Interval Translation Maps of three intervals

Denis Volk

SISSA, Trieste
IITP RAS, Moscow

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Interval Translation Maps

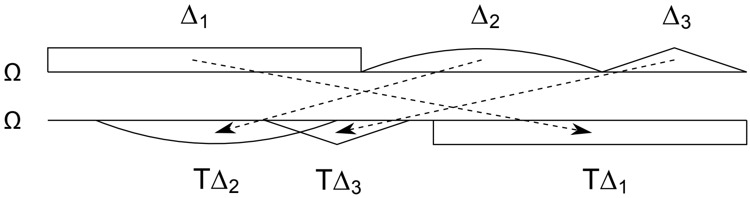


FIGURE: An Interval Translation Map of $d = 3$ intervals.

$$0 = \beta_0 < \beta_1 < \cdots < \beta_d = 1, \quad \Delta_j := [\beta_{j-1}, \beta_j),$$
$$\Omega := [0,1), \quad \Omega = \sqcup_{j=1}^{d} \Delta_j.$$

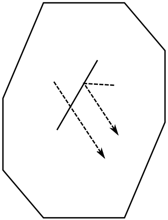
An *interval translation* $T \colon \Omega \to \Omega$ is a map given by a translation on each of Δ_j:

$$T|_{\Delta_j} : x \mapsto x + \gamma_j,$$

for some $(\gamma_1, \ldots, \gamma_d) \in \mathbb{R}^d$.

Interval Translation Maps

Introduced in 1995 by Boshernitzan, Kornfeld.



- Non-invertible generalization of Interval Exchange Transformations.
- Come from polygonal billiards with semi-permeable walls.
- Lebesgue measure is no longer invariant. New effects due to this.

Limit set

Let $\Omega_0 = \Omega$, $\Omega_n = T\Omega_{n-1}$ for $n \geq 1$.
The *limit set* X is the closure of $\cap_{n=1}^{\infty}\Omega_n$.

An interval translation map is called of *finite type* if $\Omega_{n+1} = \Omega_n$ for some n, otherwise it is called of *infinite type*.

Denote the set of infinite type ITMs by $\mathcal{S}$.

Example

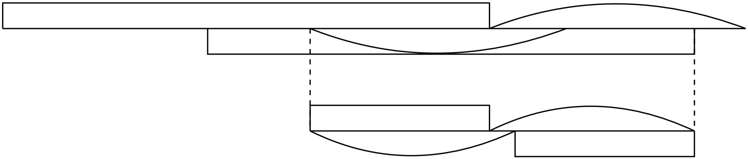


Figure: ITM of two intervals: rotation.

Finiteness results

Theorem (Boshernitzan, Kornfeld, 1995)

- $\mathrm{rk}(\beta_i, \gamma_i)_{\mathbb{Q}} \leq 2 \Rightarrow T$ *is of finite type.*
- *There exists a translation map of three intervals of infinite type.*

Theorem (Schmeling, Troubetzkoy, 1998)

- *Finite type* $\Leftrightarrow X$ *is a finite union of intervals,* $T|_X$ *is IET.*
- *Infinite type,* $T|_X$ *is transitive* $\Rightarrow X$ *is a Cantor set.*

Parameter space

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FINITENESS PROBLEM

How big is the set $\mathcal{S}$ of ITMs of infinite type?

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Finiteness Problem

How big is the set $\mathcal{S}$ of ITMs of infinite type?

Theorem (2012)

In the 5-dim space $\mathrm{ITM}(3)$*, the set $\mathcal{S}$ has zero Lebesgue measure. Moreover, from numerics (Bruin, Clack, 2011) follows*

$$4 \leq \dim_H(\mathcal{S} \cap \mathrm{ITM}(3)) \leq 4.88.$$

Strategy

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4. Infinite type $\Leftrightarrow$ $\mathcal{R}$-orbits never fall into some “Abyss” region.
5. $\mathcal{R}$ has an absolutely continuous ergodic measure m.
6. $\mathrm{m}(\mathcal{S}) = 0 \quad \Rightarrow \quad \mathrm{Leb}(\mathcal{S}) = 0.$

Inducing on a subinterval

$\Delta \subset \Omega$ is *regular* if $\forall x \in \Omega$ some $T^n x \in \Delta$, n uniformly bounded.
T_Δ is the induced map.

$\Delta \subset \Omega$ is a *trap* if it is regular and $T\Delta \subset \Delta$. Then $T_\Delta = T|_\Delta$.

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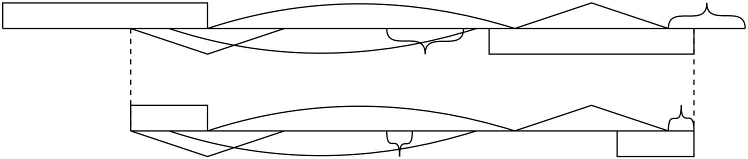
Lemma

Assume X is transitive.

- *Let T have finite type. Then for any regular Δ the map T_Δ has finite type.*
- *Let T_Δ have finite type for some regular Δ. Then T has finite type.*

DIMENSION REDUCTION



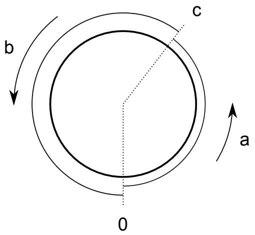
$T : \Omega \to \Omega$ is *tight* if $[\inf T\Omega, \sup T\Omega) = \Omega$.
TITM(d) is the space of tight ITMs of d intervals.

$\dim \mathrm{TITM}(d) = \dim \mathrm{ITM}(d) - 2 = 2d - 3.$

LEMMA

For any $T \in \mathrm{ITM}(d)$ there exists a trap Δ such that the map T_Δ is a tight interval translation map of r intervals, $r \leq d$.

Double rotations



Suzuki, Ito, Aihara, 2005

A *double rotation* is

$$f_{(a,b,c)}(x) = \begin{cases} \{x+a\}, & \text{if } x \in [0,c), \\ \{x+b\}, & \text{if } x \in [c,1). \end{cases}$$

Independent rotations of two complementary arcs of S^1.

$$\dim \mathrm{Rot}(2) = 3.$$

Any double rotation is an ITM of 2–4 intervals.

DOUBLE ROTATIONS

THEOREM (BRUIN, CLACK, 2011)

The set $\mathcal{S} \cap \mathrm{Rot}(2)$ *has zero Lebesgue measure.*
Moreover, numerically

$$2 \leq \dim_H(\mathcal{S} \cap \mathrm{Rot}(2)) \leq 2.88.$$

Proof by Suzuki, Ito, Aihara's renormalization in the parameter space.

Reduction to double rotations

Theorem (2012)

$\mathrm{TITM}(3) = A \cup B \cup C \ \cup \ K$*:*

- $A \cup B \cup C$ *is open and dense.*
- K *is a union of countably many hyperplanes.*

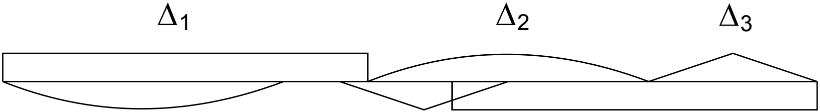
Moreover,

- *any* $T \in A$ *is a double rotation,*
- *any* $T \in B$ *is reduced to a* $\mathrm{Rot}(2)$ *via Type 1 induction,*
- *any* $T \in C$ *is reduced to a* $\mathrm{Rot}(2)$ *via Type 2 induction.*

The inductions are piecewise-invertible rational maps.

Thus zero measure sets and the Hausdorff dimension are preserved.

Combinatorics of $T\Delta_i$



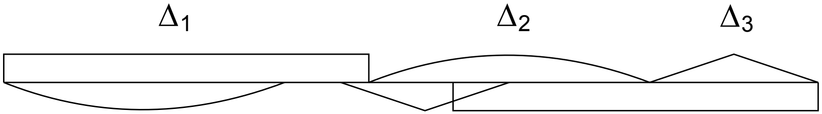
Assume $\gamma_1 > 0$ and $\gamma_3 < 0$.

Because T is tight, some interval (not Δ_1) must go to the leftmost position, and some interval (not Δ_3) must go to the rightmost position. There are 3 cases:

	A	A'	$B\&C$
Leftmost	Δ_2	Δ_3	Δ_3
Rightmost	Δ_1	Δ_2	Δ_1

The cases A and A' are mirror images of each other, so we consider only case A of these two.

Double rotation in disguise



Δ_2 goes to the leftmost position and Δ_1 goes to the rightmost position.

T is a double rotation with $c = \beta_2$ (i.e. the first arc is $\Delta_1 \cup \Delta_2$ and the second one is Δ_3) and $a = -|\Delta_1|$, $b = \gamma_3$.

INDUCTIONS

	A	A'	$B\&C$
Leftmost	Δ_2	Δ_3	Δ_3
Rightmost	Δ_1	Δ_2	Δ_1

Now Δ_1 goes rightmost, Δ_3 goes leftmost. Because of the symmetry, we can assume $|\Delta_1| \geq |\Delta_3|$.

Consider the two sub-cases: $\gamma_2 < 0$ and $\gamma_2 > 0$.

Type 1 induction

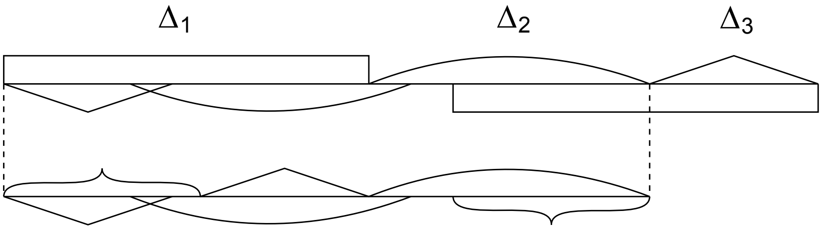


Figure: Induction to $\Delta_1 \cup \Delta_2$.

Case $\gamma_2 < 0$.

Proposition

In this case, $\Delta = \Delta_1 \cup \Delta_2$ *is regular with the return time* ≤ 2. T_Δ *is a tight ITM of three intervals which is a double rotation.*

Type 2 induction

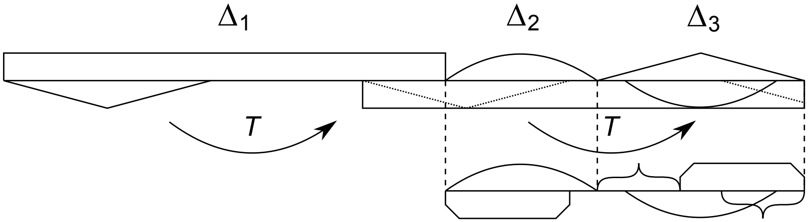


Figure: Induction to $\Delta_2 \cup \Delta_3$.

Case $\gamma_2 > 0$.

Proposition

In this case, $\Delta = \Delta_2 \cup \Delta_3$ is regular, and T_Δ is a tight ITM of three intervals which is a double rotation.

That’s it. THANK YOU!