Partially Hyperbolic Dynamics and Rigidity

Federico Rodriguez-Hertz (Penn State) – Pablo Carrasco (IMPA)

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Chapter 1

PH diffeomorphisms

Throughout this notes M will denote a closed riemannian manifold (that is, compact and without boundary) of class C^∞. We now give several definitions of our object of study, the partially hyperbolic maps.

We say that a diffeomorphism $f : M \to M$ is *partially hyperbolic in the strong sense* if there exists a continuous splitting of the tangent bundle into a Whitney sum of the form

$$TM = E^u \oplus E^c \oplus E^s$$

where neither of the bundles E^s nor E^u are trivial, and such that[1].

1. All bundles E^u, E^s, E^c are df-invariant.

2. $\lambda = \|df|E^s\| < 1 < \mu = \min_x\{m(df_x|E^u)\}$.

3. For all $x \in M$ and for all unit vectors $v^\sigma \in E^\sigma_x$ ($\sigma = s, u, c$)

$$\|d_x f(v^s)\| < \|d_x f(v^c)\| < \|d_x f(v^u)\|.$$

The bundles E^s, E^u, E^c are the *stable, unstable* and *center* bundle respectively. We also define the bundles $E^{cu} = E^c \oplus E^u$ and $E^{cs} = E^s \oplus E^c$, the *center stable* and *center unstable* bundles.

If instead of 3) we require

3') $\lambda < \min\{m(\|df|E^c\|)\} \leq \max\{\|df|E^c\|\} < \mu$

[1]Recall that for a linear map A, its conorm $m(A) := \frac{1}{\|A\|}$

we say that f is *absolutely partially hyperbolic in the strong sense.*

We will be primarily interested in partially hyperbolic diffeomorphism in the strong sense, and we will refer to them simply as "partially hyperbolic" (abbreviated as PH). Likewise we will write APH instead of "absolutely partially hyperbolic in the strong sense".

Remark 1.1. You may be wondering about the use of the word *strong* in the previous definition. There is a more flexible definition of partial hyperbolicity where one only requires the existence of a decomposition of the form $E^{cs} \oplus E^u$, where E^u is exponentially expanded under the derivative of f and dominates E^{cs} in the sense of 2). This definition is important when one tries to understand robust properties of diffeomorphisms. See [Díaz et al., 1999].

Strictly speaking the case $E^c = \{0\}$ (the Anosov case) is permitted in the definition. However here we want to study the case of "true" PH-diffeomorphisms so almost always we deal with the case when the center bundle is not trivial. That being said, you should always try to test any result for PH in the Anosov case if possible.

Remark 1.2. One should note that the definition of partial hyperbolicity does not depend on the riemannian metric used. Moreover, as in the hyperbolic case, one can find a metric where the the bundles are perpendicular and from now on we will always work such metric when discussing PH maps.

1.0.1 First Example: Products

Let $A_T : T^2 \to T^2$ denote the Thom map (i.e. the map induced by the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$) and consider the map $f = A_T \times id : T^3 \to T^3$. Clearly f is a linear map on the torus: denote by $\widetilde{A}$ the matrix inducing f. It is easy to see now that f is APH where the bundles E^s_f, E^u_f, E^c_f, coincide with the projection on the torus of the generalized eigenspaces of $\widetilde{A}$ corresponding to eigenvalues of norm smaller, equal or bigger than one respectively.

Similarly, consider $R_\alpha : T \to T$ an irrational rotation of angle $\alpha \in R \setminus Q$ in the circle ($R_\alpha(t) = t + \alpha \; mod \; 1$) and define $g = A_T \times R_\alpha : T^3 \to T^3$. Again g is PH, with the same invariant bundles as f.

Let us investigate some of the properties of the maps f and g. It will be convenient to have some definitions to make the discussion easier.

Definition 1.3.

1. We say that an immersed submanifold W^c is a center manifold if it is everywhere tangent to the center bundle. If $x \in M$ and there exist a center submanifold containing x we will denote it by[2] $W^c(x)$.

2. If for each $x \in M$ there exist a center manifold $W^c(x)$ and these submanifolds comprise a foliation (see Appendix A), we refer this foliation as the center foliation and its leaves as center leaves.

Similar definitions hold for the other invariant sub-bundles. We will carry the discussion for f and g at the same time.

1. For each x the center bundles integrate to unique center foliations with leaves

$$W^c_f(x) = W^c_g(x) = p_{T^2}^{-1}(p_{T^2}(x)) \qquad (p_{T^2} : T^3 \to T^2)$$

   All the center leaves are compact.

2. The periodic center leaves are dense.

3. Both f and g admit a generalized *Markov Partition*, in the following sense: there exist subshifts of finite type $\sigma_R : \Sigma_R \to \Sigma_R$, $\sigma_S : \Sigma_S \to \Sigma_S$ and continuous bounded-to-one surjective maps $h_R : \Sigma_R \times T \to T^3, h_S : \Sigma_S \times T \to T^3$ such that h_R semi-conjugates $\sigma_R \times id$ with f, and h_S semi-conjugates $\sigma_S \times R_\alpha$ with g.

   Recall that a $n \times n$ 0–1 matrix G defines a subshift of finite type Σ_G by

$$\Sigma_G = \{\underline{x} = (x_n)_{n\in\mathbb{Z}} : x_n \in 0, \ldots, n-1, A_{x_n,x_{n+1}} = 1\}$$

   The shift operator σ_G on Σ_G is defined by $\sigma_G(\underline{x})_n = x_{n+1}$.

[2]Note that we are fixing the submanifold: in principle, x could belong to different center manifolds.

1.0.2 Ergodicity

Given a probability space (X, μ) and a measure preserving transformation $T : X \to X$, that is[3] $T_\star\mu = \mu$, we say that T is ergodic with respect to μ if the only invariant L^2 functions are the constant ones: i.e. for every $\phi \in L^2(X)$

$$\phi(Tx) = \phi(x) \text{ a.e.}(x) \;\Rightarrow \phi = \text{const a.e.}$$

Equivalently, if $X_0 \subset X$ satisfies $T^{-1}(X_0) = X_0$ a.e. then $\mu(X_0)\mu(X_0^c) = 0$. If the measure μ is clear for the context we will just say that T is ergodic. For basic background on these concepts see [Mañé, 1983].

We will study extensively ergodicity or PH systems, providing different proofs and discussing the scope of them. We start with the maps f and g, establishing one of the main differences between them: while g is ergodic, f is not.

1. The map g is ergodic: Let ϕ be an L^2 g-invariant function, and consider the augmented matrix B given by

$$B = \begin{pmatrix} A & O \\ 0 & 1 \end{pmatrix}$$

Writing the Fourier expansion of ϕ we get

$$\phi(x) = \sum_{n \in Z} a_n \cdot e_n(x) = \phi(gx) \quad (e_n(x) = \exp(2\pi i \langle x, n\rangle))$$

Note that

$$\begin{aligned} e_n(gx) &= \exp\left(2\pi i \langle (A(x_1,x_2), x_3 + \alpha), (n_1,n_2,n_3)\rangle\right) \\ &= \exp\left(2\pi i \langle (A(x_1,x_2), x_3) + (0,0,\alpha), (n_1,n_2,n_3)\rangle\right) \\ &= \exp\left(2\pi i (\langle B(x_1,x_2,x_3), (n_1,n_2,n_3)\rangle + n_3\alpha)\right) \\ &\qquad \exp 2\pi i \langle (x_1,x_2,x_3), B^*(n_1,n_2,n_3)\rangle \exp\left(2\pi i n_3 \alpha\right) \end{aligned}$$

[3] $T_\star\mu(Y) := \mu(T^{-1}Y)$

Then

$$\phi(gx) = \sum_{n\in\mathbb{Z}} a_n e_{B^*n}(x) exp(2\pi i n_3 \alpha) = \phi(x) = \sum_{m\in\mathbb{Z}} a_m e_m(x)$$

and hence, by uniqueness of Fourier coefficients, we conclude

$$a_{B^*n} = a_n exp(2\pi i n_3\alpha) \Rightarrow |a_{B^*n}| = |a_n|$$

and thus, for every $k \in \mathbb{Z}$

$$|a_n| = |a_{(B^*)^k n}|$$

Fix n. Bessel's inequality implies

$$\infty > \|\phi\|_2^2 = \sum_{m\in\mathbb{Z}} |a_m|^2 \geq \sum_{k\in\mathbb{Z}} |a_{(B^*)^k n}|^2. \tag{1.1}$$

We have two cases:

(a) All the $\{(B^*)^k n\}$ are different. Then necessarily $a_n = 0$, since the right side series is a convergent series with all coefficients equal.

(b) There exist k such that $(B^*)^k n = n$. Observe that

$$(B^*)^k = \begin{pmatrix} (A^*)^k & O \\ 0 & 1 \end{pmatrix}$$

and $(A^*)^k v \neq v$ for any non-zero vector v (why?). We conclude that $n = (0, 0, n_3)$, and

$$a_{B^*n} = a_n exp(2\pi i n_3\alpha) \Rightarrow a_{(0,0,n_3)} = a_{(0,0,n_3)} exp(2\pi i n_3\alpha)$$

Since α is irrational $n_3\alpha$ is never an integer (unless $n_3 = 0$), hence $n_3 = 0$.

We have shown that $a_n = 0$ if $n =\neq (0, 0, 0)$: in other words $\phi(x) = a_0$ constant.

2. Referring now to the previous part, it is easy to see that the function $\phi(x) = e_{(0,0,1)}(x)$ is f-invariant. Hence $\underline{f \text{ is not ergodic}}$.

1.1 Coherence and the Stable Manifold Theorem

We come back to the study of general PH. We address here the important topic of the integrability of the invariant bundles.

The classical stable manifold theorem[4] below states that always the bundles E^s and E^u are integrable. You probably know the proof for the hyperbolic case, but the proof generalizes to our context.

Theorem 1.4. *Let f be PH of class C^r. Then*

1. *There exist $\epsilon > 0$ such that for every $x \in M$ the set*

$$W^s_\epsilon(x) := \{y \in M : \limsup_{n \mapsto \infty} \frac{d(f^n x, f^n y)}{\lambda} \leq 1, d(x,y) < \epsilon \}$$

coincides with the exponential of the graph of a C^r function

$$\sigma_s : E^s(\epsilon') \to E^{cu}$$

which is tangent to E^s_x at zero, for some $\epsilon' \approx \epsilon$.

2. *Define*

$$W^s(x) = \cup_{n \geq o} f^{-n}(W^s_\epsilon(f^n x))$$

Then for every x the set $W^s(x)$ is a immersed submanifold of M of class C^r, tangent to E^s on each of its points. Each $W^s(x)$ is homeomorphic to the Euclidean space R^s ($s = dim E^s$).

3. *The collection $\{W^s(s)\}$ is a foliation of class $C^{r,0+}$.*

You can find the proof of this result in [Hirsch et al., 1977] (or [Brin and Pesin, 1974]), and [Pugh et al., 1997]. This theorem is highly non-trivial and requires a fair amount of work to prove it in the general case that we stated.

Instead of proving the stable manifold theorem in general, we will restrict ourselves to discuss the case when the manifold is the torus and f is a small

[4] We state the Theorem for the stable bundle only since the case of the unstable follows from it by considering f^{-1}.

perturbation of a linear automorphism, just to give a flavor of how the proof works. Incidentally, in this case we will obtain a stronger result. The proof given is due to Federico and appears in [Rodriguez-Hertz, 2005], Appendix B.

Remark 1.5. Given a matrix $\widetilde{A} \in SL(n, \mathbb{Z})$ consider the linear automorphism $A : T^n \to T^n$ that it induces. If $spec(\widetilde{A}) \not\subset S^1$ then A is APH.

Setting: Let $A : T^n \to T^n$ be a linear automorphism of the torus with invariant bundles E^σ and consider the map $f = A + r$ where r is bounded and Lipschitz small. We will denote the lift of the maps to R^n by $\widetilde{f}, \widetilde{A}, \widetilde{r}$, and we will use the decomposition $R^n = E^u \oplus E^{cs}$.

Given $x \in R^n$, we are interested in looking at the graphs of Lipschitz continuous functions $x + E^u \to x + E^{cs}$ such that their "tangent" at zero is not too bad. Given one of these functions we can apply $\widetilde{f}$ and obtain the graph of a similar function, but now anchored at $\widetilde{f}(x)$. For example, if $\widetilde{r} = 0$, one can easily check that the only invariant graph precisely corresponds to the unstable bundle (i.e: $\sigma^u(v) = v$ if $v \in x + E^u$). The idea is that since $\widetilde{f}$ is a small perturbation of $\widetilde{A}$ we should obtain its unstable manifold as an invariant graph as well.

To this end, we will consider Lipschitz functions $\sigma_x : E^u \to E^{cs}$. Fixing one of these, a typical point of the graph of $x + \sigma_x$ is of the form

$$x + v + \sigma(v),\ v \in E^u.$$

If the image of the graph of σ_x under $\widetilde{f}$ is the graph of a new function ν_x, we should have

$$\widetilde{f}(x + v + \sigma_x(v)) = \widetilde{f}(x) + v' + \nu_{\widetilde{f}(x)}(v') \text{ for some } v' \in E^u \tag{1.2}$$

or

$$\widetilde{r}(x + v + \sigma_x(v)) - \widetilde{r}(x) + \widetilde{A}v + \widetilde{A}\sigma_x(v) = v' + \nu_{\widetilde{f}(x)}(v') \tag{1.3}$$

Projecting on E^u, E^{cs} we obtain

$$v' = \widetilde{r}^u(x + v + \sigma_x(v)) - \widetilde{r}^u(x) + \widetilde{A}^u v \tag{1.4}$$

$$\nu_{\widetilde{f}(x)}(v') = \widetilde{r}^{cs}(x + v + \sigma_x(v)) - \widetilde{r}^{cs}(x) + \widetilde{A}\sigma_x(v) \tag{1.5}$$

Note that since $\widetilde{A}^u$ is invertible and $\widetilde{r}$ is Lipschitz small, we can invert (1.4) and solve in (1.5) to find the general expression of $\nu_{\widetilde{f}(x)}$.

Given a point x of R^n define the space of graphs

$$G_x = \{\sigma_x : E^u \to E^{cs} : Lip(\sigma) < \infty\}$$

We equip each of these spaces with the norm

$$|\sigma_x|_* = Lip_0(\sigma_x)$$

This norm makes each G_x a Banach space. Now we consider the bundle $G = \cup_{x \in R^n} G_x$, and let

$$\Gamma(G) = \text{ space of bounded sections of } G$$

equipped with the uniform norm. As we have seen, $\widetilde{f}$ acts by graph transform in $\Sigma(G)$.

By domination of E^{cs} by E^u one can prove that this graph transform in fact is a contraction on the complete space $\Sigma(G)$, hence it has a unique fixed point $\sigma^u \in \Gamma(G)$. We then have

$$W^u_f(x) = pr(x + graph(\sigma^u_x)) \quad pr : R^n \to T^n$$

Remark 1.6. In this case we have obtained that the unstable manifold is obtained globally as a graph. This is seldom the case, and usually one can only guarantee that the unstable manifold is a local graph.

O.k., so E^s, E^u are integrable. What about the other bundles, say E^c? Things here become more difficult, and in fact the answer to the previous questions is negative in general as we will see. First some definitions.

Definition 1.7. We say that a PH system is *dynamically coherent* if the bundles E^c, E^{cs}, E^{cu} are integrable to foliations $\mathcal{W}^c, \mathcal{W}^{cs}, \mathcal{W}^{cu}$ (integrability), and $\mathcal{W}^c$ sub-foliates both $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ (coherence).

We remark that the definition requires both integrability and coherence of the center foliation with respect to the others; is not at all clear that coherence will follow from just the existence. On the other hand, K.Burns and A.Wilkinson observed that $\mathcal{W}^s, \mathcal{W}^u$ always sub-foliate $\mathcal{W}^{cs}, \mathcal{W}^{cu}$ when these exist. See [Burns and Wilkinson, 2008].

The first example of a PH diffeomorphism with non-integrable center bundle was given by Smale (based in ideas of A. Borel) in [Smale, 1967]. It is essentially an algebraic Anosov diffeomorphism on a 6 dimensional nil-manifold, where one considers the stable bundle and unstable bundles as the bundles generated by the directions of biggest contraction and biggest expansion, respectively. The center bundle is thus a smooth four dimensional bundle, with the particularity that it does not satisfy the Frobenius integrability conditions, and hence it cannot be integrable (not even locally!). See [Hertz et al., 2007] for an account of this example.

Nonetheless, it is well known (starting with the work of Anosov [Anosov, 1967]) that the invariant bundles are seldom differentiable: *they are only Hölder continuous* [Shub, 1986]. That raised the natural question of whether there could be any other obstruction to integrability of E^c when this is bundle is not C^1. This question remained open until very recently, when F.Rodriguez-Hertz, J. Rodriguez-Hertz and R. Ures gave an striking counterexample: they found a PH diffeomorphism on the 3 torus where the line bundle E^c is not integrable. Note that since $\dim E^c = 1$ by Peano's theorem there exist locally curves which integrate E^c. The problem is that these curves cannot be assembled into a foliation. We will discuss this example in section 1.3.

Why do we care so much about the existence of these foliations? Well, in fact one can derive many dynamical properties of the maps using them and during the course we will see difference instances of this. Let us see an example: we will use the properties of $\mathcal{W}^s, \mathcal{W}^u$ to establish ergodicity of Anosov diffeomorphisms with respect to a natural invariant measure.

1.1.1 Codimension one Anosov maps are ergodic (mixing)

In this section we will give a proof of the fact that codimension one Anosov Diffeomorphisms are Ergodic using properties of the invariant foliations. The idea here is two-fold: on the one hand show that geometrical properties of the foliations are useful for establishing dynamical properties, and on the other to recall some definitions. Furthermore a similar technique will be used later when we study the geodesic flow.

The setting here is: $f : M \to M$ is an Anosov with $dimE^u = 1$. We will show that f is mixing with respect to its entropy maximizing measure. We essentially follow McMullen's notes [McMullen, 2011], where the same result is proved for the Thom map.

Note: If you are not familiar with hyperbolic dynamical systems just assume that A is the Thom map. It's entropy maximizing measure is the Lebesgue measure.

Remark 1.8. The restriction of codimension 1 is not necessarily and one can prove the same result for any Anosov map by using symbolic dynamics (see [Bowen, 1975]).

We first recall the following.

Definition 1.9. Let $T : (X, \mu) \to (X, \mu)$ be a measure preserving map. One says that T is *Mixing* if for every pair of functions $f, g \in L^2(X)$ we have

$$\lim_{n \mapsto \infty} \int f(x)g(T^n x)d\mu(x) = \int f(x)d\mu(x) \int g(x)d\mu(x) \tag{1.6}$$

If T is mixing then T is ergodic, but the converse is not true. If X is a compact metric space, it suffices to show (1.6) for continuous functions.

Definition 1.10. Assume that X is a compact metric space and consider $T \in Homeo(X)$. We say that T is *uniquely ergodic* if it has only one invariant measure.

If T is uniquely ergodic then it is ergodic with respect to its invariant measure. This is a consequence of the fact that the set $\mathcal{X}_T$ of T invariant measures is a convex, ω^*-closed subset of the set of all probabilities, and the ergodic measures correspond precisely to its extremal points. The set of extremal points is non-empty (a consequence of Krein-Millman's Theorem), so if there is only one invariant measure it is necessarily extremal, and thus ergodic.

Unique ergodicity is equivalent to: there exist some measure μ such that for every $x \in X$ we have

$$\sum_{i=0}^{n-1} \delta_x \xrightarrow[n \to \infty]{w^*} \mu$$

Assume first that $f = A$ is a linear hyperbolic map acting on the torus T^n. Then A preserves the Lebesgue measure ($detA = 1$) and this measure is its entropy maximixing measure, meaning

$$h_{Leb}(A) = h_{top}(A)$$

In general the existence of a entropy maximizing measure is a delicate issue, but in the case of a hyperbolic matrix is just a direct computation.

Consider the expanding foliation of A. This is a one dimensional orientable (since E^u is generated by the unstable eigenvector) foliation, and thus its leaves are the orbits of a flow ϕ_t. We call this flow the *horocycle flow*. One can make similar definitions of ergodicity, unique ergodicity, etc. for flows. For example, the definition of ergodicity for a flow is that there are not L^2 invariant functions except for the constant ones.

Lemma 1.11. *The horocycle flow is ergodic.*

In fact the horocycle flow of an Anosov diffeomorphisms is uniquely ergodic, but for us it suffices to show ergodicity and in out context the proof is much simpler. If you are assuming that A is the Thom map skip the following proof and prove the statement directly.

Proof. First we prove that the flow the horocycle flow is minimal (all its orbits are dense) and for this, since all trajectories are translates of the orbit of 0, that the orbit of zero is dense.

It is easy to show that the set of periodic points for A is dense (they correspond to the points of the torus with rational coordinates). Now let $N = cl(orb_{\phi^t}(0)) = W^u(0)$. It suffices to show that N is open.

Take $x \in N$ and consider a small neighborhood U of x. If $p \in U$ is a periodic point, then by invariance of $\mathcal{W}^s$,$\mathcal{W}^u$it follows (draw a picture!) that $W^s(p) \cap W^s(0) \neq \emptyset$. But points in $W^s(p)$ approach p under iteration, and thus $p \in N$. We have shown that all periodic points of U are also in A. Since these points are dense in U and N is closed, we conclude $U \subset N$.

Let $v = (v_1, \dots, v_n)$ the (constant) vector field generating ϕ_t and consider the map $T : T^{n-1} \times 0 \subset T^n \to T^{n-1} \times 1$ induced by the flow. It is easy to see that $T = R_\alpha$ for some vector $\alpha \in T^{n-1}$, and since the flow is minimal, T is minimal as well.

Now we recall that the only measure on a torus which is invariant under all traslations R_β is the Lebesgue measure (the Haar measure of the torus). Take now any measure ν invariant under T. If we take any continuous function $h : \mathbb{T}^{n-1} \to \mathbb{T}^{n-1}$ we obtain

$$\int h(T^m(x))d\nu(x) = \int h(x + m\alpha)d\nu(x) = \int h(x + T^m(0))d\nu(x)$$

for every $m \in \mathbb{Z}\dot{\text{B}}$ut the orbit of 0 under T is dense, and hence we obtain for every $a \in \mathbb{T}^n$

$$\int h(x)d\nu(x) = \int h(x + a)d\nu(x) = \int h(R_a(x))d\nu(x)$$

We conclude that ν is the Lebesgue measure. In other words T is uniquely ergodic. Finally we take a continuous invariant function l under the flow: it induces an T invariant function, and hence it is constant a.e.

By invariance we conclude that l is constant, and the horocycle flow is ergodic.

□

Denote by ϕ_t the horocycle flow and let l, m be two continuous functions in T^n. We have that

$$\begin{aligned}\langle l, m \circ A^n \rangle = \int_{T^n} l(x)m(A^n x)dx &= \frac{1}{T}\int_0^T \int_{T^n} l(x)m(A^n x)dxdt \\ &= \frac{1}{T}\int_0^T \int_{T^n} l(\phi_t x)m(A^n \phi_t x)dxdt \ \forall T\end{aligned}$$

since ϕ_t preserves the Lebesgue measure. Note also that

$$A^n(\phi_t x) = A^n(x + te_u) = A^n(x) + \lambda^n te_u = \phi_{\mu^n t}(A^n x).$$

Using Fubini we switch the order of integration to get

$$\langle l, m \circ A^n \rangle = \int_{T^2} \frac{1}{T}\int_0^T l(\phi_t x)m(\phi_{\mu^n t}(A^n x))dtdx \ \forall T$$

Now l is (uniformly) continuous, so we can use the mean value theorem in the inner integral and get for T small

$$\approx \int_{T^n} l(x)\frac{1}{T}\int_0^T m(\phi_{\mu^n t}(A^n x))dtdx$$
$$= \int_{T^n} l(x)\frac{1}{\mu^n T}\int_0^{\mu^n T} m(\phi_t(A^n x))dtdx$$
$$= \int_{T^n} l(A^{-n}x)\frac{1}{\mu^n T}\int_0^{\mu^n T} m(\phi_t(x))dtdx$$

where in the last equality we have used that A also preserves the measure. Using Birkhoff's theorem[5], and since the horocycle flow is ergodic, the inner integral converges to the constant function $\int mdx$ as $n \mapsto \infty$. Thus

$$\lim_{n\mapsto\infty}\langle l, m \circ A^n\rangle = \int mdx \int l \circ A^{-n}dx = \int ldx \int mdx$$

We now consider the general case, i.e. when f is a codimension one Anosov, not necessarily linear. By the results of J. Franks [Franks, 1968] and S. Newhouse [Newhouse, 1970], it is conjugated to a linear hyperbolic matrix $A : T^n \to T^n$ (in particular $M = T^n$). Under the conjugacy, the entropy maximizing measure of A goes to the entropy maximizing measure of f, and by the previous case we conclude that this measure is mixing for f.

1.1.2 Second Example: Automorphisms of the Torus

We will fix $\widetilde{A} \in SL(n, \mathbb{Z})$ and consider $A : T^n \to T^n$ the induced linear automorphism of the torus: A is is APH. In fact A is dynamically coherent, and all invariant foliations are smooth. Furthermore the distribution $E^u \oplus E^s$ is also integrable. We first study the ergodic properties of A.

Proposition 1.12. *The following conditions are equivalent.*

1. *A is ergodic with respect to the Lebesgue measure.*
2. *The eigenvalues of $\widetilde{A}$ are not root of the unity.*

[5]See Appendix B.

3. *The orbit of any* $n \in Z^n \setminus 0$ *under* $\widetilde{A}^*$ *is unbounded.*

4. *A is mixing.*

5. *The foliation tangent to* $E^u \oplus E^s$ *is minimal.*

6. *A is K.*

7. $\widetilde{A}$ *is Bernoulli.*

For the proof we will use the following important observation:

Remark 1.13. If A a linear automorphism of the torus whose eigenvalues are all of norm equal to one, all its eigenvalues are roots of the unity. In particular a linear ergodic automorphism of the torus is PH.

Proof. Recalling that the eigenvalues of $\widetilde{A}^k$ are the k^{th}-powers of the eigenvalues of $\widetilde{A}$, one obtains that for every positive integer k

$$tr(A^k) = \sum_1^n \lambda_i^k \in \mathbb{Z}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\widetilde{A}$.

By compactness there exist positive integers $(k_l)_l^\infty$ such that

$$(\lambda_1^{k_l}, \ldots, \lambda_n^{k_l}) \xrightarrow[l \to \infty]{} (1, \ldots, 1)$$

and hence $\sum_1^n \lambda_i^{k_l} \underset{l}{\to} n$.

This implies that for big enough l the sum $\sum_1^n \lambda_i^{k_l}$ is equal to n, and since each eigenvalue has norm one we conclude that each one of them is necessarily equal to one. □

Proof. That $1 \iff 2 \iff 3 \iff 4$ follows from arguments similar to the ones used in section 1.0.1 (excercise 4). That $5 \Rightarrow 1$ will be proved in the next chapter.

We study $2 \Rightarrow 5$: Let $E = cl(E_A^u \oplus E_A^s) \subset \mathbb{T}^n$. We will show that $E = \mathbb{T}$. Note that E is the projection of $\widehat{E} = cl(E_{\widetilde{A}}^s \oplus E_{\widetilde{A}}^s)$. The set E is an invariant (additive) subgroup of the torus, compact and connected (being the projection of a connected set). Hence it is a sub-torus of $\mathbb{T}^n$. This implies that $\widehat{E}$ has a basis of integer vectors, say $\mathcal{B}$.

Now we apply Gauss elimination to obtain a triangular form of $\widetilde{A}$: in other words $\widehat{A}$ is conjugated (by an integer matriz!) to a matrix of the form

$$\Gamma = \begin{pmatrix} F & G \\ 0 & H \end{pmatrix}$$

where F, G, H are integer matrices, $F : \widehat{E} \to \widehat{E}$.

The fact that this conjugacy is made by an integer matrix implies that A is conjugated in the torus to the corresponding automorphism induced by Γ. Note that the eigenvalues of Γ are the same of the ones of $\widehat{A}$, and in particular no eigenvalue of Γ is root of the unity (since the matrices are conjugated). But then H: is not present in Γ; otherwise H would be a matrix whose eigenvalues have all norm equal to one, which implies that they are root of the unity.

So $\Gamma = F$ and $E = \mathbb{T}^n$.

For $1 \Rightarrow 6$ see Appendix C.

$1 \Rightarrow 7$ is a celebrated theorem due to Katznelson: it means that f is measurably conjugated to a Bernoulli Scheme. A Bernoulli Scheme is a system of the following form: the space consist of all bi-infinite sequences whose terms belong to some finite alphabet $\{1, \ldots, r\}$, hence the space is $\Sigma = \{1, \ldots, r\}^{\mathbb{Z}}$. The dynamics is the shift automorphism σ defined by

$$(\sigma x)_n = x_{n+1}$$

if $x = (x_n) \in \Sigma$. Finally, the invariant measure is a Bernoulli measure. Here is the definition: take a probability vector $P = (p_1, \ldots, p_r)$ and consider the measure ν_P on $\{1, \ldots, r\}$ defined by $\nu_P(i) = p_i$. A Bernoulli measure is a measure on Σ of the form $(\nu_P)^{\mathbb{Z}}$. □

We record the following properties of the ergodic automorphism A.

1. There are not any closed center leaves: if we have a closed leaf, it is a torus. Then a similar argument of $2 \Rightarrow 5$ finishes the proof.

2. The periodic leaves are dense.

Consider the matrix

$$F_{RH} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{pmatrix}.$$

Then $F_{RH} \in SL_4(\mathbb{Z})$ has one eigenvalue of norm bigger than one, one eigenvalue of norm smaller than one, and two eigenvalues on the unit circle, none of them roots of the unity. Hence F_{RH} is ergodic. In fact Federico proved in his thesis that this map was not only ergodic but stably ergodic, meaning:

Definition 1.14. Let $f \in Diff^r_{vol}(M)$. We say that F is stably ergodic if there exist a neighborhood $U \subset Diff^1_{vol}(M)$ of f such that every $g \in U$ is ergodic.

For example, in the next chapter we will give a complete proof that $\mathcal{C}^2$ Anosov diffeomorphisms are ergodic. Combining this with the fact that Anosov diffeomorphisms are $\mathcal{C}^1$ open we conclude that every $\mathcal{C}^2$ Anosov diffeomorphism is stably ergodic.

For the case of the map F_{RH} things are much more difficult, and we will not explain here the delicate proof of its stable ergodicity referring the reader to [Rodriguez-Hertz, 2005] instead. The map $F_{RH} : T^4 \to T^4$ is the F.Rodriguez-Hertz map.

1.2 The geodesic flow

We will study now another type of PH diffeomorphism, the time 1-map of the geodesic flow corresponding to surfaces of negative curvature. First we will recall some general definitions and review some basic facts of the geometry of hyperbolic surfaces.

Definition 1.15. Let M be a compact surface and $\phi = (\phi_t)$ a $\mathcal{C}^r$-flow with $r \geq 1$. We say that ϕ is an Anosov flow if there exists a continuous splitting of the tangent bundle into a Whitney sum of the form

$$TM = E^u \oplus E^c \oplus E^s$$

where neither of the bundles E^s nor E^u are trivial, E^c is the direction generated by the tangent of ϕ, and such that.

1. For every t,and for every $x \in M, d_x\phi_t E^\sigma_x = E^\sigma_x$.
2. $\lambda = \sup_{t\geq 0}\{\|d\phi_t|E^s\|\} < 1 < \mu = \inf_{t\geq 0}\{m(d\phi_t|E^u)\}$.

It is obvious that if ϕ is an Anosov flow then for every T the time-T map $\phi_T : M \to M$ is APH. We will study the map $f = \phi_1$.

The most well-known example of hyperbolic flow is the geodesic flow corresponding to a surface of negative curvature. References for this section are [McMullen, 2011] and [Rhoades,].

1.2.1 Geometry in $\mathbb{H}$

We will work with the upper-half plane equipped with the hyperbolic metric

$$ds_h^2 = \frac{1}{y^2}dxdy$$

This is a complete metric of constant sectional curvature $K_g = -1$. It is a simple exercise to verify that every element of

$$PSL_2(\mathbb{R}) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ a, b, c, d \in \mathbb{R}, ad - bc = 1\}$$

defines an isometry of $\mathbb{H}$. In fact, by Schwartz-Pick we have.

Lemma 1.16. *$Isom(\mathbb{H}, ds_h) = PSL_2(\mathbb{R})$*

By a *non-euclidean line* we mean either a a semicircle in $\mathbb{H}$ with center in $\mathbb{R}$ or a vertical semi line. It is easy to prove that elements of $PSL_2(\mathbb{R})$ preserve the family of non-euclidean lines, and in fact the group $PSL_2(\mathbb{R})$ acts transitively on the set of non-euclidean lines. One verifies directly that the y axis minimizes the distance between two points on it, hence it is a geodesic line, for example the trace of the geodesic $\gamma_{i,u_2}(t)$ determined by the point i and the unit vector $u_2 = (0, 1)$. Thus we conclude that the set of non-euclidean lines coincide with the set of geodesics in $\mathbb{H}$.

To define the geodesic flow[6], we need to work in the tangent bundle, and since geodesics have constant velocity, we will restrict ourselves to the unit

[6] Geodesics in $\mathbb{H}$ intersect!

tangent bundle (i.e. we will parametrize by arc length). The geodesic flow is then, the flow $g_t : T_1\mathbb{H} \to T_1\mathbb{H}$ given by

$$g_t(z,v) = (\gamma_{z,v}(t), \gamma'_{z,v}(t))$$

Note that by direct computation

$$\gamma_{i,u_2}(t) = ie^t.$$

Now since $PSL_2(\mathbb{R})$ consist of isometries, this group acts on $T_1\mathbb{H}$ as follows: if $A \in PSL_2(\mathbb{R})$ and we denote $L_A(z) = Az$, then

$$A \cdot (z,v) = (L_A(z), L'_A(z)v)$$

Using cross-ratios, for example, we verify that this action is transitive, and by direct computation one sees that the isotropy group of (i, u_2) is trivial, hence one can identify $PSL_2(\mathbb{R}) \approx T_1\mathbb{H}$ via

$$A \leftrightarrow A \cdot (i, u_2)$$

On the other hand,

$$(\gamma_{z,v}(t), \gamma'_{z,v}(t)) = (ie^t, ie^t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} \cdot (i, u_2)$$

and furthermore if $(z,v) = A \cdot (i, u_2)$ then $\gamma_{z,v}(t) = A\gamma_{i,u_2}(t)$. We have shown

Lemma 1.17. *Under the identification $PSL_2(\mathbb{R}) \approx T_1\mathbb{H}$ the geodesic flow is given by*

$$g_t(A) = A \cdot \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$

There are two other important flows related to g_t, which we now describe. Given a vector $v_z = (z, v)$, denote by $a-, a+$ the left and right intersections with the real line of the geodesic γ_{v_z} (if v is vertical we define $a+ = \infty$). Consider the circle $H^s(v_z)$ tangent to v at z and to $\mathbb{R}$ in $a+$. If $a+ = \infty$ define $H^s(v_z)$ to be the horizontal line passing through z. The set $H^s(v_z)$ is called the *stable horosphere* of v_z. We note that the family of stable horospheres is invariant under the action of $PSL_2(\mathbb{R})$, and this action is also transitive.

We will suppose that horospheres are oriented with the usual conventions, and we parametrize them with unit speed. Then we define the s-horocycle flow as the flow which parallel transports v_z along the horosphere. Note that this is a flow in the unit tangent bundle of $\mathbb{H}$, $h^s_t : T_1\mathbb{H} \to T_1\mathbb{H}$.

Remark 1.18. If $\pi : T_1\mathbb{H} \to T_1\mathbb{H}$ denotes the projection, then $d_{\mathbb{H}}(\pi v_z, \pi h^u_t(v_z)) = t$.

We also define the unstable horosphere as $H^u((z,v)) = H^s((z,-v))$ and the u-horocycle flow $h^u_t : T_1\mathbb{H} \to T_1\mathbb{H}$ by $h^u_t((z,v)) = -h^s_{-t}((z,-v))$.

Definition 1.19. The orbit of v_z under h^s is the stable horocycle (nt. $W^s(v_z)$) and the orbit under h^u is the ustable horocycle.

The stable-unstable horocycles are homeomorphic to lines: see the exercises .

Let $i_i = (i, u_2)$ and consider $H^s(i_i)$: we have $\pi\phi_t(H^s(i_i)) = ie^t + H^s(i_i)$, hence the length along the horizontal line decreases by e^{-t}, so we have

$$\phi_t \circ h^s_v = h^s_{e^{-t}v} \circ \phi_t$$

and hence,

$$\phi_t \circ h^u_v = h^u_{e^t v} \circ \phi_t$$

If we define E^s, E^u, E^c to be the line fields generated by the tangent vectors to h^s, h^u, ϕ_t we have then.

Proposition 1.20. *The geodesic flow on $T_1\mathbb{H}$ is a hyperbolic flow with respect to the decomposition $E^u \oplus E^c \oplus E^s$.*

Going back to our identification $PSL_2(R) \approx T_1\mathbb{H}$, note that

$$h^s_t(i_i) = (i + t, u_2) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot (i, u_2)$$

and since elements of $PSL_2(R)$ permute s-horospheres, we finally get.

Proposition 1.21. *In $PSL_2(R)$, we have*

$$h^s_t(A) = A \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$h_t^u(A) = A \cdot \begin{pmatrix} 1 & 1 \\ t & 1 \end{pmatrix}$$

The second formula follows from the first one together with the fact $(z, v) = A \cdot (i, u_2) \to (z, -v) = JA \cdot (i, u_2)$ where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Finally we prove that the product of the Riemannian area $d\lambda = \frac{1}{y^2} dx \wedge dy$ on $\mathbb{H}$ with the Lebesgue measure on S^1 is invariant by action of $PSL_2(\mathbb{R})$ on $T^1\mathbb{H} = \mathbb{H} \times S^1$.

Given $A \in PSL_2(\mathbb{R})$ suppose that $A^1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $Im(L_{A^{-1}}(z) = \frac{Im(z)}{|cz+d|^2})$, and since $L'_{A^{-1}}(z) = \frac{1}{(cz+d)^2}$ its jacobian is equal to $j(z) = \frac{1}{|cz+d|^4} = |\frac{Im(L_{A^{-1}}(z)}{Im(z)}|^2$. Let f be any continuous function with compact support on $\mathbb{H}$ and compute

$$(L_A)_*\lambda(f) = \int f(Az) \frac{1}{Im(z)^2} dLeb(z) = \int f(z) j(z) \frac{1}{Im(L_{A^{-1}}(z)} dLeb(z) = \lambda(f)$$

hence $(L_A)_*\lambda = \lambda$.

We use the coordinates (z, θ) on $T_1\mathbb{H}$. Note that the action of A in the θ coordinate is just a translation (because L_A is complex differentiable its action on vectors amounts to only rotate them). Hence we see that the *Liouville* measure

$$dLiou = d\lambda d\theta$$

is invariant under the action of $PSL_2(\mathbb{R})$.

Proposition 1.22. *The Liouville measure on $PSL_2(\mathbb{R}) = T^1\mathbb{H}$ coincides with the Haar measure.*

Proof. By the previous computations, we get that the Liouville measure is invariant by multiplication on the left by elements of $PSL_2(\mathbb{R})$. On the other hand, it is well known that $PSL_2(\mathbb{R})$ is unimodular, i.e. the left and the right Haar measures coincide, hence the claim. $\square$

Remark 1.23. Note that, in particular, the Liouville measure is preserved by ϕ_t, h_t^s, h_t^u.

1.2.2 Hyperbolic Surfaces

Let S be a compact surface of negative sectional curvature. It is a consequence of the Uniformization Theorem that in this case the universal covering of S is $\mathbb{H}$, and the fundamental group Γ of S acts by hyperbolic isometries on $\mathbb{H}$. We can thus identify $\Gamma < PSL_2(R)$ and

$$S = \Gamma \setminus \mathbb{H} = \{\Gamma z : z \in \mathbb{H}\}$$

Hence, we can see unit tangent bundle of S as the homogeneous space

$$T^1 S = \Gamma \setminus \ PSL_2(R) = \{\Gamma A : A \in PSL_2(R)\}$$

Under these identifications we have

$$g_t(\Gamma A) = \Gamma A$$

$$h_t^s(\Gamma A) = \Gamma A \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$h_t^u(\Gamma A) = \Gamma A \cdot \begin{pmatrix} 1 & 1 \\ t & 1 \end{pmatrix}$$

By Proposition 1.22, the Liouville measure induces a (finite area) measure on $T_1 S = \Gamma \setminus \ PSL_2(R)$, the Liouville measure on $T_1 S$. We assume that this measure is normalized to have area equal to one. Note that the flows g_t, h_t^s, h_t^u preserve the Liouville measure.

1.2.3 Ergodicity of the Geodesic Flow

Now we will prove that the time-one map of the geodesic flow in a surface of negative curvature is ergodic. To prove this, we will use the ergodic properties of the flow instead. Now, if a flow is ergodic it is not generally true that the time-one map is ergodic (give a counterexample). On the other hand, it does follows that if the flow is mixing then the time one map is also mixing. See exercise **??**. It would suffice then to show that if Γ is a cocompact lattice in $PSL_2(\mathbb{R})$ then $g_t : M = \Gamma/PSL_2(\mathbb{R}) \to M$ is mixing.

Consider the subgroups $G = (g_t) \cdot I, H^s = (h_t^s)I, H^u = (h_t^u)I$ of $PSL_2(\mathbb{R})$. Since (g_t) is an Anosov flow, it follows easily that $PSL_2(\mathbb{R})$ is generated by $\{G, H^s, H^u\}$.

Since the action of the geodesic and horocycle flows amounts to multiplication (on the right) by a certain matrix of $PSL_2(\mathbb{R})$, it is clear that we can extend the flows to an action of the complete $PSL_2(\mathbb{R})$. For $a \in PSL_2(\mathbb{R})$ we consider the Koopman operator $T_g : L^2(M) \to L^2(M)$ of this complete action, which is defined by

$$T_a(f) = f \circ R_a$$

where $R_a(\Gamma b) = \Gamma ab$. The Koopman operators induced by the geodesic and horocycle actions are denoted T_t^g, T_t^s, T_t^u respectively. Observe that each of the Koopman operators are unitary, and in fact the assignation $T : a \to T_a$ is a unitary representation of $PSL_2(\mathbb{R})$. Similarly for the other operators.

Lemma 1.24. *Fix elements $g = diag(x, 1/x) \in G, a \in H^+, b \in H^-$. Then*

$$\lim_{n\to\infty} g^n a g^{-n} = I \text{ if } x < 1$$

$$\lim_{n\to\infty} g^n b g^{-n} = I \text{ if } x > 1$$

Proof. Excercise. □

Proposition 1.25 (Mautner's Lemma)**.** *Consider the Koopman unitary representation $T : PSL_2 \to U(L^2(M))$ and suppose that $g, h \in SL_2(\mathbb{R})$ satisfy*

$$\lim_{n\to\infty} g^n h g^{-n} = I$$

Then if $f \in L^2(M)$ satifies $T_g(f) = f$, it also satifies $T_h(f) = f$.

Proof. We have

$$\|T_h(f) - f\|^2 = \langle T_{g^n} T_h T_{g^{-n}} f - f \rangle = \langle T_{g^n h g^{-n}} f - f \rangle$$

By SOT-continuity (exercise 8) we conclude the claim. □

Theorem 1.26. *The geodesic flow on M is ergodic.*

Proof. It is easy to verify that the flow (g_t) is ergodic if and only if

$$T_t^g(f) = f \text{ a.e. } \to f = \text{ constant a.e.}$$

Let $f \in L^2(M)$ be fixed for every T_{g_t}. By Mautner's Lemma and the Lemma before it, we conclude that f is invariant under the Koopman action of G, H^s, H^u, which generate $PSL_2(\mathbb{R})$, hence f is constant as we wanted to show. □

Now we will show that the horocycle flows are ergodic as well.

Theorem 1.27. *The horocycle flows h^s, h^u are ergodic.*

Proof. Of course it suffices to show the claim for h^s. We proceed as in the proof of the ergodicity of the geodesic flow.

Note that is enough to establish the following: if $f \in L^2(M)$ is fixed under the action of T_a for $a \in H^s$ then f is fixed under the action of T_g for $g \in G$ (as we have seen, invariance under the geodesic flow implies invariance under both horocycle flows).

Define $\psi : PSL_2(\mathbb{R}) \to \mathbb{R}$ by $\psi(a) = \langle T_a f, f \rangle$: ψ is continuous. Fix $g = diag(x, 1/x)$ and consider the matrices

$$a_n = \begin{pmatrix} 0 & \alpha_n^{-1} \\ \alpha_n & 0 \end{pmatrix}$$

$$b_n = \begin{pmatrix} 1 & x\alpha_n^{-1} \\ 0 & 1 \end{pmatrix}$$

$$c_n = \begin{pmatrix} 1 & x^{-1}\alpha_n^{-1} \\ 0 & 1 \end{pmatrix}$$

where (α_n) is a sequence of positive numbers converging to zero. Then

$$b_n a_n c_n = \begin{pmatrix} x & 0 \\ \alpha_n & x^{-1} \end{pmatrix}$$

Furthermore by invariance, $\psi(a_n) = \psi(b_n a_n c_n)$ hence

$$\lim_{n\to\infty} \psi(a_n) = \psi(g)$$

But the a_n do not depend on g and g is arbitrary, hence we conclude that ψ is constant in G, i.e. for every $g \in G$

$$\langle T_g f - f, f \rangle = 0$$

By Cauchy-Schwartz (T_g is unitary) we conclude $T_g f = f$ as we wanted to show. □

Theorem 1.28. *The geodesic flow (g_t) is mixing, hence $f = g_1$ is also mixing.*

Proof. Take two continuous observables ϕ, ψ, and let $h = h^u$. Then

$$I = \langle \phi, \psi \circ g_t \rangle \approx \langle \phi \circ h_{-s}, \psi \circ g_t \rangle = \langle \phi, \psi \circ h_s g_t \rangle$$

if s small. Since $h_s g_t = g_t h_{e^t s}$, we get

$$I \approx \langle \phi, \psi \circ g_t h_{e^t s} \rangle = \langle \phi g_{-t}, \psi h_{e^t s} \rangle$$

Taking S small and averaging over S we obtain

$$I = \frac{1}{S} \int_0^S I ds \approx \langle \phi g_{-t}, f \rangle$$

$$f(t,x) = \frac{1}{S} \int_0^S \psi(h_{e^t s} x) ds = \frac{1}{e^t S} \int_0^{e^t S} \psi(h_s x) ds$$

and thus, if t is sufficiently large by ergodicity of h^s we get

$$f(t,x) \approx \int \psi dm$$

Finally, putting everything together we obtain for large t,

$$I \approx \int \phi dm \int \psi dm$$

as we wanted to show. □

1.2.4 Consequences of the Ergodicity

Here we obtain some meaningful dynamical properties of the time one map of the geodesic flow. We start with the following.

Corollary 1.29. *The time one map of the geodesic flow is transitive.*

This follows immediately since it is mixing with respect to a measure positive on open sets.

Now will prove that there exist infinitely many closed leaves for $f = g_t$. These correspond to the periodic points of the geodesic flow. For doing that we introduce the important property of shadowing.

Definition 1.30. Let $f : M \to M$ be PH[7].

1. A sequence $\underline{x} = \{x_n\}_{-N}^{N}$ where $N \in \mathbb{N} \cup \{\infty\}$ is called a δ-pseudo-orbit for f if $d(fx_n, x_{n+1}) \leq \delta$ for every $n = -N, \dots, N-1$.
2. The pseudo-orbit $\underline{y} = \{y_n\}_{-N}^{N}$ ϵ-shadows the pseudo-orbit $\underline{x} = \{x_n\}_{-N}^{N}$ if $d(x_n, y_n) < \epsilon$ for every $n = -N, \dots, N-1$.
3. We say that the pseudo-orbit $\underline{x}$ *respects the foliation* $\mathcal{F}$ or it is *subordinate to the foliation* $\mathcal{F}$ if for every $n \in \{-N, \dots, N-1\}$, $f(x_n)$ and x_{n+1} are in the same leaf of $\mathcal{F}$.

The relevant theorem involving pseudo-orbits is the following.

Theorem 1.31 (Shadowing)**.** *Let $f : M \to M$ be a dynamically coherent PH diffeomorphism. Then there exists a constant $C(f) > 1$ only depending on f such that if δ is sufficiently small then any δ-pseudo-orbit can be $C(f)\delta$-shadowed by a $C(f)\delta$-pseudo-orbit subordinate to the foliation $\mathcal{W}^c$. That is, $f(x_n)$ and x_{n+1} lie always in the same center plaque.*

In particular, if $E^c = 0$ (i.e. the system is hyperbolic) we recover the classical shadowing theorem. Namely pseudo-orbits can be shadowed by true orbits.

This theorem is due to Hirsch, Pugh and Shub (see theorem 7A-2 in [Hirsch et al., 1977]). The version presented here appears in [Carrasco, 2010].

[7] Of course these notions can also be defined for a general homeomorphism acting on a metric space. We will not have the opportunity to use these more general notions though.

Definition 1.32. We say that a foliation $\mathcal{F}$ is *plaque expansive* if there exists $\xi > 0$ such that if $\underline{x} = \{x_n\}_n, \underline{y} = \{y_n\}_n$ are two ξ-pseudo-orbit respecting $\mathcal{F}$ and satisfying $d(x_n, y_n) < \xi$ for every $n \in \mathbb{Z}$, we have that x_n and y_n are always in the same plaque of $\mathcal{F}$. We say that a PH map is plaque expansive if $\mathcal{W}^c$ is plaque expansive.

Corollary 1.33. *Under the same hypothesis of theorem* **??**, *if f is plaque expansive then there exists $\delta_0 > 0$ such that if $0 < \delta \leq \delta_0$ then any bi-infinite δ-pseudo-orbit $\underline{x}$ can be $C\delta$-shadowed by a $C\delta$-pseudo-orbit $\underline{y}$ which respects $\mathcal{W}^c$. If $\underline{z}$ is any other $C\delta$-pseudo-orbit which $C\delta$-shadows $\underline{x}$ and respects $\mathcal{W}^c$, then y_n and z_n are always in the same plaque of $\mathcal{W}^c$.*

Corollary 1.34. *Let f be a dynamically coherent partially hyperbolic diffeomorphism and suppose that the non-wandering set of f is equal to M. Then the set of points whose central leaf is periodic is dense in M.*

Proof. Take any point x and let U be an arbitrary neighbourhood of x. Since x is non-wandering there exists a positive integer k such that $f^k(U) \cap U \neq \emptyset$.

Now define the pseudo-orbit obtained by taking the sequence $[xf(x) \dots f^n(x)]$, and copying this block one after the other (in both directions).

By 1.31 there exists a pseudo-orbit $\{y_n\}_n$ subordinate to $\mathcal{W}^c$ and close to pseudo-orbit defined before. Now consider the pseudo-orbit $\{z_n\}$ defined by shifting everything k places to the left, i.e. $z_n = y_{k+n}$.

Then $\{z_n\}$ also shadows the first pseudo-orbit , and hence, since the system is plaque expansive, z_n and y_n are always in the same plaque. In particular y_0 and $f^k(y_0)$ are in the same plaque.

This means that the leaf through y_0 is periodic.

□

Using 1.29 and the previous Corollary, we finally obtain.

Corollary 1.35. *There exist infinitely countably many closed leaves for f.*

1.3 An example of non-dynamically coherent PH diffeomorphism

Here we will be concerned with integrability of the center bundle. The arguments developed in section 1.1 show that at least for the special case when $M = T^n$ and f is a small perturbation of a linear map, then f is dynamically coherent (in fact all invariant distributions are integrable). Of course this is a very particular case, but at least for $M = T^3$ the following theorem is true.

Theorem 1.36 (Brin-Burago-Ivanov)**.** *Assume that $f : T^3 \to T^3$ is APH. Then f is dynamically coherent.*

See [Brin et al., 2009].

Even though this Theorem is fairly recent, is what was expected. The idea is that since $dimE^c = 1$ there exist, by Peano's theorem, curves tangent to E^c through every point of the manifold. Then you "only" have to assemble them into a foliation...

Of course this assembling is a very difficult problem: here we will show that in fact this is impossible in general. We will explain the following.

Theorem 1.37 (F.R.Hertz-J.R.Hertz-R.Ures)**.** *There exist a PH map $f : T^3 \to T^3$ which is not dynamically coherent.*

Proof. Let $\lambda < 1$ the stable eigenvalue of A_T (hence the unstable eigenvalue is $1/\lambda$) and let u a unit eigenvector of eigenvalue λ. Consider also a north pole-south pole function $f : T \to T$ such that

$$f(0) = 0, f(1/2) = 1/2$$

$$f'(1/2) = \sigma < \lambda < 1 < \nu = f'(0) < 1/\lambda$$

and a differentiable function $\phi : T \to \mathbb{R}$, and construct perturbation F of the Axiom-A map $A_t \times f$ by "pushing" in the stable direction of A, namely

$$F(x, \theta) = (Ax, f(\theta)) + (\phi(\theta)u, 0)$$

Note the strong unstable direction of $A_t \times f$ is unaltered by this perturbation, and in particular the strong stable manifold of the perturbation exist and coincides with the strong stable manifold of the unperturbed map. What we need to study are the other directions. Also note that the unperturbed map is not PH.

We are seeking invariant directions of the derivative of F:

$$dF_{(x,\theta)}(v,t) = (Av, f'(\theta)t) + (\phi'(\theta)tu, 0)$$

An invariant direction (inside the $u \times T$ plan) will be generated by a vector field of the form $(a(\theta)u, 1)$ for some function a, hence we need to solve

$$a(f(\theta)) = \lambda a(\theta) + \phi'(\theta) \tag{1.7}$$

We are thus led to find a solution of the cohomological equation

$$b \circ f = \lambda b + \phi \tag{1.8}$$

(the solution of (1.7) is just $a = b'$). But the solutions of this type of cohomological equation are known, and we can directly check that

$$\eta(\theta) = \frac{1}{\lambda}\sum_{1}^{\infty} \lambda^n \phi(f^{-n}\theta) \tag{1.9}$$

$$\zeta(\theta) = -\frac{1}{\lambda}\sum_{0}^{\infty} \lambda^{-n} \phi(f^{n}\theta) \tag{1.10}$$

are solutions.

For every $0 < \epsilon < 1/2$ we have $f^{-n}|_{(-\epsilon,\epsilon)} \xrightarrow{n\to\infty} = 0 = f(0)$, hence the function η is well defined and continuous in $T \setminus \{1/2\}$. Deriving term by term and using $\lambda < f'(0)$ we conclude that in fact $\eta \in \mathcal{C}^1(T \setminus \{1/2\})$. Similarly, if $\phi(1/2) = 0$ then $\zeta \in \mathcal{C}^1(T \setminus \{0\})$: we will assume that is the case.

Back to the invariant directions. Note that $\eta'(\theta)$ gets bigger as θ approaches $1/2$, and thus if we can choose ϕ so that

$$\lim_{\theta\to 1/2} \eta'(\theta) = \infty \tag{1.11}$$

we will get continuity for E^c by defining

$$E^c(\theta = 1/2) = span\{(u, 0)\} = E^s_{A_T} \times 0$$

Arguing similarly, we define

$$E^s(\theta = 0) = E^s_{A_T} \times 0$$

and we will get a continuous bundle provided that we prove

$$\lim_{\theta \to 0} \zeta'(\theta) = \infty \tag{1.12}$$

Assume for now that we have proved that these bundles are continuous. Now we want to show that $TT^3 = E^s + \oplus E^c \oplus E^u$, or what is equivalent, that the angle between E^s and E^c are not zero. What we need to show is that $\eta' \neq \zeta'$ for $\theta \neq 0, 1/2$. Note that for $\theta = 0, 1/2$ the angle is not zero, and hence it is not zero in a neighbourhood of these points. But by the cohomological equations,

$$\eta' f - \zeta' f = \lambda$$

and using the form of the dynamics of f, we conclude that the sign of $\eta' - \zeta'$ is constant in $(0, 1/2)$ and $(1/2, 1)$, and clearly non zero.

Proposition 1.38. *There exist ϕ so that*

1. *Equations* (1.11) *and* (1.12) *hold.*

2. *η' has opposite sign in* $(0, 1/2)$ *and* $(1/2, 1)$.

For example $\phi(\theta) = 1 + \cos(frm{-}epi\theta)$ works. The proof is carried in [Hertz et al., 2010].

Hence F is PH (but note that NOT APH). Finally we prove that is not dynamically coherent.

Important remark: Since the bundles only depend on θ we obtain the stable, unstable and center manifolds (provided that this last one exists) by translating a given one of the same type.

Consider the function $h : T^2 \to T^2$ given by

$$h(x, \theta) = x - \phi(\theta)u$$

Then $Fh = hA_T$ and h is clearly surjective, hence it is a semiconjugacy. Note that we have

$$l(x, \theta) = h^{-1}(h(x, \theta)) = (h(x, \theta), 0) + (\eta(\theta), \theta)$$

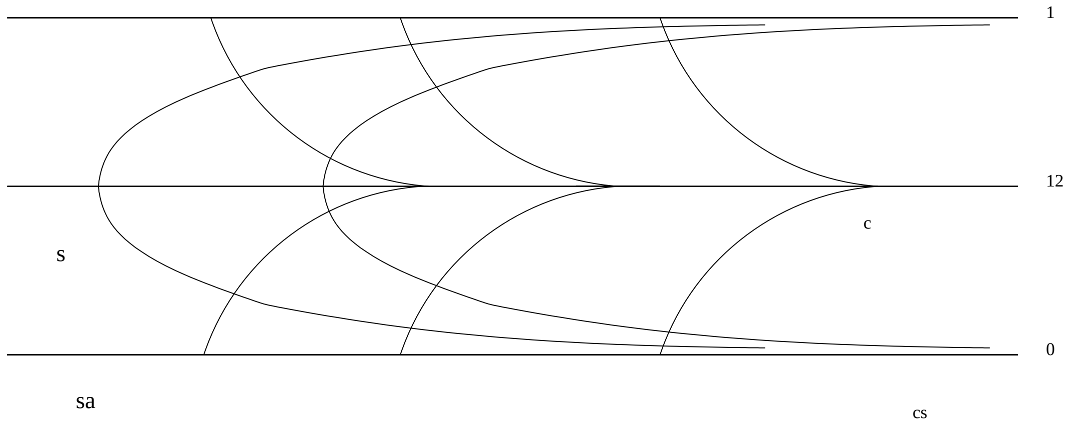

Figure 1.1: Behavior of center and stable curves.

and hence, the family of curves $\{l(x,\theta)\}$ is tangent to E^c if $\theta \neq 1/2$. For $\theta \neq 1/2$ the bundle E^c is uniquely integrable and hence its invariant curves are precisely the $l(x,\theta)$. But for $\theta = 1/2$ $E^c = E^s_A \times \{0\}$, hence its tangent curves have to be horizontal. Now we use that η' have different signs on the intervals $(0,1/2)$ and $(1/2,1)$ to conclude that this family is not a foliation near $\theta = 1/2$, hence the bundle E^c is not integrable. See Figure 1.1.

□

Important remark: The example previously constructed in fact is robust, meaning that in a neighbourhood of ot there are no dynamically coherent PH diffeomorphisms, a surprising fact. This is consequence of the existence of a *cu*-torus, a torus tangent to the center unstable direction (in the example is the invariant torus corresponding to $\theta = 1/2$).

One very important question in the theory is to characterize the obstructions to integrability. In particular the following question is still open:

Question: Does bunching imply dynamical coherence?

If the bundles E^{cs}, E^{cu} are $\mathcal{C}^1$ we have the following.

Proposition 1.39 (Burns-Wilkinson)**.** *Assume that* E^{cs}, E^{cu} *are* $\mathcal{C}^1$ *and* f *is center bunched. Then* E^{cs}, E^{cu}, E^c *are integrable.*

In fact, this proposition can be improved to the case where E^c is Lipschitz. See [Burns and Wilkinson, 2008] and references therein.

Note that we did not claim invariance of the resulting foliations, and in fact we can ask:

Question:: Assume that E^c is integrable. Does it follow that the foliation is f-invariant?

We investigate these types of questions a little bit deeper. We will explain the concept of *completeness*, and to state it we will use some notation.

Given a point $x \in M$ and a positive number $\gamma > 0$ we will denote by $W^s_\gamma(x)$ the open disc of size γ inside the leaf $W^s(x)$. Similarly for $W^u_\gamma(x)$. If L is a center manifold we define.

$$W^s_\gamma(L) = \bigcup_{x \in L} W^s_\gamma(x)$$

$$W^u_\gamma(L) = \bigcup_{x \in L} W^u_\gamma(x)$$

$$W^s(L) = \bigcup_{n \geq 0} f^{-n} W^s_\gamma(f^n L) = \bigcup_{x \in L} W^s(x)$$

$$W^u(L) = \bigcup_{n \geq 0} f^{n} W^u_\gamma(f^{-n} L) = \bigcup_{x \in L} W^u(x)$$

Note that $W^s_\gamma(L) \subset W^s(L), W^u_\gamma(L) \subset W^u(L)$ are open (with the induced topology).

Since the foliation $\mathcal{W}^c$ is normally hyperbolic one can use the results of Section 6 of [Hirsch et al., 1977] to both $W^s(L), W^u(L)$ are immersed submanifolds tangent to $E^s \oplus E^c$, $E^u \oplus E^c$ respectively. See also Proposition 3.4 in [Brin et al., 2004].

It follows by definition that for a given center leaf L the submanifolds $W^s(L)$ and $W^u(L)$ are saturated by the corresponding strong foliation (either $\mathcal{W}^s$ or $\mathcal{W}^u$).

Definition 1.40. Assume that E^c integrates to an invariant foliation $\mathcal{W}^c$. Then submanifolds $W^s(L)$ and $W^u(L)$ are said to be *complete* if they are saturated by the center foliation. The center foliation is complete if for every center leaf L the submanifolds $W^s(L)$ and $W^u(L)$ are complete

In the case of a dynamically coherent partially hyperbolic diffeomorphism, given a point $x \in L$ it follows that $W^s(L) \subset W^{cs}(x)$ is an open submanifold, and likewise $W^u(L) \subset W^{cu}(x)$ is open. Completeness of $W^s(L)$ is the same as metric completeness inside $W^{cs}(x)$.

Lemma 1.41. *Assume that f is partially hyperbolic diffeomorphism with invariant complete center foliation $\mathcal{W}^c$. Then f is dynamically coherent.*

Proof. Consider a manifold $W^s(L)$ where L is a center leaf. As we explained before, the results of section 6.1 of [Hirsch et al., 1977] imply that this is an immersed submanifold of M tangent to E^{cs}. Take $x \in W^s(L)$: by hypothesis $W^c(x) \subset W^s(L)$ and thus $W^s(W^c(x)) \subset W^s(L)$, and by the same argument $W^s(W^c(x)) = W^s(L)$. This shows that the family $\{W^s(L) : L \in \mathcal{W}^c\}$ is a partition of the manifold M into $\mathcal{C}^1$ submanifolds tangent to the continuous bundle E^{cs}, and thus it is a foliation. By hypothesis its leaves are saturated by the center foliation. Invariance follows from the fact that both $\mathcal{W}^c$ and $\mathcal{W}^s$ are invariant.

Similarly, the bundle E^{cu} integrates to an invariant foliation whose leaves are saturated by the center foliation. □

Corollary 1.42. *Assume that E^c integrates to a $\mathcal{C}^1$ foliation. Then f is dynamically coherent.*

Proof. This follows easily from the fact that if $\alpha : [0,1] \to M$ is a $\mathcal{C}^1$ curve tangent to E^c, then $\alpha([0,1]) \subset W^s(L_{\alpha(0)}) \cap W^u(L_{\alpha(0)})$, and hence $\mathcal{W}^c$ is complete. □

In principle it is not obvious at all that dynamical coherence imply completeness. You can find some discussion about this in [Carrasco, 2012].

Let us finish this part with another question. If E^{cs}, E^{cs} are integrable then E^c is integrable to the foliation obtained as intersection of cs, cu-leaves. However is we assume that E^{cs}, E^{cs}, E^c are integrable to foliations $\mathcal{W}^{cs}, \mathcal{W}^{cu}$ $\mathcal{W}^c$ there is no reason a priori that implies that the intersection of a center stable and a center unstable leaf is a leaf of $\mathcal{W}^c$. The reason is the that E^c may integrate to many foliations, so in principle we could have a center stable and a center unstable intersecting in a manifold tangent to E^c but not necessarily a leaf of $\mathcal{W}^c$. In [Hertz et al., 2010] the authors construct a center bundle that is not uniquely integrable, but in fact the unique integrability only fails at a curve. So we are led to the following question.

Question: If E^c is integrable, is it true that it has a unique foliation tangent to it?

1.4 Skew Products and Compact Foliations.

For a dynamically coherent PH map, we have seen three different types of examples in terms of the structure of its center foliation, namely

1. All center leaves are closed (Product case).
2. No center leaf is closed (Anosov case).
3. There are infinitely many closed center leaves and infinitely many non-closed center leaves (Anosov flow case).

This short list in fact resumes all known behaviour in terms of the structure of the center foliation, and thus one is led to wonder whether this list is complete, namely

Question: Are there any PH diffeomorphisms with other type of center structure?

Here we will investigate the first case, namely we will assume that f is PH with compact center foliation $\mathcal{W}^c$ (i.e. all leaves of $\mathcal{W}^c$ are compact). We will only sketch some results, and refer the reader to [Carrasco, 2012] for the complete proofs.

Example: Skew products.
Take an Anosov map $A : N \to N$ and let G a (connected) compact Lie Group. Suppose that there exist a representation $\phi : N \to Aut(G)$ and define

$$f(x, g) = (Ax, \phi(g)(x))$$

Then $f : N \times G \to N \times G$ is PH, and is called a Skew product. These types of examples have been studied thoroughly .
More in general, we can consider a PH map f such that it fibers over an Anosov map A.

In general, for a compact foliation on a compact riemannian manifold, the most (and characterizing) property is the fact of being uniformly compact.

Definition 1.43. A compact foliation $\mathcal{F}$ of a compact Riemannian manifold M said to be *uniformly compact* if the function $vol : M \to \mathbb{R}_+$ which assigns to each point x the Riemannian volume of the submanifold $L_x \subset M$ is uniformly bounded from above, i.e.

$$\sup\{vol(L) : L \text{ leaf of } \mathcal{F}\} < \infty$$

Establishing this property is the fist step to prove that f fibers over some hyperbolic map. Let us mention that this concern is not unjustified: there are examples of compact foliations on compact manifolds which are not uniformly compact. See [Sullivan, 1976].

Theorem 1.44 (D.B.A. Epstein)**.** *The following conditions are equivalent.*

1. *$\mathcal{F}$ is uniformly compact.*

2. *Every leaf L of $\mathcal{F}$ has finite holonomy.*

3. *$X/\mathcal{F}$ is Hausdorff.*

We will also need the following.

Theorem 1.45. *Let G be a group that acts effectively on a connected manifold N by homeomorphisms and such that every point has a finite orbit. Then G is finite.*

Now we are ready to start working. First we have the following.

Proposition 1.46. *Let f be a partially hyperbolic diffeomorphism with uniformly compact foliation $\mathcal{W}^c$. Then*

1. *f is dynamically coherent.*

2. *The foliation $\mathcal{W}^c$ is complete.*

Proof. By Lemma 1.42 it suffices to show that $\mathcal{W}^c$ is complete.

We denote by $\pi : M \to X = M/\mathcal{W}^c$ the projection onto the quotient of M by the leaves of the center foliation. The space N is a compact Hausdorff space (see 1.44), and it is thus metrizable where a compatible metric is given as follows: if K, K' are two center leaves then the distance between them as points in N is

$$d_X(\pi(K), \pi(K')) = \inf\{d(x, y) : x \in K, y \in K'\}$$

(see [Bourbaki, 1998]).

Fix a leaf $L \in \mathcal{W}^c$, and consider any other leaf L' such that $L' \cap W^s(L) \neq \emptyset$. We will show that given $\epsilon > 0$ there exist some N such that for every $n \geq N$ $d_X(f^n(L), f^n(L') < \epsilon$. The characterization[8] given in Theorem 6.1 in [Hirsch et al., 1977] implies that $L' \subset W^s(L)$.

By theorem **??** we can conclude that every point $F \in N$ has a neighbourhood $U(L, \delta_L)$ of radius δ_L, and since the space N is compact we can find finitely many points $F_1, \dots, F_k$ and $0 < \delta < \epsilon/2$ satisfying

$$X = \cup_{i=1}^k U(F_i, \delta).$$

Let α the Lebesgue number of the covering $\{pi^{-1}(U(F_i, \delta))\}$. By hypothesis there exist $x \in L, y \in L'$ in the same strong stable manifold. Thus, r there exist some N such that for all $n \geq N$ the points $f^N(x)$ and $f^N(y)$ are at distance less than α, and hence $d_X(f^n(L), f^n(L')) < 2\delta < \epsilon$. □

We want to prove the converse, namely that if the center foliation is complete and f is dynamically coherent with compact center foliation then $\mathcal{W}^c$ is uniformly compact. We need the following theorem.

Theorem 1.47. *Assume that f is dynamically coherent with compact center foliation $\mathcal{W}^c$ and such that for every f-periodic leaf L, $|G(L)| < \infty$. Then $\mathcal{W}^c$ is uniformly compact.*

The proof of this theorem is given in [Carrasco, 2012].

Since we are assuming dynamical coherence, for a given leaf L we can talk about its holonomy group when restricted to the center stable and center unstable manifolds where it is contained. For any point $x \in L$ we denote

$$G^s(L) = G(L|W^{cs}(x)), G^u(L) = G(L|W^{cu}(x))$$

[8] This is similar to the characterization given in the Classical Stable Manifold Theorem.

Dynamical coherence also implies that if D is a small transverse disc to $\mathcal{W}^c$ one can define coordinates on it by using the three transverse foliations. See figure 1.2.

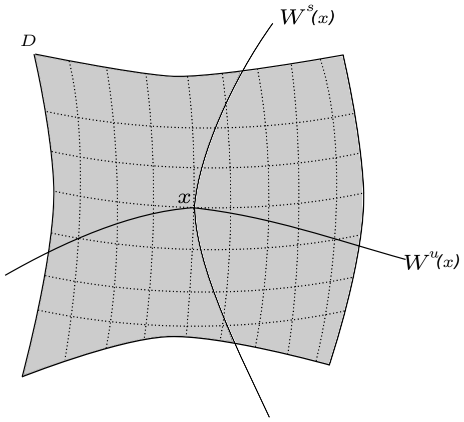

Figure 1.2: Transverse coordinates in D.

Using these coordinates, one can prove.

Proposition 1.48. *The group $G(L)$ is finite if and only if $G^s(L), G^u(L)$ are finite. In this case $G(L)$ is isomorphic to $G^u(L) \times G^s(L)$.*

Corollary 1.49. *Assume that $dimM = 3$ (thus all E^s, E^u, E^c are one dimensional. Then the foliation $\mathcal{W}^c$ is uniformly compact. Moreover, if E^c, E^{cs}, E^{cu} are oriented then all center leaves are without holonomy and f fibers over a map $g : \mathbb{T}^2 \to \mathbb{T}^2$ which is conjugate to a hyperbolic automorphism.*

Note that the condition of E^c, E^{cs}, E^{cu} being oriented can be achieved by passing to a finite covering of M.

Proof. It is not hard to see that an open surface foliated by circles is homeomorphic to either a cylinder or a Mbius band, hence it follows that the holonomy of every leaf is finite. If the bundles are oriented all center stables and center unstables are homeomorphic to cylinders hence G^s, G^u are trivial, and thus all holonomy groups are trivial. In this case $\pi : M \to M/\mathcal{W}^c$

is a fibration[9] and one concludes that $M/\mathcal{W}^c$ is a smooth manifold and $g : M/\mathcal{W}^c \to M/\mathcal{W}^c$ is a differentiable map preserving two invariant transverse foliations $\mathcal{U}, \mathcal{S}$ such that g uniformly expands $\mathcal{U}$ and uniformly contracts $\mathcal{S}$. Hence $M/\mathcal{W}^c = \mathbb{T}^2$ and g is conjugated to an Anosov diffeomorphism. □

Proposition 1.50. *Assume that f is dynamically coherent and $\mathcal{W}^c$ is complete. Then $\mathcal{W}^c$ is uniformly compact.*

Suppose that L is a periodic center leaf, and $W^s(L)$ is complete. Then for every leaf $L' \subset W^s(L)$ we have

1. *L' is a finite covering of L.*
2. *The group $G^s(L')$ is finite.*

Sketch. It suffices to show that for a given periodic leaf L, $G^s(L), G^u(L)$ are finite. Consider then $N = W^s(L)$. Since N is complete, it is not hard to prove that $N = W^{cs}(x)$ where x is any point in L. Fix x and let $W = W^s(x)$. By iterating one verifies that $W \cap L = \{x\}$.

We now use completeness: if $L' \subset N$ is any other center leaf then under iteration is approaching L (figure 1.3). Hence we conclude that L' is a covering of L where the covering projection is given by the stable projection.

This mechanisms allows us to extend any representative $g : D \subset W \to W$ of a holonomy germ to the whole W. Here is how: for $y \in W$ consider the lift $\widetilde{\alpha}_y$ of the curve α to the leaf $W^c(y)$, such that $\widetilde{\alpha}_y(o) = y$. Then define $\widehat{g}(y) = \widetilde{\alpha}_y(1)$. This procedure clearly defines a continuous extension of g to the whole W, with the property that for every point $y \in W$, the points y and $g(y)$ are in the same center leaf. In particular we can think of the holonomy group of L as a group H of homeomorphisms of the manifold W. Again completeness imply that for any L' we have $|L' \cap W| < \infty$, and thus the orbit of any point $y \in W$ under the action of H is finite. Theorem 1.45 implies then that H is finite, as we wanted to show.

□

By Corollary 1.42 a $\mathcal{C}^1$ foliation is complete, hence we get.

[9] This is consequence of Reeb's Stability Theorem.

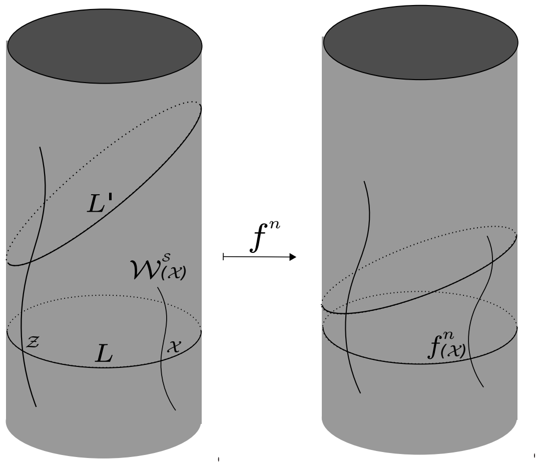

Figure 1.3: Under iterations L' aproaches L.

Corollary 1.51. *Assume that E^c integrates to a $\mathcal{C}^1$ compact foliation $\mathcal{W}^c$. Then $\mathcal{W}^c$ is uniformly compact.*

There are other cases where a compact center foliation is known to be compact, namely we have the following two theorems.

Theorem 1.52. *Assume that f is dynamically coherent and E^c is a line bundle which integrates to a compact foliation $\mathcal{W}^c$. Then $\mathcal{W}^c$ is uniformly compact.*

Definition 1.53. A partially hyperbolic diffeomorphism f is said to be *center isometric* if $\|df|E^c\| = 1$.

It is a result of M. Brin that in this case f is dynamically coherent. See [Brin, 2003].

Theorem 1.54. *Let $f : M \to M$ be a center isometric partially hyperbolic diffeomorphism whose center foliation $\mathcal{W}^c$ is compact.*

Then $\mathcal{W}^c$ is uniformly compact.

See [Carrasco, 2012].

As we saw with Corollary 1.49, at least in 3 dimensions uniform compactness imply that our map fibers over some hyperbolic map. This is also true (with the same proof) if all holonomy is trivial. We can put together this two facts and ask

Question: If $\mathcal{W}^c$ is uniformly compact, does there exist a finite covering of the manifold such that the corresponding foliation is without holonomy?

Chapter 2

Hopf's Method

2.0.1 Absolute Continuity

Here we investigate an important property of the invariant foliations: its absolute continuity. What we are seeking is the following Fubini type claim: if $\mathcal{F}$ is a foliation on M and $S \subset M$ is measurable, then $Leb(S) = 0$ if and only if $S \cap L$ has zero Lebesgue measure inside L for almost every leaf L (again here we are referring to the Lebesgue measure). To make prove this claim we first recall some definitions.

2.0.2 Lebesgue measure in a manifold

Here we recall the definition of the Lebesgue measure class on a manifold M. A measure ν on M is a smooth measure if there exist an atlas $\{\phi_i, U_i\}$ of M such that for every i there exist a smooth bounded-away from zero positive function $f_i \in \mathcal{C}^\infty(\phi_i(U_i), \mathbb{R}^n)$ such that

$$(\phi_i)_*\nu = f_i dx$$

where dx stands for the Lebesgue measure on $\mathbb{R}^n$.

Note that if $\widetilde{\phi_i} : \widetilde{U_i} \to \mathbb{R}^n$ is any other chart, by the change of variables formula

$$(\widetilde{\phi}_i)_*\nu = f_i \circ \phi_{\widetilde{ii}} |det(d\phi_{\widetilde{ii}})| dx$$

where $\phi_{\widetilde{ii}} = \phi_i \circ \widetilde{\phi_i^{-1}}$, and thus the definition of smooth measure is independent of the atlas used.

Using partitions of the unity is easy to see that all smooth measures are equivalent, hence we define

Definition 2.1. The *Lebesgue class* on a manifold is the equivalence class of all smooth measures in the manifold.

We remark that every manifold can be equipped with a smooth measure (i.e. the Lebesgue class is non-empty): see excercise 5.

It makes sense then to speak of null sets with respect to smooth volumes: we call these *Lebesgue null sets*. We define.

Definition 2.2. A measurable bijection $h : M \to M$ is *absolutely continuous* if it induces a bijection in the collection of Lebesgue null sets of M.

If h is abs. cont. and ν is a smooth volume on M then $h_*\nu \ll \nu$ and hence by Lebesgue-Radon-Nikodym theorem there exist some density J such that

$$d(h_*\nu) = J d\nu \tag{2.1}$$

Further properties of J are of course desirable.

Example: Assume that M is orientable and let ω a non-vanishing n-form (a volume form). If h is a $\mathcal{C}^r$-diffeo we have

$$J(x) = |det_\omega d_x h|$$

and in particular, $J(x)$ is continuous and positive everywhere.

We will be applying these notions to holonomy maps corresponding to invariant foliations. In particular we will deal with maps $h : D \to D'$ defined on small completely transverse disks D, D' to $\mathcal{F}$ (the Lebesgue measure induced by the Riemannian metric of M).

We record here the following lemma which will be used in the next section. See [Mañé, 1983] for the proof.

Lemma 2.3. *Let* $(h_n)_{n\geq 0} \subset Emb^1(D, D')$ *and denote* J_n *the Jacobian of* h_n*. Assume that there exist a topological embedding* $h : D \to D'$ *and a positive continuous function* $J : D \to \mathbb{R}_+$ *such that*

$$h_n \rightrightarrows h$$
$$J_n \rightrightarrows J$$

Then h *is absolutely continuous and* $dh_* m = J dm$*.*

Note the similitude of this lemma with the well known fact that $\mathcal{C}^r(D, D')$ is $\mathcal{C}^r$ closed.

2.0.3 Absolute continuity of the unstable foliation

Definition 2.4. A foliation $\mathcal{F}$ is *absolutely continuous* if every holonomy transport map $h_{x,y} : D_x \to D_y$ is absolutely continuous.

Since the pseudo-group of holonomy transports is countably generated, it suffices to prove that holonomies transports are absolutely continuous for small discs.

Here is the main theorem of this section:

Theorem 2.5. *If* f *is PH then* $\mathcal{W}^u$ *is absolutely continuous.*

Would E^u be a C^1 bundle the result will be immediate, but this is seldom the case (see [Anosov, 1967]). The best that we can hope for is that E^u is θ-Hölder , as explained in [Pugh et al., 1997] for example. The natural idea here is to try to approximate $\mathcal{W}^u$ by smooth local foliations $\mathcal{F}_n$ prove that their holonomy transports $h_n : D \to D'$ converge uniformly to h, and their Jacobians converge uniformly to some map $J : D \to \mathbb{R}$. Then using Lemma 2.3 we could conclude that h is absolutely continuous.

However this is impossible to do in practice. The only structure that for sure we could approximate is the bundle E^u. Note that if we do approximate E^u by $\mathcal{C}^r$-bundles E_n, these bundles will not be necessarily integrable. To circumvent this difficulty we will work with some version of a local foliation: a plaquation. The proof that we present here using plaquations is the one of [Pugh and Shub, 1972]. There are other ways to prove the absolute continuity of $\mathcal{W}^u$ but the use of plaquations is a typical technique of partially hyperbolic dynamics.

Proof.

We fix small cs-dimensional discs $D_p \subset D, D'_{p'} \subset D'$ transverse to $\mathcal{W}^u$, and consider the holonomy transport map $h : D_p \to D'_{p'}$. We assume that $D_p, D'_{p'}$ are small enough so that h can be extended to an embedding defined on bigger discs. As explained before, absolute continuity is a local fact so is no loss of generality to assume that D, D' are contained in a common foliation chart.

Take any $\mathcal{C}^r$ approximation E to E^u and consider

$$\mathcal{P} = \{P_x = exp_x(E_x(r))\}_{x \in M}$$

where $r > 0$ will be specified later. In principle we only require r to be less than the injectivity radius of exp. We note that $\mathcal{P}$ is a collection of u-dimensional discs centered at each point $x \in M$ and such that they vary continuously with respect to its center: this is precisely a plaquation. We omit the formal definition (the reader should be able to provide by him/herself).

The point is that we can also define holonomy transports using $\mathcal{P}$ (provided that $D_p, D'_{p'}$ are close enough). In principle the holonomy maps are much worse behaved that in the case of a foliation. However we have the following:

Lemma 2.6. *Let $0 < \alpha < \pi/2$ be given. Then we can choose r so that if A, B are two transverse discs to $\mathcal{P}$ so that*

1. $max\{ang(TA, E^{cs}), ang(TB, E^{cs})\} < \alpha$
2. $x \in A \Rightarrow d(x, B \cap P_x) < r$

then the holonomy transport $g_{x,x'} : A_x \subset A \to B_{x'} \subset B$ is $\mathcal{C}^r$-immersion.

(Recall that we are using a metric so that E^u is perpendicular to E^{cs}).

Proof. The fact that $g_{x,x'}$ is $\mathcal{C}^r$ is immediate. Consider then y close to p and denote $y' = g(y)$. We want to show that $d_y g_{x,x'} : T_y A_x \to T_{y'} B_{x'}$ is bijective. Note that for y close to x we have $g_{x,x'} = g_{y,y'}$, so with no loss of generality we can assume $y = x$.

If $y = x = x'$ then the claim is clear, i.e. we have injectivity in the diagonal in a neighbourhood of (x, x'). The map g and its derivative depends continuously on x, x' and on the discs $A_x, B_{x'}$; since $\{V_p \subset T_p M : ang(V_p, E^{cs}) \leq \alpha\}, M$ are compact, bijectivity on the diagonal extends to a r-neighbourhood of the diagonal. □

We will use the lemma with $\alpha = ang(E^{cs}, E^{\perp})$. Now we take advantage of the fact that f preserves $\mathcal{W}^u$.

Remark 2.7. Since f preserves the foliation $\mathcal{W}^u$, so it is enough to prove absolute continuity for a holonomy transport $H : D_{f^{-n}p} \to D'_{f^{-n}}$ with $n \geq 0$. Of course as we pre-iterate the pre-image of the discs D, D' become larger, but again using that abs. cont. is local, we see that it suffices to show absolute continuity for discs contained in the same foliation chart. As we pre-iterate, the tangent bundles of $D_p, D_{p'}$ converge uniformly to the bundle E^{cs} (this is a consequence of the λ-lemma; alternatively see [Hirsch et al., 1977]), so the pre-iterates of the discs are still in hypothesis of the previous lemma.

Define now the following family of plaquations: $\mathcal{P}^n_x = f^n \mathcal{P}_{f^{-n}x}$.

Fact: There exist $\epsilon > 0$ such that if $\mathcal{P}^n(\epsilon)$ denotes the plaquation $\mathcal{P}^n$ restricted to the discs of size ϵ then

$$\mathcal{P}^n(\epsilon) \rightrightarrows W^u_\epsilon(x)$$

See [Hirsch et al., 1977] for the proof.

We will assume that $d(x, x' < \epsilon/2$. Finally let $g_n : D_x \to R_{x'_n} \subset D_{x'}$ be the holonomy transport of $\mathcal{P}^n$. Here we mean $x'_n = g_n(x), R_{x'_n} = g_n(D_x)$. Then

$$g_n \rightrightarrows h$$

Lemma 2.8. *Each g_n is an embedding.*

Proof. By 2.6 and the previous remarks, each g_n is a $\mathcal{C}^r$-immersion. With no loss of generality we can assume that all g_n and h are defined in a slightly bigger disc $\widehat{D}$ and that $\widehat{g_n} \rightrightarrows \widehat{h}$ (a hat denotes the extensions). Also the extension $\widehat{h}$ of h can be taken as being an homeomorphism.

Note that all maps are local homeomorphisms, so we can compute their degree using their local degrees. To this end, take $B \subset D_{x'}$ a compact neighborhood of hD_x inside $\widehat{h}\widehat{D}$. For each $y \in B$ we have that $deg_y(\widehat{h}) = 1$, and hence by uniform converge $deg_y(\widehat{g}_n) = 1$ if n is sufficiently large. This proves injectivity of g_n. □

Denote for $y \in D_p$ $y'_n = g_n(y)$ and let $p_n = f^{-n}p, q_n = f^{-n}(x'_n), D_n = f^{-n}D_p, D'_n = f^{-n}D_{p'}$. Then we can write $g_n = f^n \circ g_{p_n,q_n} f^{-n}$ where g is the holonomy transport corresponding to the plaquation $\mathcal{P}$. In particular $q_n \in \mathcal{P}_{p_n}$ and furthermore since f^{-n} contracts the u-distance and $\mathcal{P}$ is close to $\mathcal{W}^u$, there exist a sequence of positive numbers ϵ_n converging to zero so that $q_n \in \mathcal{P}_{p_n}(\epsilon_n)$.

For any y in D_p we have

$$\begin{aligned} Jac_y(g_n) &= |\det(df^n|T_{f^{-n}y'_n}D'_n)\det(dg_{p_n,q_n}|T_{f^{-n}y'_n}D_n)\det(df^{-n}|T_yD_p)| \\ &= \det(dg_{p_n,q_n}|T_{f^{-n}y'_n}D_n)|\frac{\det(df^{-n}|T_yD_p)}{\det(df^{-n}|T_{y'_n}D'_{p'})}| \end{aligned}$$

The first term converges uniformly to 1 (since $D_n, D'_n \rightrightarrows E^cs, d(p_n,q_n) \to 0$)). We then have

Lemma 2.9. *Define $L_n : D_p \to \mathbb{R}$ by*

$$L_n(y) = |\frac{\det(df^{-n}|T_yD_p)}{\det(df^{-n}|T_{y'_n}D'_{p'})}|$$

$$J(y) = \lim_{n\to\infty} |\frac{\det(df^{-n}|T_yD_p)}{\det(df^{-n}|T_{y'}D'_{p'})}|$$

Then $J(y$ is well defined, the convergence is uniform and $L_n \rightrightarrows J$.

After proving this lemma we can use lemma 2.3 to conclude the proof of the theorem.

Proof.

Particular case: $y = p, T_pD_p = E^{cs}_p, T_{p'}D_{p'} = E^{cs}_{p'}$.

Denote $d^{cs}f = df|E^{cs}$. We are trying to prove that

$$\exists \lim_n \frac{\det(d^{cs}_p f^{-n})}{\det(d^{cs}_{p'} f^{-n})} = \lim_n \prod_{k=0}^{n} \frac{\det(d^{cs}_{f^{-k}p} f^{-1})}{\det(d^{cs}_{f^{-k}p'} f^{-1})}$$

or equivalently, taking *log* that the series

$$\sum_{k=0} |\log\det(d^{cs}_{f^{-k}p} f^{-n}) - \log\det(d^{cs}_{f^{-k}p'} f^{-n})| \tag{2.2}$$

is finite.

As stated before, it is known that E^{cs} is Hölder , and since the function $x \to log(\det(d_x f^{-1})$ is C^1 in the manifold we conclude that its cs-restriction $x \to log(\det(d_x^{cs} f^{-1})$ is θ-Hölder . Then, the series of (2.2) is dominated by

$$\sum_{k=0}^{\infty} Cd(f^{-k}p, f^{-k}p')^\theta \leq C \sum_{k=0}^{\infty} (|d^u f^{-1}|)^\theta)^k < \infty$$

and thus converges as well (note that $|d^u f^{-1}|^\theta < 1$).

General Case: Consider $\pi : TM = E^u \oplus E^{cs} \to E^{cs}$ the projection, and note that π commutes with df. Then for any point $y \in D_p$ we can write

$$df^{-n}|T_y D_p = (\pi|T_{f^{-n}y} D_n)^{-1} \circ d^{cs} f^{-n}|E^{cs} \circ \pi|T_y D_p$$

and hence

$$L_n(y) = \frac{\det(\pi|T_{f^{-n}y'} D'_n)}{\det(\pi|T_{f^{-n}y} D_n)} \frac{\det(d_y^{cs} f^{-n}|E^{cs})}{\det(d_{y'_n}^{cs} f^{-n}|E^{cs})} \frac{\det(\pi|T_y D_p)}{\det(\pi|T_{y'_n} D'_{p'})}$$

Again since TD_n, TD'_n converge uniformly to E^{cs}, the first quotient converges uniformly to 1 and furthermore using that $g_n \rightrightarrows h$ and $D'_{p'}$ is $\mathcal{C}^1$ we get that

$$\pi|T_{y'_n} D'_{p'} \rightrightarrows \pi|T_{y'} D'_{p'}$$

Hence, we just need to prove

$$L'_n(y) := \frac{\det(d_y^{cs} f^{-n}|E^{cs})}{\det(d_{y'_n}^{cs} f^{-n}|E^{cs})} \rightrightarrows J'(y) := \lim_{n\to\infty} \frac{d_y^{cs} f^{-n}|E^{cs}}{d_{y'}^{cs} f^{-n}|E^{cs}}$$

By the particular case $J'(y)$ is well defined and the convergence is uniform.

Note

$$L'_n(y) = \frac{d_y^{cs} f^{-n}|E^{cs}}{d_{y'}^{cs} f^{-n}|E^{cs}} \frac{\det(d_{y'}^{cs} f^{-n}|E^{cs})}{\det(d_{y'_n}^{cs} f^{-n}|E^{cs})}$$

The first term converges uniformly to $J'(y)$ so we need to prove that the second term converges uniformly to 1. Equivalently we can prove that the series whose general term is

$$\sum_{k=0}^{n-1} |\log(\det(d^{cs}_{f^{-k}y'_n} f^{-1}|E^{cs})) - \log(\det(d^{cs}_{f^{-k}y'} f^{-1}|E^{cs}))| \leq C \sum_{k=0}^{n-1} d(f^{-k}y'_n, f^{-k}y')^{k\theta} \quad (2.3)$$

is uniformly convergent in y to zero. Observe that the points y'_n, y' are not in the same unstable manifold in general, so we need to be a little more careful in the estimates than in the particular case (and furthermore we need to find the limit of the series)

What it is true however is that $f^{-n}y' \in W^u_\epsilon(f-ny)$ so $\epsilon_m \leq m(df^u)^{-n}$, and since $\mathcal{P}$ is almost tangent to E^u we can also assume that $f^{-n}y'_n \in \mathcal{P}_{f-ny}(\epsilon_n)$ if n is sufficiently big. Hence if n is sufficiently big we have $d(f^{-n}y, f^{-n}y'_n) \leq m(df^u)^{-n}$, and thus

$$d(f^{-k}y'_n, f^{-k}y') = d(f^{n-k}(f^{-n}y'_n), f^{n-k}(f^{-n}y'))$$

As we pre-iterate the discs D'_k become tangent to E^{cs} so we get for some constant independent of k, n

$$d(f^{-k}y'_n, f^{-k}y') = d(f^{n-k}(f^{-n}y'_n), f^{n-k}(f^{-n}y')) \leq C'm(df^u)^{-n}|d^{cs}f|^{n-k}$$

Putting everything together we estimate

$$\sum_{k=0}^{n-1} d(f^{-k}y'_n, f^{-k}y')^{k\theta} \leq C' \sum_{k=0}^{n-1} (m(df^u)^{-n}|d^{cs}f|^{n-k})^\theta = \frac{1-|d^{cs}f|^{n\theta}}{1-|d^{cs}f|^\theta} m(df^u)^{-n}$$

which converges to zero as n goes to infinity, independently of y.

□

□

Now suppose that f is a PH diffeomorphism which preserves a smooth volume ν, and consider a foliated cube $U = \{P(x) = W^u(x) \cap U\}_{x \in U}$. Since the plaques of U have $\mathcal{C}^1$ tubular neighbourhoods we can assume with no loss of generality that $U = \{D(x)\}_{x \in U}$, where $D(x)$ is a disc of dimension $c+s$ transverse to $\mathcal{W}^u$ and such that this partition is $\mathcal{C}^1$. Denote by H the partition whose atoms are the plaques $P(x)$ and T the partition whose atoms are the discs $D(x)$.

Now consider the disintegrations of ν with respect to this partitions and denote ν_H and ν_T the corresponding quotient measures. Note that we can identify U/H by any of the discs $D(x)$. We fix one of them, $D = D(x_o)$ and we will think ν_H as a measure on D. We then have

Proposition 2.10. *If we denote by m the Lebesgue measure we have.*

1. *ν_x^H is equivalent to $m_{P(x)}$.*
2. *ν_H is equivalent to m_D.*

Proof. This is Theorem 7.8[1] in [Pesin, 2004]. Here is the proof.

It is easier to work with local coordinates: take a diffeomorphism ($\mathcal{C}^1$!) $x : U \to (-1,1)^u \times (-1,1)^{c+s}$ that sends the partition T to the vertical partition of discs $\{a \times (-1,1)^{c+s}\}$. Note that under this diffeomorphism each horizontal plaque is mapped to a completely transverse submanifold to the vertical partitions. We will continue using the same notation for both of this partitions in the x coordinates. Also, it is no loss of generality to assume that $D = D(0)$. Let $m = u+c+s$ the dimension of the manifold M.

If $A \subset U$ has positive measure, then for some positive differentiable function $\rho : U \to \mathbb{R}$ we have

$$\mu(A) = \int \rho(x)\chi_A(x)dx$$

Also

$$\mu(A) = \int_{(-1,1)^u} dx^1 \cdots dx^u \int_D (x)\rho(x)\chi_A(x)dx^{u+1} \cdots dx^m \tag{2.4}$$

Denote the points of U by $x = (y,z)$, and consider the holonomy $h_x : D \to D(y)$. By absolute continuity of the horizontal foliation, there exist a positive measurable function $J(z|y)$ such that the previous equation can be written as

[1]Pablo wants to thank Y. Pesin for clarifying the proof for him.

$$\mu(A) = \int_{(-1,1)^u} dx^1 \cdots dx^u \int_D \rho(h_x(z))\chi_A(h_x(z))J(z|y)dx^{u+1} \cdots dx^m \quad (2.5)$$

Now notice that if we fix $y \in (-1,1)^u$, the trace of $h_x(z)$ is precisely $V(x)$, hence we can write after using Fubini to interchange the order of integration

$$\mu(A) = \int_D dx^{u+1} \cdots dx^m \int_{V(x)} \rho(h_x(z))\chi_A(h_x(z))J(z|y)dx^1 \cdots dx^u \quad (2.6)$$

This proves both of the claims. □

Corollary 2.11. *Using the previous notation, if $X \subset M$ has full measure then for almost every $x \in M$ we have: for ν_x^H almost every $y \in P(x)$, $y \in X$.*

A similar statement, of course, holds for the stable foliation.

2.0.4 Hopf's Method - Anosov Case

To illustrate this method we will assume for now that $E^c = 0$. We fist establish a general criterion for ergodicity.

Proposition 2.12 (Birkhoff's converse)**.** *Let M be a compact metric space and $T : M \to M$ a measurable map preserving a measure ν. Suppose that for every continuous function $\phi : M \to \mathbb{R}$ we have*

$$\phi^+(x) = \lim_{n\to\infty} A_n(\phi)(x) = const.(= \int \phi d\nu)$$

Then T is ergodic.

Proof. Excercise. □

It is a consequence of Birkhoff Theorem that if T is invertible then for almost every $x \in M$ we have $\phi^+(x) = \phi^-(x)$ where

$$\phi^-(x) = \lim_{n\to-\infty} -A_n(\phi)(x)$$

We will denote this common value by $\phi^*(x)$.

Now we come back to our setting: $f : M \to M$ Anosov preserving a smooth volume ν.

Lemma 2.13. *Suppose that ϕ is continuous and x is such that $\phi^+(x)$ is defined. Then for every $y \in W^s(x)$ we have that $\phi^+(y)$ is defined and equal to $\phi^+(x)$. Similarly if $\phi^-(x)$ is defined and $y \in W^u(x)$ then $\phi^-(y)$ is defined and equal to $\phi^-(x)$.*

Proof. Excercise. □

Theorem 2.14. *[Anosov] f is ergodic.*

Proof.

1. Let $X \subset M$ the set where $\phi^+(x) = \phi^-(x) = \phi^*(x)$: X has full measure. Consider a foliated cube U corresponding to the unstable foliation and use the notation that we used for Theorem 2.11. Then for almost every $x \in M$ we have that ν_x^H-almost every $y \in W^u(x) \cap U$ is in X. Then $A(x) = \cup_{y \in W^u(x) \cap U, y \in X^c} W^s(x) \cap U$ has zero measure. Hence, and using the similar statement for the stable foliation we conclude that for almost every $x \in M$ its stable plaque $W^s_{loc}(x)$ intersects $A(x)$ for almost every point $y \in W^s_{loc}(x)$. I.e., almost every point x has a neighbourhood N such that for almost every $y \in N$ the point $z = W^u(x) \cap W^s(y) \cap N \in X$. Hence $\phi^* x = \phi^*(z)$; in other words ϕ^* is locally constant.

2. Now notice that given two points in the manifold we can join them by a u-s path (meaning a piecewise C^1 path consisting of pieces tangent to either E^u or E^s). In other words, we can "join" neighbourhoods where ϕ^*, and thus we conclude that ϕ^* is constant. Proposition 2.12 finishes the proof.

□

Remark 2.15. Note that the proof of the previous Theorem consist of two pieces: "local ergodicity" and "accesibility". We will say more about this later when we study Hopf method for general PH.

2.0.5 Hopf's Method - Partially Hyperbolic Case

Now we want to apply a similar procedure por general PH diffeos. Of course the presence of the center foliation complicates considerably the problem, since we cannot guarantee that Birkhoff's averages of continuous functions are constant for points in the same center leaf (provided that this leaf even exist!). In other words, we have lost the accessibility referred at the end of last section. Before going into that it will be convenient to rephrase some of the results of the previous part in a somewhat different language. We follow here the approach of [Burns et al., 1999].

Hypothesis: Throughout this part f is PH and preserves a smooth measure μ.

We will be working with the Borel σ-algebra $\mathcal{B}_M$ of M. If $\mathcal{A} \subset \mathcal{B}_M$ is a sub σ-algebra we will denote by $\widehat{\mathcal{A}}$ its "saturation" with respect to the sets of $\mathcal{B}_M$ of measure zero, namely

$$\widehat{\mathcal{A}} = \{B \in \mathcal{B}_{\mathcal{M}} : \exists A \in \mathcal{A} : \mu(B \triangle A) = 0\}$$

If you consider the proof of Lemma 2.13, you will see that in fact you do not need the points to belong to the same strong stable manifold for the conclusion to be true, but only that these points asymptotically converge to each other: in other words that they are in the same weak stable manifold

$$W^{ws}(x) = \{y \in M : \limsup_{n\to\infty} d(f^n x, f^n y) = 0\}$$

Similarly we can define the weak unstable manifold. Note that $\mathcal{F}^{ws} = \{W^{ws}(x)\}_x$ and $\mathcal{F}^{wu} = \{W^{us}(x)\}_x$ are partitions of M. In the case when f is Anosov we have $W^{ws}(x) = W^s(x)$ and $W^{us}(x) = W^u(x)$, but if f is PH we can only assert the inclusions

$$W^s(x) \subset W^{ws}(x)$$
$$W^u(x) \subset W^{wu}(x).$$

We denote by $\mathcal{S}$,$\mathcal{U}$ the σ-algebras consisting of sets saturated by weak stable and weak unstable leaves, and by $\mathcal{SS}, \mathcal{SU}$ the corresponding σ-algebras consisting of sets saturated by strong stable and strong unstable leaves. Note that $\mathcal{S} \subset \mathcal{SS}$ and $\mathcal{U} \subset \mathcal{SU}$, but in general the opposite inclusions do not hold.

Proposition 2.16 (Hopf's Method - Revised version)**.**

If $\widehat{\mathcal{S}} \cap \widehat{\mathcal{U}} = \nu$ *(being* ν *the trivial* σ*-algebra) then* f *is ergodic.*

Proof. If f were nor ergodic, there would be a continuous function ϕ and a real number c such that

$$0 < \mu\{\phi^+ < c\} < 1$$

Consider the sets

$$A = \{x : \exists \phi^+(x) \text{ and } \phi^+(x) < c\}$$

$$B = \{x : \exists \phi^-(x) \text{ and } \phi^-(x) < c\}$$

Then $A \in \mathcal{S}, B \in \mathcal{U}$ and $\mu(A \triangle B) = 0$ (Birkhoff), hence

$$A \cap B \in \widehat{\mathcal{S}} \cap \widehat{\mathcal{U}} = \nu$$

But since $0 < \mu(A) < 1$, we conclude $A \cap B \notin \nu$, a contradiction. □

Now we go back to the problem of accessibility.

Definition 2.17. Fix a point $x \in M$. We say that $y \in M$ is *accessible from* x if there exist a piecewise C^1 curve $\alpha : x \mapsto y$ everywhere tangent to E^s or E^u. The set of points y accessible from x is called the accessibility class of x and denoted $ACC(x)$.

We then say that f is

1. *accessible* if there is only one accessible class.
2. *essentially accessible* if given a borel set A consisting of accessibility classes, then A has either full or null measure.

Example 1: The map $f = A_T \times R_\alpha : T^3 \to T^3$ is not essentially accessible. Note that given $p = (x_p, y_p, z_p), q = (x_q, y_q, z_q) \in T^3$ then

$$d(f^n p, f^n_q) \xrightarrow[n\to\infty]{} 0 \iff z_p = z_q \text{ and } (x_q, y_q) \in W^s_{a_T}((x_p, y_p))$$

By the dynamical characterization of the stable foliation we conclude that the stable manifold of p is the set $W^s_{a_T}((x_p, y_p)) \times \{z_p\}$. Similarly, the unstable manifold of p is $W^u_{a_T}((x_p, y_p)) \times \{z_p\}$. From here follows that f is not essentially accessible.

Example 2: The time one map f of the geodesic flow on a surface of negative curvature is accessible (See [Grayson et al., 1994]).

Denote by X, X^u, X^s the unit vector fields generating the flows g, h^u, h^s. We first note the following.

Lemma 2.18. $[X^s, X^u] = X$.

Proof. Let $Z = [X^s, X^u]$. Then we can write $Z = aX^s + bX + cX^u$ for some differentiable functions a, b, c. It follows that

$$(g_t)_*Z = [(g_t)_*X^s, (g_t)_*X^u] = [e^{-t}X^s, e^tX^u] = Z = ae^{-t}X^s + bX + ce^tX^u$$

hence $a = c = 0$. □

Now since $span\{X^s, X, X^u\} = TM$, for each $p \in M$ the map $\Phi : (-1,1)^3 \to M$ given by

$$\Phi(t, v, w) = h^s_v h^u_w g_t(p)$$

is a diffeomorphism onto an open neighbourhood of p containing a ball $B_p(r_p)$ of radius r_p. By compactness there exist $r > 0$ such that $r_p \geq r$. We know that the geodesic flow is transitive[2]: let γ be a dense orbit.

Lemma 2.19. *There exist $L > 0$ such that any two $B_p(r), B_q(r)$ can be joined by an arc of γ of time length $\leq L$.*

The proof is an exercise.

Using Lemma 2.18 we see that there exist a constant $c > 0$ such that if $q = g_t(p)$ for some $|t| \leq c$ we have $q = h^u_{-w} h^s_{-} v h^u_w h^s_v(p)$ for some $|v|, |w| \leq 1$. From here it is obvious.

Example 3: The F.Rodriguez-Hertz map is essentially accessible.

This follows since $E = E^s \oplus E^u$ is an irrational plane, and hence dense. Note that since E is integrable, F_{RH} cannot be accessible.

[2] Transitivity of the Geodesic Flow can be established directly without reference to its ergodicity.

We now go back to the general setting. Note that essential accessibility is equivalent to

$$\widehat{\mathcal{SS} \cap \mathcal{SU}} = \nu.$$

On the other hand we always have

$$\widehat{\mathcal{S}} \cap \widehat{\mathcal{U}} \subset \widehat{\mathcal{SS}} \cap \widehat{\mathcal{SU}}$$
$$\widehat{\mathcal{SS} \cap \mathcal{SU}} \subset \widehat{\mathcal{SS}} \cap \widehat{\mathcal{SU}}.$$

Hence to prove ergodicity for an essentially accessible PH map it suffices to show, by 2.16, the inclusion

$$\widehat{\mathcal{SS}} \cap \widehat{\mathcal{SU}} \subset \widehat{\mathcal{SS} \cap \mathcal{SU}}.$$

The following important Theorem due to C. Pugh and M. Shub was one the foundational results in the ergodic theorem of

Theorem 2.20 (C.Pugh-M.Shub)**.** *Let f be a $\mathcal{C}^2$ PH diffeo preserving a smooth volume and satisfying:*

1. *f is dynamically coherent.*
2. *f is center bunched , namely* $\sup_x \frac{\|d_x^c f\|}{m(d_x^c f)} \approx 1$.
3. *f is essentially accessible.*

Then $\widehat{\mathcal{SS}} \cap \widehat{\mathcal{SU}} \subset \widehat{\mathcal{SS} \cap \mathcal{SU}}$, and thus f is ergodic.

This is a very important Theorem in the ergodic theory of PH maps, not only for the result per se but for the tools developed to prove it. You can find the original proof in [Pugh and Shub, 1999]. This Theorem has been improved by K. Burns and A. Wilkinson ([Burns and Wilkinson, 2010]) to drop the condition of dynamical coherence. See also Appendix C.

Corollary 2.21. *The following maps are ergodic.*

1. *The time-one map of the geodesic flow corresponding a surface of constant negative curvature.*
2. *The F.R.Hertz map.*

Exercises

1. Prove that E^s, E^u are *uniquely integrable*, i.e. if $\alpha : [0,1] \to M$ is a $\mathcal{C}^1$ curve such that $\alpha'(t) \in E^s_{\alpha(t)}$ $\forall t$ then $\alpha([0,1]) \subset W^s(\alpha(0))$.

2. Probe that irrational rotations are uniquely ergodic. Hint: First show that it suffices to prove weak converge of the measures $\sum_{i=0}^{n-1} \delta_x$ for a dense set of $C(S^1)$.

3. Prove that for an automorphism of the Torus the periodic points are dense.

4. Prove $1 \iff 2 \iff 3 \iff 4$ in 1.12.

5. Given a paracompact smooth manifold M prove that its Lebesgue class is non-empty as follows:

   (a) Suppose that you are given an atlas $\{\phi_i, U_i\}$ and a collection of smooth bounded-away from zero positive function $f_i \in \mathcal{C}^\infty(\phi_i(U_i), \mathbb{R}^n)$ satisfying for all i, j such that $U_i \cap U_j \neq \emptyset$

   $$f_j = f_j \circ \phi_{ji} |det(d\phi_{ji})| \quad \phi_{ji} = \phi_j \circ \phi_i^{-1}$$

   Use Riesz-Markov theorem to show that there exist a smooth measure ν on M such that

   $$(\phi_i)_* \nu = f_i dx$$

(b) Equip M with a Riemannian metric g and show that the local coordinates of the metric (g_{ij}) with respect to some atlas satisfy the compatibility condition of the previous part.

6. Prove Lemma 1.24.
7. Prove Lemma 2.19.
8. Let G be a locally compact group and $\mathcal{H}$ be a Hilbert space. We say that a representation $T : G \to \mathcal{B}(\mathcal{H})$ is a unitary representation if

   (a) Each T_g is unitary.

   (b) T is continuous when $\mathcal{B}(\mathcal{H})$ is equipped with the SOT-topology.

   Prove that the Koopman representation for a measure preserving action is a unitary representation.

9. Let M be a compact metric space and $F \subset M$ a closed subset. Suppose that $\phi : M \to N$ is a local homeomorphism such that is 1 to 1 in F. Prove that there exist an open neighbourhood of U of F in M such that ϕ is 1 to 1 on U.
10. Prove Proposition 2.12.
11. Prove Lemma 2.13. Hint: ϕ is uniformly continuous.
12. Prove that for all dynamical coherent examples given in the text, the center foliation is complete.

Appendix A: Foliations.

Let M be a m dimensional smooth manifold. A foliation on M of codimension q is a decomposition of M of the form $\mathcal{F} = \{L\}$ where

1. Each L is an immersed connected submanifold of M of dimension $p = m - q$: these submanifolds are called the leaves of the foliation.

2. $M = \sqcup_{L \in \mathcal{F}} L$.

3. The bundle $T\mathcal{F} = \sqcup_L TL$ is a continuous subbundle of TM.

Sometimes in the literature these are called $\mathcal{C}^{r,0}$ foliations (being r the degree of differentiability of the leaves). More in general, the foliation is of class $\mathcal{C}^{r,0}$ if $T\mathcal{F}$ is subbundle of TM of differentiability class $\mathcal{C}^s$ (Warning: this is seldom the case in our context that $T\mathcal{F}$ is smooth).

There is another equivalent and useful way to define foliations. Given an atlas $\{\phi, U_i\}$ of M we say that is a foliation atlas if

1. For each i, $\phi(U_i) = [-1,1]^p \times [-1,1]^q$.

2. The change of variables maps $\phi_{ij} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ are of the form
$$\phi_{ij}(s,t) = (\alpha(s,t), \beta(t))$$
where α is smooth and β is continuous.

The differential structure induced by a foliation atlas is also called a foliation.

The sets of the form $\phi([-1,1]^p, t)$ are called plaques. We define an equivalence relation on M saying that two points x, y are equivalent if they there exist a chain of plaques $P_0, \ldots, P_n$ (i.e. $P_i \cap P_{i+1} \neq \emptyset$) such that $x \in P_0, y \in P_n$. We then see that the equivalence classes of this relation give a foliation of

M in the first sense. Conversely, given a foliation is not hard to construct a foliation atlas. Note that for a foliation atlas of a foliation $\mathcal{C}^{r,s}$, the number r is precisely the degree of differentiability of the maps α, and the number s is the degree of differentiability of the maps β.

Suppose that $\mathcal{F}$ is a foliation of M and let x, y be in the same plaque. Consider a chart $\phi : U \to [-1, 1]^p \times [-1, 1]^q$ of the foliation atlas such that $x, y \in U$ and take two completely transverse discs D_x, D_y centered at x, y respectively. We can then define a local homeomorphism $h_{y,x} : D_x \to D_y$ such that $z, h_{y,x}(z)$ are in the same plaque. More in general, if two points belong to the same leaf then there exist a chain of plaques joining them, and thus we can define for each such chains, local homeomorphisms by concatenating the $h_{y,x}$ constructed before. The maps thus constructed are called *holonomy transports* and the collection of all such maps is the *holonomy pseudo-group*. It can be proved that the holonomy pseudo-group is countably generated.

In particular one can consider holonomy transports obtained by fixing a point x and a completely transverse disc D. In this case one obtains a map from a small neighbourhood $U \subset D$ of x to D. One verifies that the germ of these type of maps $h : U \to D$ only depends of the homotopy class of the loop (with endpoints fixed) used to define it. In other words, for each element of $\pi_1(L_x, x)$ one obtains a germ of diffeomorphisms of D at x. The assignation

$$H : \pi_1(L_x, x) \to Germ_x(D)$$

is in fact a representation, and is called the *holonomy representation* of L_x at x. Choosing a different base-point $y \in L_x$ amounts to change the holonomy representation by an equivalent one. It is usual to be a little imprecise and just refer these representations as *the* holonomy representation (while in fact is a equivalence class of representations). The image[3] of the holonomy representations is called the *holonomy group* of L and its denoted by $G(L)$. One says that L has finite holonomy (no holonomy) if $G(L)$ is finite (trivial). Sometimes it is necessary to work with *saturated sets* of a foliated manifold, namely sets which consists of full leaves. In this case one can restrict the holonomy to the set: if $E \subset M$ is saturated we will denote this restriction to $G(L|E)$.

See [Candel and Conlon, 2000] for general background on foliation theory.

[3]Again, a conjugacy class of a group instead of a group.

Appendix B: Birkhoff's Ergodic Theorem.

Let (X, μ) be a probability space and $T : X \to X$ a measure preserving transformation.

Theorem 2.22. *If $f \in L^1(X)$ then*

$$\exists \lim_{n\to\infty} \frac{1}{n} \sum_{0}^{n-1} f(T^i x) =: \widetilde{f}(x)$$

a.e.(x) and in $L^1(X)$. The function $\widetilde{f}$ satisfies

1. *$\widetilde{f} \in L^1(X)$.*
2. *$\int \widetilde{f} d\mu = \int f d\mu$.*

The function $\widetilde{f}(x)$ is clearly T-invariant, and in fact, it is equal to the conditional expectation of f with respect to the invariant σ-algebra of T ($= \{A \text{ measurable } : T^{-1}A = A\}$).

Remark 2.23.

1. If furthermore T is ergodic we have $\widetilde{f} = \int f d\mu$.
2. If T is invertible then a.e.(x)

$$\widetilde{f}(x) = \lim_{n\to\infty} \frac{1}{|n|} \sum_{0}^{n-1} f(T^{-i}x) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{-n}^{n} f(T^i x)$$

Appendix C: The K-property.

Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be measure preserving and invertible. We say that T is K if there exist a sub-algebra $\mathcal{A} \subset \mathcal{B}$ such that

1. $T^{-1}\mathcal{A} \subset \mathcal{A}$.
2. $\sigma(\cup_{n\in\mathbb{Z}} T^n\mathcal{A})\mathcal{B}$.
3. $\cap_{n\geq 0} T^{-n}\mathcal{A} = \nu$.

Theorem 2.24 (Brin-Pesin)**.** *Assume that f is a volume preserving PH such that $\widehat{\mathcal{SS}} \cap \widehat{\mathcal{SS}} = \nu$. Then f is K.*

See [Brin and Pesin, 1974].

Corollary 2.25. *Let f be a $\mathcal{C}^2$ center bunched essentially accessible PH diffeo. Then f is K.*

This follows directly from Theorem 2.20.

When studying the ergodic properties, there is some sort of hierarchy among the different properties being the most significant ones the following:

f is Bernoulli $\Rightarrow$ f is K $\Rightarrow$ f is mixing (of all orders) $\Rightarrow$ f is weak-mixing $\Rightarrow$ f is ergodic .

It is an open question to show whether the conditions of Theorem 2.20 imply the stronger level in the hierarchy, namely that f is Bernoulli. This is known for Anosov systems by the work of Sinai and Bowen, and for ergodic automorphisms of the torus (Katznelson's Theorem [Katznelson, 1971]).

Appendix D: Disintegration of measures.

A probability space $(X, \mathcal{B}, m)$ is a *Lebesgue space* if it is measure isomorphic to the interval $[0,1]$ with a Lebesgue-Stieljes measure. This is a very common class: for example if X is a compact metric space then it is a Lebesgue space with any probability defined on its Borel σ-algebra. We will only deal with Lebesgue spaces.

Given a partition $\eta = \{\eta_i\}$ of X by measurable sets, we say that the partition is *measurable* if $(X/\eta, \pi_*\mathcal{B}, m_\eta = \pi_* m)$ is a Lebesgue space, where $\pi : X \to X/\eta$ is the canonical map and $\pi_*\mathcal{B} = \{A \subset X/\eta : \pi^{-1}A \in \mathcal{B}\}$.

If η is measurable, there exist a full set $X' \subset X$ and family of probability measures $\{m_x^\eta\}_{x \in X'}$ on X satisfying:

1. Each m_x^η is supported in $\eta(x)$, the atom containing x.

2. For every $A \subset X$ measurable, the set $A \cap \eta_i$ is measurable for m_η-a.e. $\eta_i \in X/\eta$ and

$$m(A) = \int_{X/\eta} m_x^\eta(A \cap \eta_i) dm_\eta(i) = \int_X m_x^{\eta(x)}(A \cap \eta(x)) dm(x)$$

This is a celebrated theorem due to Rohklin. You can find the proof in [Rokhlin, 1962].

We record the following important consequences.

i) Uniqueness: If $\{\mu_x\}_{x \in X''}$ is another family family of measures defined for the full set X'' and satisfying 1 and 2, then $m_x^\eta = \mu_x$ m-a.e.(x).

ii) Let $f : X \to \mathbb{R}$ be an integrable function and define $f_{\eta_i}(x) = f(x)$ if $x \in \eta_i$. Then

$$\int_X f dm = \int_{X/\eta} dm_\eta(i) \int_{\eta_i} f_{\eta_i}(x) dm_x^\eta(x)$$

Moreover, if $\widehat{\eta}$ denotes the σ-algebra generated by η, then

$$E(f|\widehat{\eta})(x) = \int_{\eta(x)} f_{\eta(x)}(x) dm_x^{\eta(x)}$$

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