Birkhoff sums of i.e.t.'s: KZ cocycle (5th lecture)

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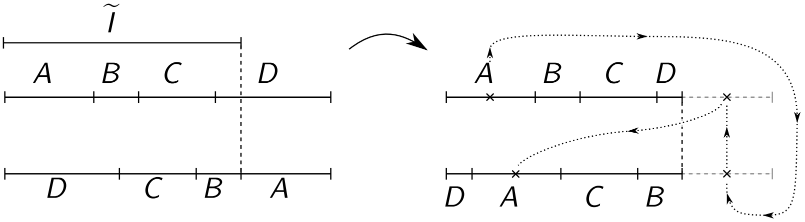
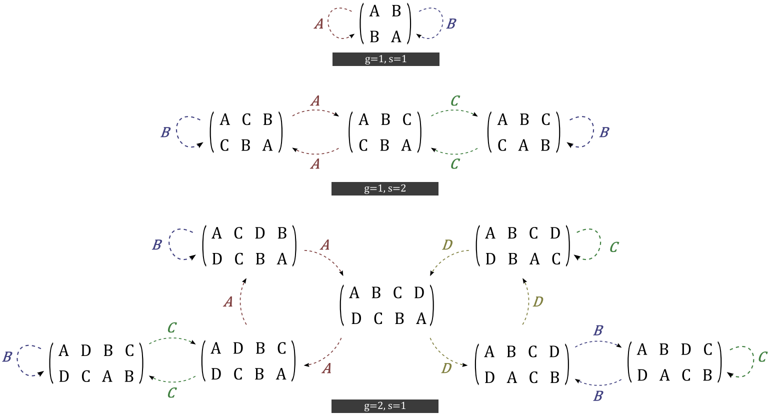
Basic step of Rauzy-Veech algorithm for i.e.t.'s

Yesterday we saw that the prototype of basic step of RV algorithm for i.e.t.'s is :



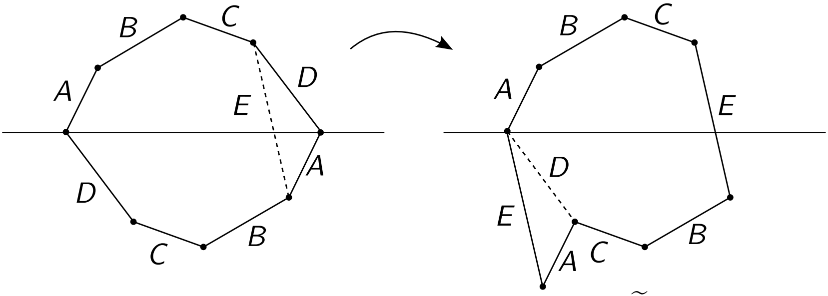
Rauzy diagrams

Also, we saw that, at the combinatorial level, RV algorithm consists of applying a top or bottom operation $\widetilde{\pi} = R_t(\pi)$ or $\widetilde{\pi} = R_b(\pi)$, and this led to Rauzy diagrams:



Basic step of RV algorithm for suspensions of i.e.t.'s I

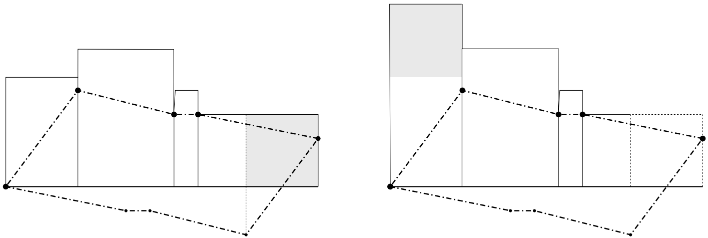
Finally, we saw that the RV algorithm for susp. $(\pi, \lambda, \tau) \mapsto (\widetilde{\pi}, \widetilde{\lambda}, \widetilde{\tau})$ geometrically is:



$M(\pi, \lambda, \tau)$ $\qquad$ $M(\widetilde{\pi}, \widetilde{\lambda}, \widetilde{\tau})$

Basic step of RV algorithm for suspensions of i.e.t.'s II

and



How do the length and transl. data evolve under RV alg.?

By studying Rauzy diagrams we got an idea on how comb. data evolve under top and bottom operations.

But, what about length and translation data?

Iteration of Rauzy-Veech algorithm I

Starting with $T = T_0$ i.e.t. w/o conn., we have a seq. T_n of i.e.t.'s whose comb. datum $\pi^{(n)}$ belong to a ∞-complete path γ_T.

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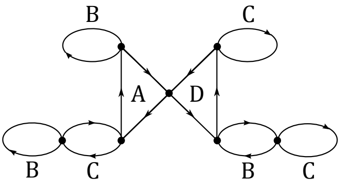
The length $\lambda^{(j)}$ and transl. $\delta^{(j)}$ (and eventual susp. data) for $T^{(j)}$ are related by products of matrices in $SL(\mathbb{Z}^{\mathcal{A}})$:

- firstly, for γ arrow w/ winner α and loser β, let $B_\gamma = \mathrm{Id} + E_{\beta\alpha}$ where $E_{\beta\alpha}$ is the elementary matrix $(E_{\beta\alpha})_{nm} = \delta_{n\beta}\delta_{m\alpha}$;
- secondly, for $\underline{\gamma} = \gamma_1 \dots \gamma_n$ finite path, let $B_{\underline{\gamma}} = B_{\gamma_n} \dots B_{\gamma_1}$;
- then, for $m \leq n$, if we let $\gamma(m,n)$ be the finite path connecting $\pi^{(m)}$ to $\pi^{(n)}$, then $\lambda^{(m)} = \lambda^{(n)} B_{\gamma(m,n)}$ and $\delta^{(n)} = B_{\gamma(m,n)} \delta^{(m)}$.

Iteration of Rauzy-Veech algorithm II

Exercise

Compute B_γ of the path $D \to B^2 \to D \to C \to D \to A^3$ on our "preferred" genus 2 Rauzy diagram (see below), calculate its characteristic polynomial and determine the moduli of its eigenvalues.



Remark

Later we'll see that this B_γ is not "typical" (wrt RV algorithm) ...

Discrete KZ cocycle

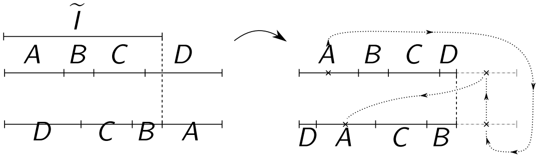
The B_γ's are in some sense a *discrete* version of the (extended) *Kontsevich-Zorich (KZ) cocycle*.

We'll comment more on this later (when relating B_γ with the *continuous* time version), but for today let's make the following *fundamental* observation (verified by induction on $n-m$):

Theorem

The (non-neg.) coeff. of $B_{\gamma(m,n)}$ at pos. α, β is exactly the *time* a pt. of $I_\alpha^{(n)}$ spends in $I_\beta^{(m)}$ under T_m-iterates before returning to $I^{(n)}$.

E.g.:



Birkhoff sums and KZ cocycle I

An *equivalent* formulation of previous theorem is: given φ a function on I, consider the *special Birkhoff sum*

$$S^{(n)}\varphi(x) = \sum_{0\leq i<r(x)} \varphi(T^i(x))$$

where $x \in I^{(n)}$ and $r(x) = \min\{j \geq 1 : T^j(x) \in I^{(n)}\}$.

Now, given $w \in \mathbb{R}^{\mathcal{A}}$, consider the piecewise cst. fct. $\sum\limits_{\alpha\in\mathcal{A}} w_\alpha \chi_{I^t_\alpha}$.

The special Birkhoff sum $S^{(n)}w$ is cst. on $I^{(n),t}_\alpha$, i.e., $S^{(n)}w \in \mathbb{R}^{\mathcal{A}}$.

In this language, the previous theorem implies that

$$S^{(n)}w = B_{\gamma(0,n)}w$$

Birkhoff sums and KZ cocycle II

In other words, KZ cocycle is intimately related to special Birkhoff sums of i.e.t.'s! (as it was promised in the title of this course)

For now, we will not develop further this observation. Instead, we will discuss further properties of the matrices B_γ.

Symplecticity of discrete KZ cocycle I

Since $B_{\underline{\gamma}}$'s allow to relate the length and transl. vectors before and after applying the RV algorithm, one can prove that

$$\Omega_{\pi'} = B_{\underline{\gamma}} \Omega_{\pi} {}^t B_{\underline{\gamma}}$$

Furthermore, one can show that $\exists$ coherent choice of basis on $\ker(\Omega_\pi)$ (for each $\pi \in \mathcal{D}$) such $B_{\underline{\gamma}}^{-1}$ acts as the identity from $\ker(\Omega_\pi)$ to $\ker(\Omega_{\pi'})$. In other words, the "exciting" part of the action of KZ cocycle happens in $\mathbb{R}^{\mathcal{A}}/\ker(\Omega_\pi) \simeq \mathrm{Im}(\Omega_\pi)$.

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Remark

For $s = 1$ (i.e., $\exists!$ marked pt), $\ker(\Omega_\pi) = \{0\}$ (as $d = 2g + s - 1$).

Symplecticity of discrete KZ cocycle II

The restriction of B_γ to $\mathrm{Im}(\Omega_\pi)$ is called *restricted* KZ cocycle, and, as we just saw, it is the interesting part of B_γ.

Recall (from 3rd lecture) that Ω_π is an antisymm. matrix of rank $2g$. Thus, restr. KZ cocycle is *symplectic* because it preserves the non-deg. altern. form on $\mathrm{Im}(\Omega_\pi)$ induced by Ω_π.

Sketch of application to unique ergodicity of i.e.t.'s I

Motivated by his studies of minimality of i.e.t.'s, M. Keane conjectured (in 1977) that *almost every i.e.t.* is *uniquely ergodic*.

Here, "almost every" means for a full Leb. measure set of length data $\lambda = (\lambda_\alpha) \in \mathbb{R}^{\mathcal{A}}$.

Keane's conjecture was proved by H. Masur and W. Veech (independently) in 1982. A sketch of proof assuming some properties of the RV algorithm and KZ cocycle goes as follows.

Sketch of application to unique ergodicity of i.e.t.'s II

Let T i.e.t. w/o conn. and $\gamma = \gamma(T) = \gamma_1 \dots \gamma_n \dots$ the corresp. ∞-complete path in a Rauzy diagram $\mathcal{D}$. Define

$$\mathcal{C}(\gamma) = \bigcap_n (\mathbb{R}_+)^{\mathcal{A}} B_{\gamma_n} \dots B_{\gamma_1}$$

By the features of $B_{\underline{\gamma}}$'s, one can show that the convex cone $\mathcal{C}(\gamma)$ consists of all length data w/ path γ.

Sketch of application to unique ergodicity of i.e.t.'s III

Also, $\mathcal{C}(\gamma) \simeq \mathcal{M}(T)$ (the cone of T-inv. measures): given T_λ with length data $\lambda \in \mathcal{C}(\gamma)$, let u, u_λ be the largest sing. of T, T_λ, and define

$$H : T_\lambda^n(u_\lambda) \mapsto T^n(u)$$

Since γ is the path of T and T_λ in $\mathcal{D}$, one can check that H is *increasing*.

Then, since T and T_λ have no conn., by Keane's thm, the orbits $\{T^n(u)\}$ and $\{T_\lambda^n(u_\lambda)\}$ are *dense* and H can be uniquely extended into a *homeomorphism* from I to I_λ with

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i.e., i.e.t.'s are *conjugated* when they *share* a (irrat.) rot. nb. γ.

Sketch of application to unique ergodicity of i.e.t.'s IV

Now, since T_λ is an i.e.t., it preserves the Lebesgue measure *Leb*. Thus, T preserves the measure $\mu = H_*(Leb)$. This induces a map

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I.e.: T i.e.t. w/o conn. is *uniq. erg.* $\iff$ the cone $\mathcal{C}(\gamma)$ is 1-*dim*.

Sketch of application to unique ergodicity of i.e.t.'s V

The 1st tool to get that $\mathcal{C}(\gamma) = \bigcap_n (\mathbb{R}_+)^{\mathcal{A}} B_{\gamma_n} \dots B_{\gamma_1}$ is 1-dim. is the following *spectral gap* property:

Proposition

Let $\underline{\gamma}$ a k-complete path (i.e., a concat. of k compl. paths). If $k \geq 2d-3$, then all entries of $B_{\underline{\gamma}}$ are *positive*.

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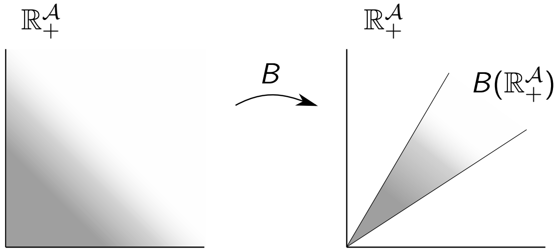
Remark

We call this *spectral gap* because of *Perron-Frobenius thm* that top eigenv. of positive matrices are simple (i.e., its multip. is 1).

Sketch of application to unique ergodicity of i.e.t.'s VI

Spectral gap is useful as posit. matrices contract the cone $(\mathbb{R}_+)^{\mathcal{A}}$:



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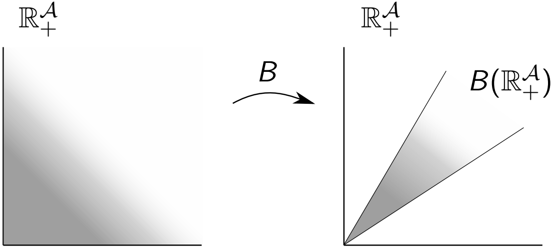

Since γ is ∞-complete, it *seems* that the proof is complete from the spectral gap. However, this is *not* the case because we need the matrices $B_{\gamma(0,n)}$ to *uniformly* contract $(\mathbb{R}_+)^{\mathcal{A}}$ infinitely many times.

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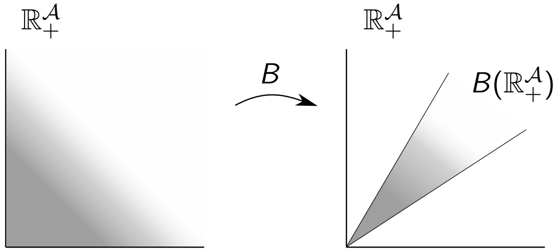

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At this point, the idea to achieve this is to appeal to *recurrence* properties of RV algorithm.

Sketch of application to unique ergodicity of i.e.t.'s VII

Since γ is ∞-complete, we can write $\gamma = \underline{\gamma}_1 \dots \underline{\gamma}_n \dots$ where each $\underline{\gamma}_n$ is a $2d-3$-complete path.

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If ("some version of") RV algorithm has nice *recurrence* properties at "typical points", some initial segment $\gamma(s) = \underline{\gamma}_1 \dots \underline{\gamma}_s$ is likely to repeat itself infinitely many times in γ, i.e., we can write

$$\gamma = \gamma(s)\gamma_1\gamma(s)\gamma_2 \dots \quad (\text{where } \gamma_i \text{ is concat. of some } \underline{\gamma}'_n s)$$

I.e., for appropriate choices of $k(n)$,

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Since $B_{\gamma(s)}$ contracts *uniformly* $(\mathbb{R}_+)^{\mathcal{A}}$, this permites to conclude (for a typical γ, T) that $\mathcal{C}(\gamma)$ is 1-dim. in this situation.

Teich. and moduli spaces are adequate parameter spaces

In order to make sense of “recurrence” in the previous argument, we need to introduce the Teich. and moduli spaces of transl. surf. (as they're the “correct” parameter spaces for the joint study of i.e.t.'s and translation flows).

Teichmüller spaces

For simplicity of notation, we'll think of transl. surf. (M, Σ, κ) as the data (M, ω) of a Riemann surf. struct. M and an Ab. diff. ω.

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Definition

The *Teichmüller space* Q_g of transl. surf. of genus $g \geq 1$ is the set of transl. surf. (M, ω) *modulo* the action of the group $\mathrm{Diff}_0^+(M)$ of orient.-pres. diffeos of M which are *isotopic* to the id..

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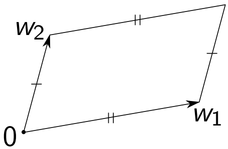
More concretely, $(M, \omega) = (M', \omega')$ in Q_g *iff* $\exists\ \rho : M \to M'$ bihol., $\rho^*(\omega') = \omega$, and ρ is isot. to id..

Example: Teich. space of torii I

Consider a torus $M = \mathbb{C}/(\mathbb{Z}w_1 \oplus \mathbb{Z}w_2)$.



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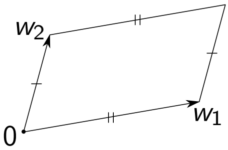

The map $z \mapsto (1/w_1)z$ is a bihol. between M and $\mathbb{C}/\Lambda(w)$ where $\Lambda(w) := \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot w$ and $w = w_2/w_1$.

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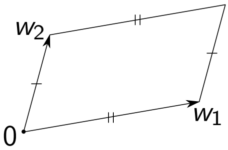

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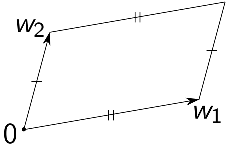

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I.e., $(M, w \cdot dz) = (\mathbb{C}/\Lambda(w), dz)$ in the Teich. space Q_1.