Maximizing measures for irreducible countable Markov shifts

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Given metric space X, a dynamical system $T : X \to X$ and a real function (potential) $f : X \to \mathbb{R}$, the main problem in *Ergodic Optimization* is to guarantee the existence and to describe the support of the *maximizing measures* for the system, that is, to describe the set of measures satisfying:

$$\alpha(f) := \sup_{\mu \in \mathcal{M}_T} \int f \, d\mu$$

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Settings:

1. X compact or X non-compact (Polish Space) [our case].

2. The potential $f : X \to \mathbb{R}$ always continuous or more (Lipschitz, Hölder, summable variation, Locally Hölder etc).

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Examples and results when X is compact:

- Maximizing measures always exist.

- When X is compact since the potential f is always assume continuous by compactness (of M_T) there exists a probability measure m in M_T such that $\alpha(f) := \sup_{\mu \in \mathcal{M}_T} \int f \, d\mu = \int f \, dm$

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Examples:

1. $X = S^1$, $T(x) = 2x \; mod \; 1$.
2. $X = \Sigma = \{1, ..., k\}^{\mathbb{N}}$ (full shift), $T = \sigma$ (shift map).

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Main Conjecture

Roughly:

Generically in the space of Lipschitz potentials with X compact and T with suitable properties the maximizing measure is unique and supported in an periodic orbit.

Partial results: G. Contreras, A. Lopes and P. Thieullen (2001), T. Bousch (2000), A. Quas and J. Siefken (to Appear in ETDS).

Important Reference: O. Jenkinson - "Ergodic Optimization - DCDS 2006.

Our Setting: non-compact Markov shifts

Given an infinite matrix $\mathbf{A} : \mathbb{N} \times \mathbb{N} \to \{0,1\}$, we call by $\Sigma_{\mathbf{A}}(\mathbb{N})$ the subset of $\Sigma(\mathbb{N}) := \mathbb{N}^{\mathbb{N}}$ of allowable sequences, that is:

$$\Sigma_{\mathbf{A}}(\mathbb{N}) := \{x \in \Sigma(\mathbb{N}),\, \mathbf{A}(x_i, x_{i+1}) = 1 \,\forall i \geq 0\}\,.$$

Fixed $\lambda \in (0,1)$, we define a metric on $\Sigma_{\mathbf{A}}(\mathbb{N})$ by $d(x,y) = \lambda^k$, where k is the first coordinate where $x_k \neq y_k$.(Polish Space)

The matrix $\mathbf{A}$ is *finitely primitive*, when there exist a finite subset $\mathbb{F} \subseteq \mathbb{N}$ and an integer $K_0 \geq 0$ such that, for any pair of symbols $i, j \in \mathbb{N}$, one can find $\ell_1, \ell_2, \ldots, \ell_{K_0} \in \mathbb{F}$ satisfying

$$\mathbf{A}(i,\ell_1)\mathbf{A}(\ell_1,\ell_2)\cdots\mathbf{A}(\ell_{K_0},j) = 1.$$

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$\pi : \Sigma_{\mathbf{A}}(\mathbb{N}) \to \mathbb{N}$ is the projection of the first coordinate, that is $\pi(x) = \pi(x_0 x_1 x_2 \ldots) = x_0$.

Given a function $f : \Sigma_{\mathbf{A}}(\mathbb{N}) \to \mathbb{R}$ the j-th variation of f is

$$V_j(f) := \sup\{f(x) - f(y)\,,\ \pi(\sigma^i(x)) = \pi(\sigma^i(y)) \text{ for } i = 0, \ldots, j-1\}\,,$$

We say that f has *bounded variation*(summable variation), when

$$V(f) := \sum_{j=1}^{\infty} V_j(f) < \infty$$

$f : \Sigma_{\mathbf{A}}(\mathbb{N}) \to \mathbb{R}$ is called *locally Hölder continuous* when there exists a constant $H_f > 0$ such that, for all integer $j \geq 1$, we have

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$$\lim_{i\to\infty} \sup f|_{[i]} = -\infty\,,$$

$[i] := \{x \in \Sigma_{\mathbf{A}}(\mathbb{N}),\ \pi(x) = i\}$ is the cylinder beginning with i.

This condition is satisfied when we have for example:

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Theorem (O. Jenkinson, R. D. Mauldin and M. Urbański 05')

The family of Gibbs measures $(\mu_{\beta f})_{\beta \geq 1}$ has at list one weak accumulation point as $\beta \to \infty$. Any accumulation point μ is a maximizing measure for f, and $\lim_{\beta \to \infty} \int f d\mu_{\beta f} = \int f d\mu$.

Proof: Prohorov 's theorem and use that the measures $\mu_{\beta f}$ are equilibrium states.

Theorem (I. D. Morris 07')

Assuming the conditions above we have that there exists a finite set $\mathcal{A} \subset \mathbb{N}$ such that

$$\beta_f = \max_{\substack{\mu \in \mathcal{M}_\sigma \\ supp\mu \subseteq \Sigma_{\mathbf{A}}(\mathcal{A})}} \int f \, d\mu.$$

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Suppose $f : \Sigma_{\mathbf{A}}(\mathbb{N}) \to \mathbb{R}$ is continuous and bounded above.

A sub-action (for the potential f) is a function $u \in C^0(\mathbf{\Sigma})$ verifying $(f + u - u \circ \sigma)(\mathbf{x}) \leq m(f), \ \forall \, \mathbf{x} \in \mathbf{\Sigma}$.

Proposition

Assume $\Sigma_{\mathbf{A}}(\mathbb{N})$ is a finitely primitive Markov subshift on a countable alphabet. Let $f : \Sigma_{\mathbf{A}}(\mathbb{N}) \to \mathbb{R}$ be a bounded above and locally Hölder continuous potential such that $\inf_{i \in \mathbb{F}} f|_{[i]} > -\infty$. Then there exists an nonnegative, bounded and locally Hölder continuous function $u : \Sigma_{\mathbf{A}}(\mathbb{N}) \to \mathbb{R}_+$ verifying

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Let $\eta > 0$ be a real constant. As f is coercive and u is bounded, there exists $\mathcal{I} \in \mathbb{Z}_+$ such that

$$\sup_{i>\mathcal{I}} (f + u - u \circ \sigma - m(f))|_{[i]} < -\eta.$$

In particular, $\mu(\cup_{i>\mathcal{I}}[i]) = 0$ and then $\mathrm{supp}(\mu) \subset \cup_{i\leq\mathcal{I}}[i]$.

Since $\mathrm{supp}(\mu)$ is a σ-invariant set

$$\mathrm{supp}(\mu) \subset \cap_{k\geq 0}\sigma^{-k}\left(\cup_{i\leq\hat{I}}[i]\right) = \Sigma_{\mathbf{A}}(\mathcal{I}).$$

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R. Bissacot and E. Garibaldi (Bull. Braz. Math. Soc. 2010)

Theorem

Suppose $\Sigma_{\mathbf{A}}(\mathbb{N})$ is shift with $\mathbf{A}$ finitely primitive. Let $f : \Sigma_{\mathbf{A}}(\mathbb{N}) \to \mathbb{R}$ be a bounded above, coercive and locally Hölder continuous potential satisfying $\inf_{i \in \mathbb{F}} f|_{[i]} > -\infty$.
Then, there exists an integer $\mathcal{I} > I_{\mathbb{F}}$ such that

$$m(f) = \max_{\substack{\mu \in \mathcal{M}_\sigma \\ supp\mu \subseteq \Sigma_{\mathbf{A}}(\mathcal{I})}} \int f \, d\mu.$$

In particular, maximizing measures do exist. Furthermore, there exists a compact σ-invariant set $\Omega \subseteq \Sigma_{\mathbf{A}}(\mathcal{I})$ such that $\mu \in \mathcal{M}_\sigma$ is f-maximizing if, and only if, μ is supported in Ω.

Key Fact:The invariant measures supported in periodic orbits are dense on the set of ergodic measures. (Parthasarathy- 1961)

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Let $\Sigma_{\mathbf{A}}(\mathbb{N})$ *be a finitely primitive Markov subshift on a countable alphabet. Assume* $f : \Sigma_{\mathbf{A}}(\mathbb{N}) \to \mathbb{R}$ *is a bounded above and locally Hölder continuous potential verifying* $\inf_{i \in \mathbb{F}} f|_{[i]} > -\infty$*. Suppose there exists an integer* $\mathcal{I} > I_{\mathbb{F}}$ *such that*

$$\sup_{i > \mathcal{I}} f|_{[i]} < \inf_{i \in \mathbb{F}} f|_{[i]} - \left[Var(f) + K_0 \left(\sup f - \inf_{i \in \mathbb{F}} f|_{[i]} \right) \right].$$

(Main)Earlier references of Finitely Primitive case:
Ergodic optimization for noncompact dynamical systems.
- O. Jenkinson, R. D. Mauldin and M. Urbański - (DS-07')

Ergodic optimization for countable alphabet subshifts of finite type.
- O. Jenkinson, R. D. Mauldin and M. Urbański - (ETDS-06')

Zero Temperature limits of Gibbs-Equilibrium states for countable alphabet subshifts of finite type.
- O. Jenkinson, R. D. Mauldin and M. Urbański - (JSP-05')

Including some results with weaker hypothesis. Bounded Variation instead of locally Hölder continuous, for example.

Connection with phase transitions

Renewal shifts

Example: Let $A = (a_{ij})_{i,j \in \mathbb{N}}$ be the transition matrix such that there exists an increasing sequence of naturals $(d_i)_{i \in \mathbb{N}}$ for which

$$a_{11} = a_{i+1,i} = a_{1,d_i} = 1, \ \forall\, i \in \mathbb{N}$$

and the others coefficients are zero.

Theorem (O. Sarig - 2001)

Let Σ be a Renewal Shift and f a locally Hölder potential such that $\sup f < \infty$. *Then there exists a constant $t_c \in (0, \infty]$ such that*

- *For $0 < t < t_c$ there exists an equilibrium probability measure μ_t corresponding to tf. For $t > t_c$ there is no equilibrium probability measures corresponding to tf;*
- *$P(tf)$ is real analytic on $(0, t_c)$ and linear on (t_c, ∞). At t_c, it is continuous but not analytic.*

Theorem (G. Iommi - 2007)

Let Σ be a Renewal Shift and f a locally Hölder potential such that $\sup f < \infty$. Then

- *For $t_c = \infty$, then there exists maximizing measures μ_t for tf.*
- *If $t_c < \infty$, then there are no maximizing measures for f and $M = \alpha(f)$, where M is the slope linear part of the pressure $P(tf)$.*

We say that $\mathbf{A}$ is *irreducible* when for any $(i, j) \in \mathbb{N}^2$ there exists a natural number $k(i, j)$ and a word $y_1 y_2 \dots y_k$ such that $i y_1 y_2 \dots y_k j$ is an allowable word: $\mathbf{A}(i, y_1) = 1$, $\mathbf{A}(y_i, y_{i+1}) = 1$ for $i = 1, \dots, k-1$ and $\mathbf{A}(y_k, j) = 1$.

Theorem (R. Bissacot and R. Freire - arxiv)

Let σ be the shift on $\Sigma_{\mathbf{A}}(\mathbb{N})$ with $\mathbf{A}$ irreducible and let be $f : \Sigma_{\mathbf{A}}(\mathbb{N}) \to \mathbb{R}$ be a function with bounded variation and coercive. Then, there is a finite set $\mathcal{A} \subset \mathbb{N}$ such that $\mathbf{A}|_{\mathcal{A}\times\mathcal{A}}$ is irreducible and

$$\alpha(f) = \sup_{\mu \in \mathcal{M}_\sigma(\Sigma_{\mathbf{A}}(\mathcal{A}))} \int f \, d\mu \, .$$

Furthermore, if ν is a maximizing measure, then

$$\mathit{supp}\,\nu \subset \mathcal{M}_\sigma(\Sigma_{\mathbf{A}}(\mathcal{A}))\, .$$

Theorem

Let Σ be a subshift on $\Sigma_{\mathbf{A}}(\mathbb{N})$ with $\mathbf{A}$ irreducible, $f : \Sigma_{\mathbf{A}}(\mathbb{N}) \to \mathbb{R}$ be a function with bounded variation. Given $\epsilon > 0$ assume there are naturals $l_2 > l_1 > 0$ such that

$$\sup f|_{[j]} < \alpha(f) - \epsilon \quad \forall j \geq l_1 ,$$

and

$$\sup f|_{[j]} < \min\{C_1, C_2\} \quad \forall j \geq l_2 ,$$

where C_1 and C_2 are constants (depending on l_1).

$$\begin{aligned} C_1 =& - \left(P_0 |min\ f|_{\Sigma_{\mathbf{A}}(\mathcal{A}_1)}| + (P_0 - 1)|m(f)| + 2V(f)\right) , \\ C_2 =& m(f) - \epsilon - V(f) . \end{aligned}$$

Then, there is a finite set $\mathcal{A} \subset \mathbb{N}$ such that $\mathbf{A}|_{\mathcal{A}\times\mathcal{A}}$ is irreducible and

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If we consider the alphabet $\mathcal{I}_1 := \{0, 1, \ldots, l_1 - 1\}$ we still have a problem that maybe there are no allowable sequences only with such symbols and, besides, the shift does not need to be irreducible when restrict to such sequences. So we complete $\mathcal{I}_1$ to a finite alphabet $\mathcal{A}_1$ in the following manner:

We choose, for each pair i, j in $\mathcal{I}_1$, one word $w = w(i, j)$ connecting i to j. Notice there is such a word since $\mathbf{A}$ is irreducible. We denote by P_0 the length of the longest of such connecting words.

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Key Fact (again): The invariant measures supported in periodic orbits are dense on the set of ergodic measures.

$$\alpha(f) = \sup_{\mu \in \mathcal{M}_{\sigma-Per}(\Sigma_{\mathbf{A}}(\mathbb{N}))} \int f \, d\mu \,.$$

Proof:
The Ergodic Decomposition theorem implies that

$$\alpha(f) = \sup_{\mu \in \mathcal{M}_{\sigma-erg}(\Sigma_{\mathbf{A}}(\mathbb{N}))} \int f \, d\mu \,,$$

where $\mathcal{M}_{\sigma-erg}(\Sigma_{\mathbf{A}}(\mathbb{N}))$ is the set of ergodic invariant probability measures and use the density of invariant measures supported in periodic orbits on the set of ergodic measures.

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Naive idea:

Since the potential f decays to $-\infty$ when the symbols grow, we can restrict ourselves to periodic orbits whose symbols are all small.

Lemma

Given $\epsilon > 0$, there is $I_1 \in \mathbb{N}$ such that if x starts in $i \geq I_1$ then $\beta_m(x) := \frac{1}{m} S_m f(x) = \frac{1}{m} \sum_{j=0}^{m-1} f(\sigma^j(x)) < \alpha(f) - \epsilon$ for any $m \in \mathbb{N}$.

In particular, if $x \in \text{Per}(\sigma)$ we have $\beta(x) < \alpha(f) - \epsilon$.

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In particular, if $x \in \text{Per}(\sigma)$ we have $\beta(x) < \alpha(f) - \epsilon$.

Proof: Since f is coercive, there is $l_1 \in \mathbb{N}$ such that

$$\sup f|_{[j]} < \alpha(f) - \epsilon \qquad \text{for all } j \geq l_1 \,.$$

We have that

$$\beta_m(x) = \frac{1}{m} S_m f(x) = \frac{1}{m} \sum_{j=0}^{m-1} f(\sigma^j(x)) \leq \frac{1}{m} \sum_{j=0}^{m-1} \sup f|_{[\pi(\sigma^j(x))]} \,,$$

since $\pi(\sigma^j(x)) \geq i \geq l_1$ for all $j = 0, \ldots, m-1$, we get

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Let $x \in \Sigma_{\mathbf{A}}(\mathbb{N})$ and $w = x_\ell \dots x_{\ell+m}$ be a word appearing on x.

Definition

1. The average of the word $w = x_\ell \dots x_{\ell+m}$ on the orbit x is

$$\kappa(\ell, m|x) = \kappa(w|x) := \frac{1}{m+1} \sum_{j=0}^{m} f(\sigma^{\ell+j}(x))\,;$$

2. if $r < m$ we define

$$\kappa_r(\ell, m|x) := \frac{1}{r+2} \left[f(\sigma^{\ell+m}(x)) + \sum_{j=0}^{r} f(\sigma^{\ell+j}(x)) \right].$$

Fact 1:
Let $x \in \Sigma_{\mathbf{A}}(\mathbb{N})$ be a periodic orbit for σ such that $\beta(x) \geq \beta_f - \epsilon$ and $x \notin \Sigma_{\mathbf{A}}(\mathcal{A}_2)$.

Then, there is at least one word $x_\ell \ldots x_{\ell+m}$ appearing in x such that

1. $\kappa(\ell, m|x) \geq \beta(x)$;
2. $x_\ell < l_1$, $x_{\ell+m} \geq l_2$; and
3. $x_{\ell+j} < l_2$ for all $j \in \{0, \ldots, m-1\}$.

Fact 2:
Let $x_\ell \dots x_{\ell+m}$ be the word given by fact 1 and $r < m$ be the greatest integer such that $x_{\ell+r} \in \mathcal{I}_1$. Then

$$\kappa(\ell, m|x) \le \kappa_r(\ell, m|x)\,.$$

The Theorem is consequence of the following result:

Lemma

Let $x \in \Sigma_{\mathbf{A}}(\mathbb{N})$ be any periodic orbit for σ such that $x \notin \Sigma_{\mathbf{A}}(\mathcal{A}_2)$ and $\beta(x) \ge \alpha(f) - \epsilon$. Then, there is a periodic orbit $z \in \Sigma_{\mathbf{A}}(\mathcal{A}_2)$ such that $\beta(z) > \beta(x)$.

Theorem

Let σ be the full shift on $\Sigma(\mathbb{R}^+)$ and $f : \Sigma(\mathbb{R}^+) \to \mathbb{R}$ be a bounded above function with bounded variation. Given $\epsilon > 0$ assume there are real numbers $l_2 > l_1 > 0$ such that

$$\sup f|_{[j]} < \alpha(f) - \epsilon \quad \forall\, j \geq l_1\,,$$

for some $\epsilon > 0$ fixed and

$$\sup f|_{[j]} < \min f|_{\Sigma([0,l_1])} - V(f) \quad \forall\, j \geq l_2\,.$$

Then, we have that

$$\alpha(f) = \sup_{\mu \in \mathcal{M}_\sigma(\Sigma([0,l_2]))} \int f \, d\mu\,.$$

Furthermore, if ν is a maximizing measure, then

$$\text{supp}\ \nu \subset \mathcal{M}_\sigma(\Sigma([0,l_2]))\,.$$

In the case that f is coercive, we have the following:

Corollary

Let σ be the full shift on $\Sigma(\mathbb{R}^+)$ and $f : \Sigma(\mathbb{R}^+) \to \mathbb{R}$ be a bounded above function with bounded variation and coercive. Then, there is $l > 0$ such that

$$\alpha(f) = \sup_{\mu \in \mathcal{M}_\sigma(\Sigma([0,l]))} \int f \, d\mu \, .$$

Furthermore, if ν is a maximizing measure, then

$$\text{supp } \nu \subset \mathcal{M}_\sigma(\Sigma([0,l])) \, .$$