{\pi} = R_b(\pi)$.

Definition

A *Rauzy class* $\mathcal{R}$ is a set of irreducible combinatorial data R_t, R_b-invariant and minimal with this property. A *Rauzy diagram* $\mathcal{D}$ is a graph having a Rauzy class $\mathcal{R}$ as a set of vertices, and oriented arrows (of top or bottom type) joining π to its image under R_t and R_b.

Rauzy diagrams

The combinatorics $\widetilde{\pi} = (\widetilde{\pi}_t, \widetilde{\pi}_b)$ depends only on $\pi = (\pi_t, \pi_b)$ and the *type* (top or bottom) of the step of RV algorithm, that is, $\widetilde{\pi} = R_t(\pi)$ or $\widetilde{\pi} = R_b(\pi)$.

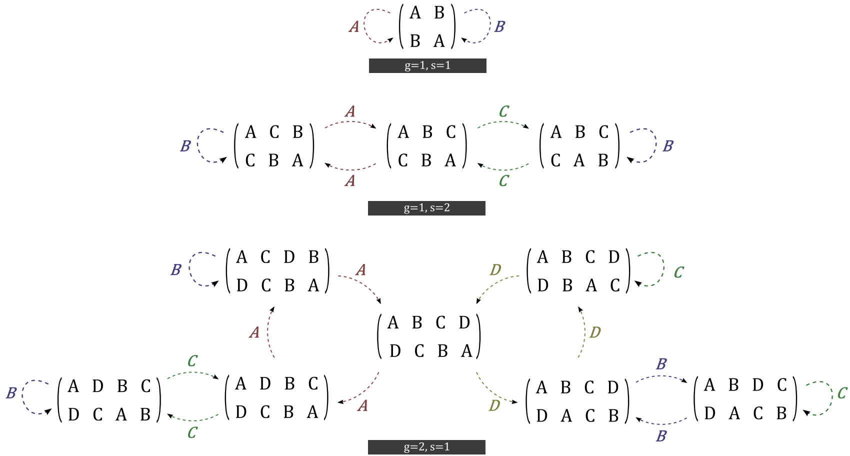
Definition

A *Rauzy class* $\mathcal{R}$ is a set of irreducible combinatorial data R_t, R_b-invariant and minimal with this property. A *Rauzy diagram* $\mathcal{D}$ is a graph having a Rauzy class $\mathcal{R}$ as a set of vertices, and oriented arrows (of top or bottom type) joining π to its image under R_t and R_b.

Definition

The *winner* of an arrow of top/bottom type starting at (π_t, π_b) is $\alpha_t = \pi_t^{-1}(d)/\alpha_b = \pi_b^{-1}(d)$, while the *loser* is α_b/α_t.

Some examples of Rauzy diagrams



Cardinality of Rauzy classes

In a recent work, Vincent Delecroix computed recurrence formulas for the cardinality of Rauzy classes: using his formulas (and/or SAGE) and denoting by $R(g)$ the cardinality of the *largest* Rauzy class of genus g, one has:

- for genus $g = 2$, $R(g) = 15$,
- for genus $g = 3$, $R(g) = 2177$,
- for genus $g = 4$, $R(g) = 617401$, and
- for genus $g = 5$, $R(g) = 300296573$
- can you guess the next value for $g = 6$? :)

Complete paths and i.e.t.'s

Definition

A finite path $\underline{\gamma}$ in a Rauzy diag. $\mathcal{D}$ is *complete* if every $\alpha \in \mathcal{A}$ wins at least once on an arrow of $\underline{\gamma}$. An infinite path γ in $\mathcal{D}$ is ∞-*complete* if every $\alpha \in \mathcal{A}$ wins in infinitely many arrows of γ (i.e., it is the concatenation of infinitely many complete paths).

Complete paths and i.e.t.'s

Definition

A finite path $\underline{\gamma}$ in a Rauzy diag. $\mathcal{D}$ is *complete* if every $\alpha \in \mathcal{A}$ wins at least once on an arrow of $\underline{\gamma}$. An infinite path γ in $\mathcal{D}$ is ∞-*complete* if every $\alpha \in \mathcal{A}$ wins in infinitely many arrows of γ (i.e., it is the concatenation of infinitely many complete paths).

Theorem

An infinite path in a Rauzy diagram comes from an i.e.t. *if and only if* it is ∞-complete.

Complete paths and i.e.t.'s

Definition

A finite path $\underline{\gamma}$ in a Rauzy diag. $\mathcal{D}$ is *complete* if every $\alpha \in \mathcal{A}$ wins at least once on an arrow of $\underline{\gamma}$. An infinite path γ in $\mathcal{D}$ is ∞-*complete* if every $\alpha \in \mathcal{A}$ wins in infinitely many arrows of γ (i.e., it is the concatenation of infinitely many complete paths).

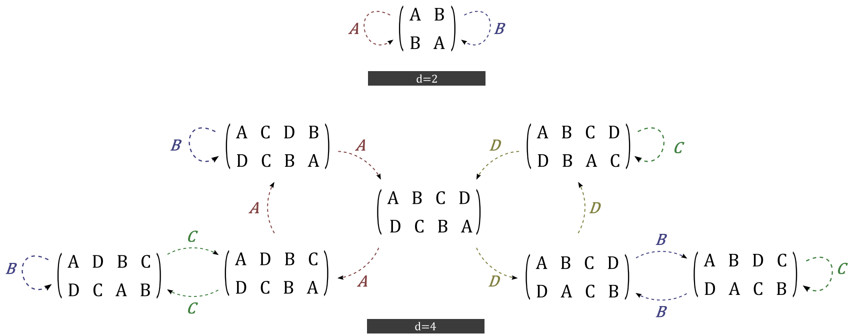
Theorem

An infinite path in a Rauzy diagram comes from an i.e.t. *if and only if* it is ∞-complete.

Remark

A ∞-complete path of an i.e.t. w/o conn. is a *rotation number*...

∞-complete paths in some concrete examples



For $d = 2$, a ∞-complete path is easy to describe: it uses all (two) arrows, i.e., it has the form $A^{a_1}B^{a_2}A^{a_3}\dots$. However, for $d = 4$, we see that $\exists$ ∞-complete paths not using all arrows.

Basic step of Rauzy-Veech algorithm for suspensions I

Given $\pi = (\pi_t, \pi_b)$, recall that *suspensions vectors* $\tau = (\tau_\alpha)$ satisfy $\sum_{\pi_t(\alpha)<k} \tau_\alpha > 0$ and $\sum_{\pi_b(\alpha)<k} \tau_\alpha < 0$ for each $1 < k \leq d$.

Basic step of Rauzy-Veech algorithm for suspensions I

Given $\pi = (\pi_t, \pi_b)$, recall that *suspensions vectors* $\tau = (\tau_\alpha)$ satisfy $\sum_{\pi_t(\alpha)<k} \tau_\alpha > 0$ and $\sum_{\pi_b(\alpha)<k} \tau_\alpha < 0$ for each $1 < k \leq d$.

These conditions define a cone $\Theta_\pi \subset \mathbb{R}^{\mathcal{A}}$.

Basic step of Rauzy-Veech algorithm for suspensions I

Given $\pi = (\pi_t, \pi_b)$, recall that *suspensions vectors* $\tau = (\tau_\alpha)$ satisfy $\sum_{\pi_t(\alpha)<k} \tau_\alpha > 0$ and $\sum_{\pi_b(\alpha)<k} \tau_\alpha < 0$ for each $1 < k \leq d$.

These conditions define a cone $\Theta_\pi \subset \mathbb{R}^{\mathcal{A}}$.

Let $\widetilde{\tau}_\alpha = \begin{cases} \tau_\alpha, & \text{if } \alpha \neq \alpha_t \\ \tau_{\alpha_t} - \tau_{\alpha_b}, & \text{if } \alpha = \alpha_t \end{cases}$ if $\widetilde{\pi} = R_t(\pi)$ has top type.

Basic step of Rauzy-Veech algorithm for suspensions I

Given $\pi = (\pi_t, \pi_b)$, recall that *suspensions vectors* $\tau = (\tau_\alpha)$ satisfy $\sum_{\pi_t(\alpha)<k} \tau_\alpha > 0$ and $\sum_{\pi_b(\alpha)<k} \tau_\alpha < 0$ for each $1 < k \leq d$.

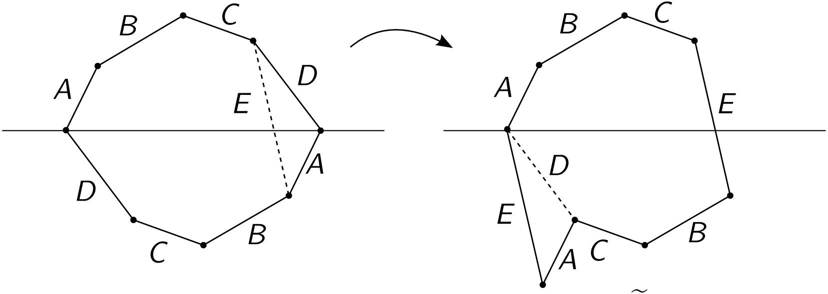
These conditions define a cone $\Theta_\pi \subset \mathbb{R}^{\mathcal{A}}$.

Let $\widetilde{\tau}_\alpha = \begin{cases} \tau_\alpha, & \text{if } \alpha \neq \alpha_t \\ \tau_{\alpha_t} - \tau_{\alpha_b}, & \text{if } \alpha = \alpha_t \end{cases}$ if $\widetilde{\pi} = R_t(\pi)$ has top type.

One can show that $\tau \mapsto \widetilde{\tau}$ sends Θ_π to $\Theta_{\widetilde{\pi}} \cap \{\sum_\alpha \widetilde{\tau}_\alpha < 0\}$.

Basic step of Rauzy-Veech algorithm for suspensions II

The RV algorithm for susp. is $(\pi, \lambda, \tau) \mapsto (\widetilde{\pi}, \widetilde{\lambda}, \widetilde{\tau})$. Geometrically:



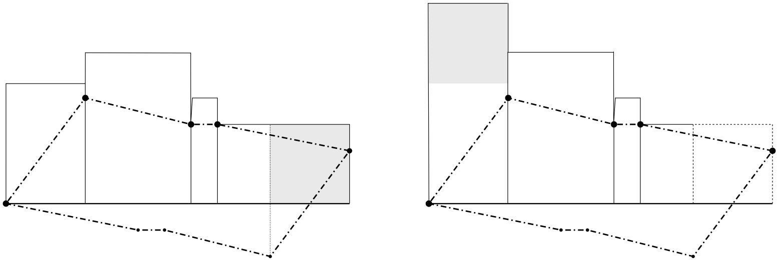
$M(\pi, \lambda, \tau)$ $\quad$ $M(\widetilde{\pi}, \widetilde{\lambda}, \widetilde{\tau})$

Remark

Above we have a top case, but, as usual, bottom case is similar.

Basic step of Rauzy-Veech algorithm for suspensions IIa

The RV algorithm in terms of Veech's construction is:



Basic step of Rauzy-Veech algorithm for suspensions III

From previous figures, one can see that the transl. surf. $M(\pi, \lambda, \tau)$ and $M(\widetilde{\pi}, \widetilde{\lambda}, \widetilde{\tau})$ are *canonically* isomorphic.

In terms of this isomorphism, the homology classes $[\zeta_\alpha]$, $[\theta_\alpha]$ and $[\widetilde{\zeta}_\alpha]$, $[\widetilde{\theta}_\alpha]$ (introduced by the end of yesterday's lecture) are related via the formulas (in the top case):

$$[\widetilde{\zeta}_\alpha] = \begin{cases} [\zeta_\alpha], & \text{if } \alpha \neq \alpha_t \\ [\zeta_{\alpha_t}] - [\zeta_{\alpha_b}] & \text{if } \alpha = \alpha_t \end{cases}, \ [\widetilde{\theta}_\alpha] = \begin{cases} [\zeta_\alpha], & \text{if } \alpha \neq \alpha_b \\ [\theta_{\alpha_t}] + [\theta_{\alpha_b}], & \text{if } \alpha = \alpha_b \end{cases}$$