Volume of Transitive Sets of Partially Hyperbolic Systems

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Outline

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Introduction

Definition (Transitive Set)

- *Let M be a closed manifold, $f : M \to M$ be a diffeomorphism.*
- *$x \in M$ is a transitive point of f if $\mathcal{O}_f(x) = \{f^n x : n \in \mathbb{Z}\}$ is dense in M.*
- $\mathrm{Tran}_f =:$ *the set of transitive points of f.*
- *f is transitive if $\mathrm{Tran}_f \neq \emptyset$.*

Remark

The set Tran_f is always residual whenever it is nonempty. So topologically large.

Motivations

Let m be a normalized volume measure on M: $m(M) = 1$.

Note

A topologically large set is not necessarily measure-theoretically large.

Examples

The set of Liouville numbers in $\mathbb{R}$: residual but zero measure.
More specifically:

$$E = \bigcap_{k\geq 1} \bigcup_{n\geq 1} B(r_n, \frac{1}{k \cdot 2^n}).$$

Introduction

Question

Is Tran_f *measure-theoretically large? More specifically, is* $m(\mathrm{Tran}_f) > 0$*?*
Or even $m(\mathrm{Tran}_f) = 1$*?*

Motivations

Bad case: $m(\mathrm{Tran}_f) = 0$

Bad example should be ample...
We (the author) do not know any concrete diffeomorphism which is transitive with $m(\mathrm{Tran}_f) = 0$.

Good ones?

- m is f-invariant and ergodic.
- general ergodicity (even m is not preserved).

Sometime maps with $m(\mathrm{Tran}_f) = 1$ are called weakly ergodic.

Outline

Anosov case

Definition

A diffeomorphism f is said to be Anosov if

- *a continuous splitting $TM = E^s \oplus E^u$,*
- *Df contracts E^s,*
- *Df expands E^u.*

Examples

- $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$.
- Hyperbolic matrix $A \in \mathrm{SL}(d, \mathbb{Z})$ acts on $\mathbb{T}^d$.
- General hyperbolic automorphism acts on nilmanifolds.
- C^1-small perturbations.

Anosov systems

Theorem (Sinaĭ 1968)

Let $f : M \to M$ be C^2 transitive Anosov. $\exists!$ Gibbs measure μ_+.

- $\text{supp}(\mu_+) = M$.
- $m(B(\mu_+, f)) = 1$: $\frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k x} \to \mu_+$ as $n \to +\infty$.
- $B(\mu_+, f) \subset \text{Tran}_f$.

So for C^2 transitive Anosov: $m(\text{Tran}_f) = 1$ (large, pretty good).

Result I

Theorem (Z. DCDS 2012)

$f \in \mathrm{PH}^2(M)$ be accessible, $\mu \ll m$ Absolutely Continuous Invariant Probability. Then

- $\mathrm{supp}(\mu) = M$ *and μ-a.e. x has a dense orbit.*
- *In particular $m(\mathrm{Tran}_f) > 0$.*

Accessible P.H.D.

Definition

A diffeo f is partially hyperbolic if

- $\exists$ *a continuous splitting* $TM = E^s \oplus E^c \oplus E^u$,
- Df *contracts* E^s *and expands* E^u,
- $E^s \prec E^c \prec E^u$.

Definition

$f \in \mathrm{PH}(M)$ *is said to be accessible if* $\forall x, y \in M$, $\exists su$*-path connecting* x *and* y.

Theorem (Dolgopyat-Wilkinson 2003)

Accessibility is C^1 *open and dense in* $\mathrm{PH}(M)$.

Hopf decomposition

Definition

Let $m(E) > 0$. E is wandering if $f^i E \cap f^j E = \emptyset$ for every $i \neq j$.
Let D_f be the 'maximal' wandering subset of f.
Put $D_f = \emptyset$ if there is no wandering set.

D_f is called the *dissipative part* of f. $C_f = M \backslash D_f$ is called the *conservative part* of f. $\{C_f, D_f\}$ is the Hopf decomposition of (m, f).

Remark

- m is not assumed to be invariant.
- $f_* m = m$ implies $C_f = M$ by Poincaré Recurrence Theorem.

Result II

Theorem (Z. arXiv:1204.0409)

$f \in \mathrm{PH}^2(M)$ be accessible. Then $m(\mathrm{Tran}_f) \geq m(C_f)$. In particular

- *either $D_f = M$: completely dissipative.*
- *or f is transitive and $m(\mathrm{Tran}_f) > 0$.*

Outline

Center Bunching

Definition

$f \in \mathrm{PH}(M)$ is center bunched if the hyperbolicity of $Df|_{E^s}$ and $Df|_{E^u}$ is stronger than the nonconformity of $DF|_{E^c}$:

$$Df|_{E^s} \prec \frac{m(DF|_{E^c})}{\|DF|_{E^c}\|} < 1 < \frac{\|DF|_{E^c}\|}{m(DF|_{E^c})} \prec Df|_{E^u}.$$

An open condition.

Center bunching

Theorem (Burns-Wilkinson 2010)

Let $f \in \mathrm{PH}^2(M)$ be center bunched. Then every bi-ess. saturated set is ess. bi-saturated.

- s-saturated $\rightsquigarrow$ ess. s-saturated
- u-saturated $\rightsquigarrow$ ess. u-saturated
- bi-saturated $\rightsquigarrow$ ess. bi-saturated
- bi-ess. saturated

Center bunched, accessible P.H.

Proposition

1. *For acip* μ: $\{x : \frac{d\mu}{dm}(x) > 0\}$ *is bi-ess. saturated.*
2. C_f *is bi-ess. saturated.*

Corollary

Let $f \in \mathrm{PH}^2(M)$ *be center bunched and accessible.*

1. *For acip* μ: $\mu \approx m$*. Hence exactly one acip and ergodic.*
2. *More generally, either* $C_f = \emptyset$ *or* $C_f = M$*. In the later case* f *is ergodic.*

Another Motivation

The (2) provides a partially generalization of the following classical dichotomy.

Theorem (Gurevich-Oseledets 1973)

Let f be C^2 Anosov. Then

- *either $\mu_+ \neq \mu_-$: then $C_f = \emptyset$ and $D_f = M$.*
- *or $\mu_+ = \mu_-$: then $\mu = \mu_+$ is an smooth measure and f is ergodic.*

Outline

Proofs

Let $f \in \mathrm{PH}(M)$, $\mathcal{W}^s$ be the foliation tangent to E^s and $W^s(x, \delta)$ be the local leaf.

Definition (Stable basis)

Let $B_n^s(x) = f^n W^s(f^{-n}x, \delta)$ for each $n \geq 0$. The collection

$$\mathcal{S} = \{B_n^s(x) : x \in M, n \geq 0\}.$$

Let $A \subset M$. A point x is an $\mathcal{S}$-density point of A if

$$\lim_{n \to +\infty} \frac{m_s(B_n^s(x) \cap A)}{m_s(B_n^s(x))} = 1. \tag{5.1}$$

Proofs

Volume of Transitive Sets

P. Zhang

Denote by A_d^s the set of $\mathcal{S}$-density points of A.

Theorem (Xia 2006)

$\mathcal{S}$ is a density basis w.r.t. m. That is, $m(A\Delta A_d^s) = 0$ every measurable subset A.

Bounded distortion estimates, absolute continuity of $\mathcal{W}^s$.

Two recurrence theorems

Recurrence on $E_\mu = \{x : \frac{d\mu}{dm}(x) > 0\}$ for acip:

Poincaré Recurrence Theorem

For every $A \subset E_\mu$, a.e. $x \in A$ will return to A infinitely many times.

Recurrence on C_f:

Halmos Recurrence Theorem

For every $A \subset C_f$, a.e. $x \in A$ will return to A infinitely many times.

Proofs

Again using bounded distortion estimates, we have

Lemma

Let A be f-invariant. If $\frac{m_s(B_n^s(x) \cap A)}{m_s(B_n^s(x))} \sim 1$, then $\frac{m_s(W^s(f^{-n}x,\delta) \cap A)}{m_s(W^s(f^{-n}x,\delta))} \sim 1$.

Applying Recurrence Theorem: every f-invariant subset $A \subset E_\mu$ (respectively, $A \subset C_f$) is ess. s-saturated. Similarly it is ess. u-saturated.

Proofs

Proposition (Burns-Dolgopyat-Pesin 2002)

ϵ-accessibility implies ϵ-transitivity.

Assume f accessible. Then $\forall \epsilon > 0$, a.e. $x \in E_\mu$ (respectively, $x \in C_f$) has ϵ-dense orbit.

Assume accessibility + center bunching. Then every f-invariant E, being *bi-ess. sat.*, is *ess. bi-sat.* and hence has volume 1 or 0.

This finishes the proof.

Thank You!