Lagrange Spectra in Teichmüller Dynamics

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Translation Surface

It is a surface of the form $X = P/\partial P$, where P is a polygon with sides in ∂P pairwise parallel, with same length and same orientation.

-Holomorphic one-form $w = dz$.

-Flat metric $dz \otimes d\bar{z}$ with conical singularities, whose conical angle is multiple of 2π.

Veech Construction Set $\mathcal{A} := \{A, B, C, D\}$.

-Consider data $\pi \in \mathrm{Sym}(\mathcal{A})$, $\lambda \in \mathbf{R}_+^{\mathcal{A}}$ and $\tau \in \mathbf{R}^{\mathcal{A}}$.

-Form a polygon P with sides $\zeta_\alpha = \lambda\alpha + i\tau_\alpha$, whose order in ∂P is given by π.

-Obtain a surface $X = X(\pi, \lambda, \tau)$, which is a *suspension* of the IET $T = (\pi, \lambda)$.

The (stratum of the) **Moduli space** is the set $\mathcal{H}_g(k_1, \dots, k_r)$ of those X with

$\mathrm{genus}(X) = g$ (and $\mathrm{Area}(X) = 1$).

Angles $2k_1\pi, \dots, 2k_r\pi$ at conical singularities.

$\mathcal{H}_g(k_1, \dots, k_r)$ is an Affine orbifold. Not compact. Local coordinates are $\zeta_A, \zeta_B, \zeta_C, \zeta_D$.

Action of $\mathrm{SL}(2,\mathbf{R})$.

-Fix A in $\mathrm{SL}(2,\mathbf{R})$. For any X take a polygon P with $X = P/\partial P$.
-Define $A \cdot X := AP/\partial AP$. It does not depend on P but just on X and A.

Teichmüller flow: $\mathcal{F}_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$.

Stable manifold: $\mathcal{U}_s := \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$.

Rotation of vertical direction: $\mathcal{R}_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Theorem. [Masur,Veech]: *There exists an unique $\mathcal{F}_t$-invariant smooth measure μ on $\mathcal{H}_g(k_1,\dots,k_r)$. Moreover μ is finite and ergodic.*
Hence: *The generic orbit is unbounded*

-A *Saddle connection* for the translation surface X is a geodesic segment $\gamma : [0,1] \to X$ with $\gamma^{-1}(\text{ conical singularities }) = \{0,1\}$.
-Let $\mathrm{Hol}(X)$ be the set of *periods* $v = \int_\gamma w$ of saddles connections γ of X. Define:

$$\mathrm{Sys}(X) := \min\{|v|; v \in \mathrm{Hol}(X)\}.$$

-*Malher criterion:* $\mathcal{K}$ is compact in the stratum iff $\exists \delta > 0$ such that $\mathrm{Sys}(X) \geq \delta$ for any X in $\mathcal{K}$.

Notions of bounded type.

Let $X = (\pi, \lambda, \tau)$ be suspension of $T = (\pi, \lambda)$.

-*Dynamics:* $s(X) := \liminf_{t \to \infty} \mathrm{Sys}(\mathcal{F}_t \cdot X)$.

-*Flat geometry:* $a(X) := \liminf_{v \to \infty} |\Re(v)| \cdot |\Im(v)|$, where $v \in \mathrm{Hol}(X)$.

-*Arithmetics:* $\mathcal{E}(T) := \liminf_{n \to \infty} n \cdot \mathcal{E}_n(T)$,
where $\mathcal{E}_n(T) := |I|^{-1} \min\{|u - u'|; u, u' \text{ sing. of } T^n : I \to I\}$ (Boshernitzan).

We say that X (equivalently T) is of bounded type if these quantities are positive.

Generically: $a(X) = 0$, $s(X) = 0$, $\mathcal{E}(T) = 0$.

Example: **(M.)** If $n\varphi(n)$ is decreasing monotone and $\sum_1^\infty \varphi(n) = +\infty$ then for a generic T there are arbitrary large n with

$$\mathcal{E}_n(T) < \varphi(n).$$

Lemma. [Vorobets identity]

$$\mathcal{E}(T) = a(X) = s^2(X)/2.$$

Lagrange spectra.

Let $\mathcal{I}$ be a closed and $\mathrm{SL}(2,\mathbf{R})$-invariant subset of some stratum $\mathcal{H}$.

-Example: $\mathcal{I} = \mathrm{SL}(2,\mathbf{R}) \cdot X$ for a *Veech surface* X.
-Example: $\mathcal{I} = \mathcal{C}$ connected component of $\mathcal{H}$.

Eskin-Mirzakhani: *all these loci are nice, i.e. they are affine sub-orbifold of strata.*

Definition For any invariant locus $\mathcal{I}$ its *Lagrange Spectrm* is the set

$$\mathcal{L}(\mathcal{I}) := \{a^{-1}(X); X \in \mathcal{I}\}.$$

-Symmetries: $a(\mathcal{F}_t \cdot X) = a(X) = a(\mathcal{U}_s \cdot X)$.

-Parametrization: if X is a Veech surface, for the slope α we set:

$$L_X(\alpha) := a^{-1}(\mathcal{R}_{\arctan(\alpha)}).$$

Then $\mathcal{L}(\mathrm{SL}(2,\mathbf{R}) \cdot X) = \{L_X(\alpha); \alpha \in \mathbf{R}\}$.

Genus one.

Set $\mathbf{T}^2 := (\mathbf{R}/\mathbf{Z})^2$. The moduli space of flat tori is $\mathcal{H}_1(0) = \mathrm{SL}(2,\mathbf{R}) \cdot \mathbf{T}^2$. The spectrum $\mathcal{L}(\mathcal{H}_1(0))$ is the classical Lagrange spectrum $\mathcal{L}$, indeed we have:

$$L_{\mathbf{T}^2}(\alpha) = \limsup_{p,q\to\infty} \frac{1}{q \cdot |q\alpha - p|},$$

Denote $\alpha = a_0 + [a_1, a_2, \dots, a_n, \dots]$ the continued fraction expansion. **Nice formula for re-normalization.**

$$L_{\mathbf{T}^2}(\alpha) = \limsup_{n\to\infty} [a_{n-1}, \dots, a_1] + a_n + [a_{n+1}, \dots].$$

i) $\min \mathcal{L} = \sqrt{5} = L_{\mathbf{T}^2}(\frac{\sqrt{5}-1}{2})$ (Dirichlet)

ii) $\mathcal{L} \cap (0,3)$ is discrete. (Markov).

iii) $\mathcal{L}$ is closed and the values $L_{\mathbf{T}^2}(\beta)$ with β quadratic irrational are dense (Cusick). *Idea:* Shadowing + Closing Lemma for the full shift over $(\mathbf{N}^*)^{\mathbf{Z}}$.

iv) Exists $c > 0$ such that $[c, +\infty) \subset \mathcal{L}$ (Hall). *Idea:* Sum of cantor sets.

v) $t \mapsto \dim_H \mathcal{L} \cap (0,t)$ is a continuous Cantor stair function. (Moreira)

Square-tiled surface:

A *Square-tiled surface* is a translation surface X admitting a ramified covering $\rho : X \to (\mathbf{R}/\mathbf{Z})^2$, ramified over 0.

-If γ is a saddle connection for X, let $m(\gamma)$ be the degree of the map $t \mapsto \rho(\gamma(t))$.
-For a rational slope p/q on X, set

$$m(p/q) = \min\{m(\gamma); \gamma \text{ has slope } p/q\}.$$

Formula for re-normalization. Let X be a *primitive* square-tiled surface with N squares. Then $\exists M > 0$ such that, if $L_{\mathbf{T}^2}(\alpha) > M$ then

$$L_X(\alpha) = N \cdot \limsup_{n\to\infty} \frac{[a_{n-1}, \ldots, a_1] + a_n + [a_{n+1}, \ldots]}{m^2(p_n/q_n)}.$$

Theorem A. [Hubert-M.-Ulcigrai] *For any square-tiled surface X the Lagrange Spectrum $\mathcal{L}(\mathrm{SL}(2,\mathbf{R}) \cdot X)$ has an Hall's ray.*

Idea: the factor $m(p_n/q_n)$ does not matter.

Connected components of strata.

Represent X by data (π, λ, τ).
Set $w_{\pi,\beta,\alpha} := \left(\sum_{\pi^b(\xi)<\pi^b(\beta)} e_\xi\right) - \left(\sum_{\pi^t(\xi)<\pi^t(\alpha)} e_\xi.\right)$
The minimal area among *diagonals* is

$$a(\pi, \lambda, \tau) := \min_{\beta,\alpha} |\langle \lambda, w_{\pi,\beta,\alpha} \rangle| \cdot |\langle \tau, w_{\pi,\beta,\alpha} \rangle|.$$

Iterate the Rauzy-Veech induction: $(\pi, \lambda, \tau) \mapsto (\pi^{(r)}, \lambda^{(r)}, \tau^{(r)})$.

Reormalized formula

$$a(X) = \liminf_{r\to\infty} a(\pi^{(r)}, \lambda^{(r)}, \tau^{(r)}).$$

Theorem B. [Hubert-M.-Ulcigrai] *Let $\mathcal{C}$ be any connected component of any stratum.*
-The Lagrange Spectrum $\mathcal{L}(\mathcal{C})$ is closed.
-The values $a^{-1}(X)$ for X with periodic Rauzy-Veech induction are dense in $\mathcal{L}(\mathcal{C})$.

Idea: sub-shift over $\Gamma^{\mathbf{Z}}$, where Γ is the set of positive paths in the Rauzy diagram.

Remarks.

Any connected component of any stratum contains square-tiled surfaces.

Crollary 1. *For any connected component $\mathcal{C}$ the Lagrange Spectrum $\mathcal{L}(\mathcal{C})$ satisfies iii) and iv).*

Theorem. [Maucourant] *Let g_t be the geodesic flow on a non-compact M with sectional curvature less than -1. Let $\phi : M \to \mathbf{R}_+$ be proper. Then the set of values*

$$L(p,v) := \limsup_{t\to\infty} \phi\big(g_t(p,v)\big)$$

with $p \in M$ and $v \in T_pM$ is closed. Moreover the values on periodic points are dense.

Maucourant's Theorem does not apply to strata, but it applies to Veech surfaces, hence we have:

Crollary 2. *For any square-tiled surface X the Lagrange Spectrum $\mathcal{L}(\mathrm{SL}(2,\mathbf{R}) \cdot X)$ satisfies iii) and iv).*

Thank you!