Linear response formula for equilibrium states in non-uniformly expanding dynamics

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(joint with T. Bomfim and P. Varandas)

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Some thermodynamical formalism concepts

M compact metric space

$f : M \to M$ continuous

$\phi : M \to \mathbb{R}$ continuous potential

Equilibrium states: f-invariant probability measures $\mu_{f,\phi}$ that attain the topological pressure

$$P_{\text{top}}(f,\phi) = \sup \left\{ h_\mu(f) + \int \phi d\mu \colon \mu \in \mathcal{M}_1(M) \right\}$$

Linear response formula: Weak differentiability of the equilibrium state $\mu_{f,\phi}$ with respect to dynamical system f and/or potential ϕ (notion introduced in dynamical systems by Ruelle)

Continuity and linear response formula: overview

Some recent contributions include:

- Hyperbolic sets, SRB measures: Ruelle 1997
- Partially hyperbolic systems, SRB measures: Dolgopyat 2004
- Unimodal, piecewise expanding maps: Smania-Baladi 2008-2011

$$f \mapsto \mu_{f,SRB} \text{ varies differentiably with } f$$

- Local diffeomorphisms with prevalence of expanding behavior: Castro-Varandas (Preprint 2011)

$$(f,\phi) \mapsto \mu_{f,\phi} \text{ vary continuously with } (f,\phi)$$

Setting

M compact Riemannian manifold
$f : M \to M$ $C^{r+\alpha}$-local diffeomorphism
$\phi : M \to \mathbb{R}$ $C^{r+\alpha}$ potential

Assumptions:

(H1) There is a set $\mathcal{A} \subset M$ where $f|_{\mathcal{A}}$ fails to be expanding, but is not very contractive.

(H2) All points $x \in M$ have some preimage outside a nhood of $\mathcal{A}$.

(P') There exists $\varepsilon = \varepsilon(f) > 0$ such that

$$\sup \phi - \inf \phi < \varepsilon \quad \text{and} \quad \max_{\ell \leq r} \|e^{\phi}\|_{\ell} < \varepsilon \, e^{\inf \phi}$$

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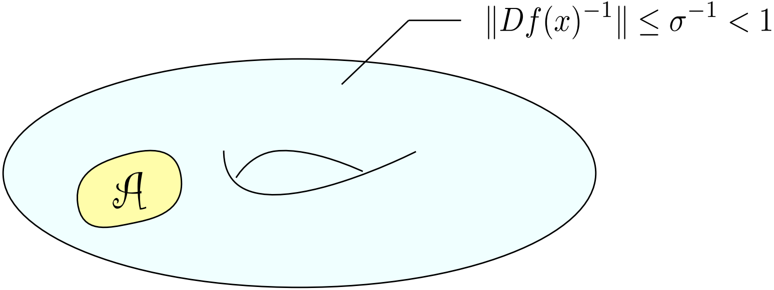
Thm: There exists a unique equilibrium state $\mu_{f,\phi}$ and varies continuously with f and ϕ


$\|Df(x)^{-1}\| \leq \sigma^{-1} < 1$
$\mathcal{A}$

Some Fundamental Questions:

Question 1: Is topological pressure $(f, \phi) \mapsto P_{\text{top}}(f, \phi)$ a continuous or differentiable map on f and ϕ?

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Question 2: Do equilibrium states $(f, \phi) \mapsto \mu_{f,\phi}$ vary continuously or differentiably? That is, given a smooth observable $g : M \to \mathbb{R}$ is

$$(f, \phi) \mapsto \int g \, d\,\mu_{f,\phi} \quad \text{differentiable?}$$

Some Fundamental Questions:

Question 3: Do Lyapunov exponents and metric entropy

$$(f,\phi) \mapsto h_{\mu_{f,\phi}}(f)$$

$$(f,\phi) \mapsto \lambda^+_{f,\phi} = \int \log \|Df(x)\| \, d\mu_{f,\phi}$$

$$(f,\phi) \mapsto \lambda^-_{f,\phi} = \int \log \|Df(x)^{-1}\|^{-1} \, d\mu_{f,\phi}$$

$$(f,\phi) \mapsto \int \log |\det Df(x)| \, d\mu_{f,\phi}$$

vary continuously or differentiably?

Question 4: Do (local) large deviation principles hold for Hölder continuous observables ψ, that is, does exist a convex function $I_{f,\psi}$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_{f,\phi}\Big(x \in M : \frac{1}{n}\sum_{j=0}^{n-1} \psi(f^j(x)) \in [a,b]\Big) \leq - \inf_{s\in[a,b]} I_{f,\phi,\psi}(s)$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_{f,\phi}\Big(x \in M : \frac{1}{n}\sum_{j=0}^{n-1} \psi(f^j(x)) \in (a,b)\Big) \geq - \inf_{s\in(a,b)} I_{f,\phi,\psi}(s)$$

If so, is the rate function $(f,\phi) \mapsto I_{f,\phi,\psi}$ a continuous or differentiable map on f and ϕ?

Some Examples

1. Maneville-Pommeau

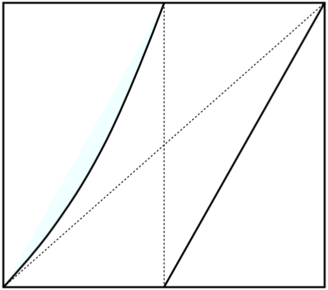

Figura: Variation of tangency order $1 + \alpha$

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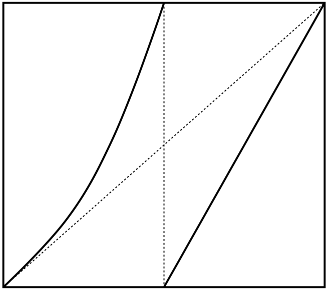

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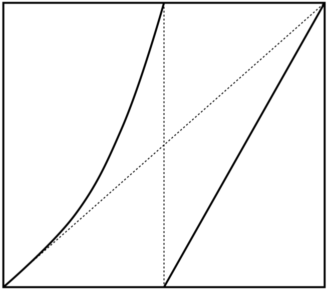

Figura: Variation of tangency order $1+\alpha$

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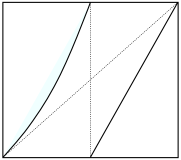

$x \mapsto x(1 + 2^\alpha x^\alpha)$

Do you know that:

1. $(1, +\infty) \ni \alpha \mapsto \mu_{\alpha,\mathrm{max}}$ varies continuously with α

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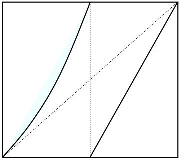

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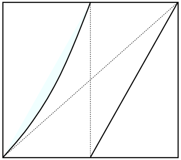

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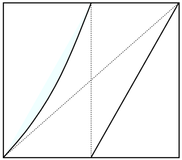

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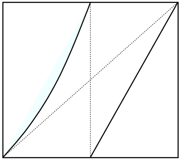

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4. $(1, +\infty) \ni \alpha \mapsto h_{\mu_{f_\alpha,\phi}}(f_\alpha)$ varies continuously with α
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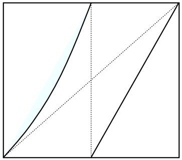

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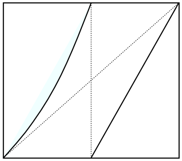

$$x \mapsto x(1 + 2^{\alpha} x^{\alpha})$$

If we adapt to be C^{1+}-differentiable in the circle:

1. $(1, +\infty) \ni \alpha \mapsto \mu_{\alpha,\text{max}}$ varies differentiably with α
2. $(1, +\infty) \ni \alpha \mapsto \lambda(\mu_{\alpha,\text{max}})$ varies continuously with α
3. $(1, +\infty) \times \mathcal{W} \ni (\alpha, \phi) \mapsto P(f_{\alpha}, \phi)$ is differentiable
4. $(1, +\infty) \ni \alpha \mapsto h_{\mu_{f_{\alpha},\phi}}(f_{\alpha})$ varies continuously with α
5. $(0, 1] \times \mathcal{W} \ni (\alpha, \phi) \mapsto P(f_{\alpha}, \phi)$ is continuous
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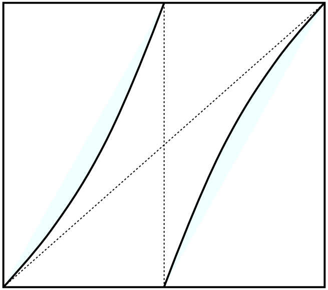
Some Examples

2. Bifurcation of circle expanding maps



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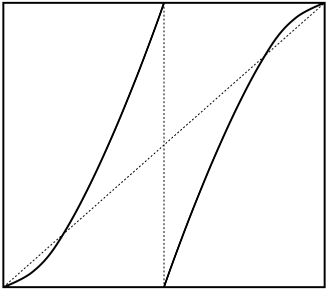

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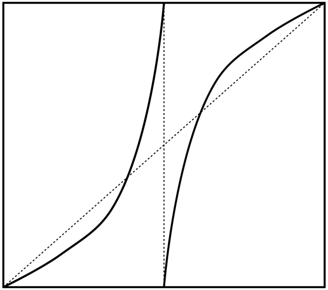
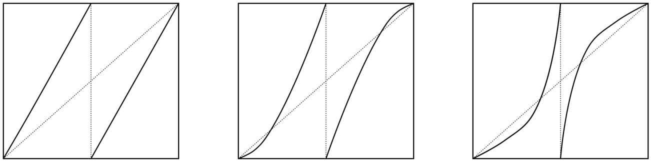


Some Examples

2. Bifurcation of $C^{2+\alpha}$-circle expanding maps



Do you know that:

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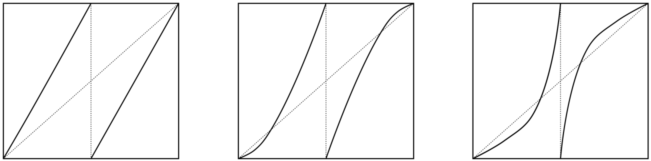

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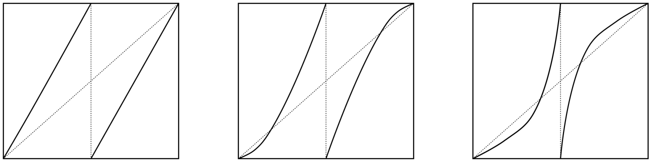

Do you know that:

1. $(f, \phi) \mapsto P_{\text{top}}(f, \phi)$ is differentiable with respect to f and ϕ
2. $f \mapsto \lambda(\mu_{f,\max})$ varies continuously
3. $f \mapsto \dim_H(\mu_{f,\max}) = \inf \{ \dim_H(K) : \mu_{f,\max}(K) = 1 \}$ varies continuously

Examples

3. Bifurcation of expanding maps on $\mathbb{T}^n$

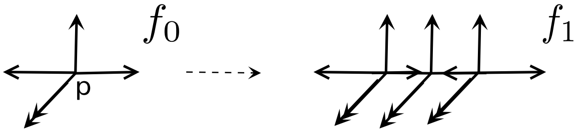

Figura: Local pitchfork perturbation in neighborhood of fixed point

Examples

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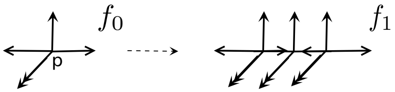

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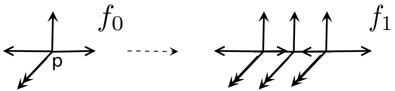

Do you know that:

1. $(f, \phi) \mapsto P_{\text{top}}(f, \phi)$ is differentiable with respect to f and ϕ
2. $f \mapsto \lambda^+(\mu_{f,\max})$ and $f \mapsto \lambda^-(\mu_{f,\max})$ vary continuously

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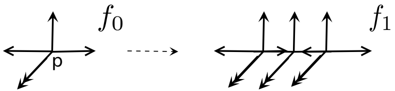

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2. $f \mapsto \lambda^+(\mu_{f,\max})$ and $f \mapsto \lambda^-(\mu_{f,\max})$ vary continuously
3. the entropy function $f \mapsto h_{\mu_{f,\max}}(f)$ is continuous

Statement of the main results

Theorem A: Let $\mathcal{W} \subset C^{r+\alpha}(M,\mathbb{R})$ open subset satisfying (P') with uniform constants. Then the following functions are analytic:

- The transfer operators $\mathcal{W} \ni \phi \mapsto \mathcal{L}_\phi \in L(C^{r+\alpha}(M,\mathbb{R}))$;
- The spectral radius function $\mathcal{W} \ni \phi \mapsto \lambda_\phi = \exp(P_{\text{top}}(f,\phi))$;
- The invariant density function $\mathcal{W} \ni \phi \mapsto h_\phi \in C^{r+\alpha}(M,\mathbb{R})$;
- The conformal measure function, in particular for any fixed $g \in C^{r+\alpha}(M,\mathbb{R})$ the map $\phi \mapsto \int g \, d\nu_\phi$ is analytic; and
- For any fixed $g \in C^{r+\alpha}(M,\mathbb{R})$ the map $\phi \mapsto \int g \, d\mu_\phi$ is analytic.

Statement of the main results

Theorem B: For any $n \geq 1$ and fixed $g \in C^{r+\alpha}(M, \mathbb{R})$ it holds for all $H \in C^{r+\alpha}(M, \mathbb{R})$

- $D_\phi \lambda_{\phi\ |\phi_0} \cdot H = \lambda_{\phi_0} \cdot \int h_{\phi_0} \cdot H\, d\nu_{\phi_0};$
- $D_\phi h_{\phi\ |\phi_0} \cdot H = h_{\phi_0} \cdot \int \left[(I - \tilde{\mathcal{L}}_{\phi_0\ |_{E_0}})^{-1} (1 - h_{\phi_0}) \right] \cdot H\, d\nu_{\phi_0};$
- $D_\phi \int g d\nu_{\phi\ |\phi_0} \cdot H = \int (I - \tilde{\mathcal{L}}_{\phi_0\ |E_0})^{-1} \big(g - \int g d\nu_{\phi_0} \cdot h_{\phi_0} \big) \cdot H d\nu_{\phi_0};$
- $$D_\phi \int g\, d\mu_{\phi\ |\phi_0} \cdot H = \int (I - \tilde{\mathcal{L}}_{\phi_0\ |E_0})^{-1} \left(g h_{\phi_0} - h_{\phi_0} \int g\, d\mu_{\phi_0} \right) \cdot H d\nu_{\phi_0} + \int g\, d\mu_{\phi_0} \cdot \int \left[(I - \tilde{\mathcal{L}}_{\phi_0\ |E_0})^{-1} (1 - h_{\phi_0}) \right] \cdot H d\nu_{\phi_0}.$$

Statement of the main results

Theorem C: Assume $r \geq 1$ and $\alpha > 0$. If $\phi \in C^{r+\alpha}(M, \mathbb{R})$ satifies (P') and $\mathcal{F}^{r+\alpha}$ is open subset of local diffeomorphisms such that (H1), (H2) hold with uniform constants. Then for $g \in C^{r+\alpha}(M, \mathbb{R})$:

- $Diff^{r+\alpha}_{loc} \ni f \mapsto \mathcal{L}^n_f(g) \in C^{r+\alpha}(M, \mathbb{R})$ is differentiable;
- The pressure $f \mapsto P_{\text{top}}(f, \phi)$ is differentiable;
- If $\phi \equiv 0$ the maximal entropy measure function $\mathcal{F}^{r+\alpha} \ni f \mapsto \mu_f$ is differentiable and

$$D_f \mu_f(g)_{|f_0} \cdot H = \sum_{i=0}^{\infty} \int D_f \tilde{\mathcal{L}}_f(\tilde{\mathcal{L}}^i_{f_0}(P_0(g))) \cdot H d\mu_{f_0}$$

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Rmk: Precise formulas the derivatives are obtained!

Statement of the main results

Corollary 1: Assume $r \geq 1$ and $\alpha > 0$, $\mathcal{F}^{r+\alpha}$ open subset of $\mathcal{C}^{r+\alpha}$ local diffeomorphisms so that (H1), (H2) hold and $\mathcal{W}^{1+\alpha}$ open set of potentials satisfying (P'), with uniform constants. Given a differentiable map $f \mapsto \phi_f \in \mathcal{W}^{1+\alpha}$ then the pressure function $f \mapsto P_{\text{top}}(f, \phi_f)$ is differentiable. In particular, if t is small enough the pressure function

$$\mathcal{F}^{2+\alpha} \ni f \mapsto P_{\text{top}}(f, -t \log \|Df^{\pm 1}\|)$$

is differentiable.

Statement of the main results

Corollary 2: Assume $r \geq 1$ and $\alpha > 0$, $\mathcal{F}^{r+\alpha}$ open subset of $\mathcal{C}^{r+\alpha}$ local diffeomorphisms so that (H1), (H2) hold and $\mathcal{W}^{1+\alpha}$ open set of potentials satisfying (P'), with uniform constants. Then

$$(f,\phi) \mapsto h_{\mu_{f,\phi}}(f)$$

and the Lyapunov exponent functions

$$f \mapsto \int \log \|Df(x)\| \, d\mu_{f,\phi} \quad \text{and} \quad f \mapsto \int \log \|Df(x)^{-1}\|^{-1} \, d\mu_{f,\phi}$$

and

$$f \mapsto \int \log |\det Df(x)| \, d\mu_{f,\phi}$$

are continuous.

Statement of the main results

Theorem E: Let ψ be a Hölder not cohomologous to a constant. Exists interval $J \subset \mathbb{R}$ so that for any $f \in \mathcal{F}$ and $[a, b] \subset J$

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_\phi \left(x \in M : \frac{1}{n} S_n\psi(x) \in [a, b] \right) \leq - \inf_{s\in[a,b]} I_{f,\psi}(s)$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_\phi \left(x \in M : \frac{1}{n} S_n\psi(x) \in (a, b) \right) \geq - \inf_{s\in(a,b)} I_{f,\psi}(s)$$

Moreover, rate function $(s, f) \mapsto I_{f,\psi}(s)$ is continuous on $J \times \mathcal{F}$.

Strategy

1. Consider the Ruelle-Perron-Frobenius operator on the space of smooth observables $\mathcal{L}_{f,\phi} : C^{r+\alpha}(M,\mathbb{R}) \to C^{r+\alpha}(M,\mathbb{R})$ given by

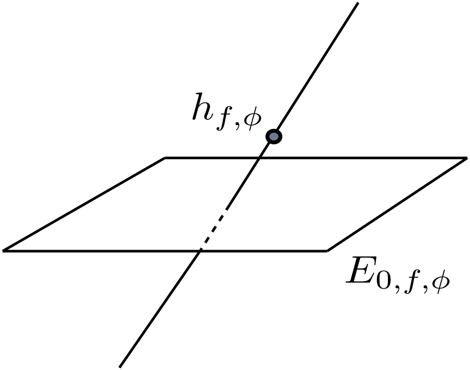
$$\mathcal{L}_{f,\phi}(g)(x) = \sum_{f(y)=x} e^{\phi(x)} g(y)$$

2. From [CV11], $\mathcal{L}_{f,\phi}$ has a **spectral gap**: there exists $\lambda_{f,\phi} > 0$,
   i. *Leading eigenfunction:* $h_{f,\phi} \in C^{r+\alpha}(M,\mathbb{R})$ such that
   $$\mathcal{L}_{f,\phi}\, h_{f,\phi} = \lambda_{f,\phi}\, h_{f,\phi}$$
   ii. *Leading eigenmeasure:* $\nu_{f,\phi} \in \mathcal{M}_1(f)$ such that
   $$\mathcal{L}^*_{f,\phi}\, \nu_{f,\phi} = \lambda_{f,\phi}\, \nu_{f,\phi}$$
   iii. *Spectral decomposition and projection:*

$$C^{r+\alpha}(M,\mathbb{R}) = \left\{ \ell h_{f,\phi} : \ell \in \mathbb{R} \right\} \oplus \left\{ g \in C^{r+\alpha}(M,\mathbb{R}) : \int g \, d\nu_{f,\phi} = 0 \right\}$$

$\lambda^{-1}_{f,\phi} \mathcal{L}_{f,\phi}$ is a contraction in $E_{0,f,\phi}$

Strategy



The transfer operators $(f, \phi) \mapsto \mathcal{L}_{f,\phi}$ do not vary continuously!

Strategy for differentiability of the pressure

1. We prove that for any $x \in M$

$$\log \lambda_{f,\phi} = P_{\text{top}}(f,\phi) = \lim_{n\to+\infty} \frac{1}{n} \log \mathcal{L}^n_{f,\phi}(1)(x)$$

2. For any $g \in C^{r+\alpha}$ the map $(f,\phi) \mapsto \mathcal{L}_{f,\phi}(g)$ is differentiable
3. Strong spectral gap property (obtained by projective cones method) guarantees uniform convergence of the derivatives of the terms in the previous expression.

Thank you!