Birkhoff sums for interval exchange maps: the Kontsevich-Zorich cocycle (VI)

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Our reference cases will be $g = 1, s = 1, \kappa_1 = 1$ and $g = 2, s = 1, \kappa_1 = 3$.

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We only consider translation surface structures which are compatible with the preferred orientation.

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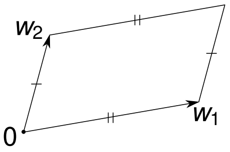

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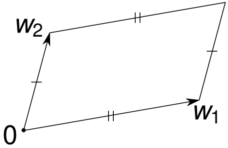

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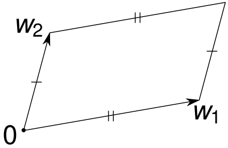

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We thus have a fibered structure over the "classical" Teichmüller space $\mathbb{H}$ of genus 1, the 1-form being specified by the additional parameter $w_1 \in \mathbb{C}^*$.

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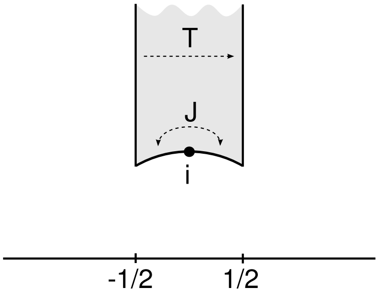
A fundamental domain is the familiar picture



with $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

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As $H^1(M, \Sigma, \mathbb{C})$ is a $\mathbb{C}$-vector space of dimension $d := 2g + s - 1$, it follows that $\mathcal{Q}(M, \Sigma, \kappa)$ is naturally a *affine complex manifold* of dimension d.

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The associated measure is the *Masur-Veech measure* on $\mathcal{Q}(M, \Sigma, \kappa)$.

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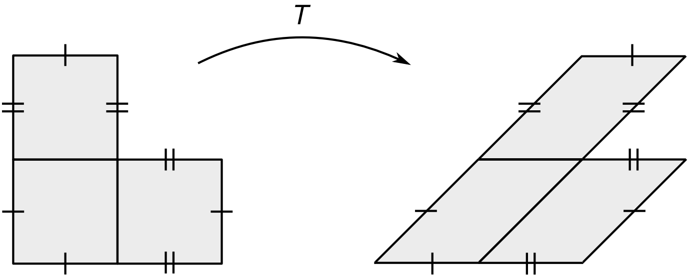

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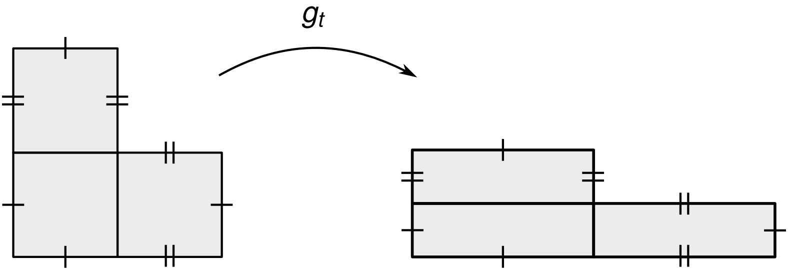
- The action of $SL(2, \mathbb{R})$ on $\mathcal{Q}(M, \Sigma, \kappa)$ or $\mathcal{M}(M, \Sigma, \kappa)$ preserves the Masur-Veech measure.

The Teichmüller flow

The *Teichmüller flow* $\mathcal{T}^t$ is the action of the diagonal subgroup $g^t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ on $\mathcal{Q}(M, \Sigma, \kappa)$ or $\mathcal{M}(M, \Sigma, \kappa)$.



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- The right endpoint of S either belongs to Σ, or to a vertical separatrix segment which does not meet S.

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Consider a *marked* translation surface M, i.e a translation surface with a preferred rightwards horizontal separatrix from a point O of Σ.

Assume that there is an initial open segment S from O on this separatrix with the following properties

- S meets every vertical connection (if any).
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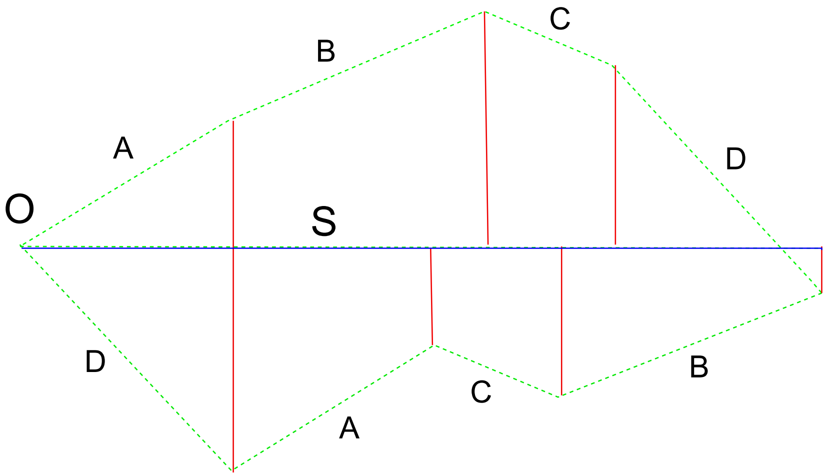
A sufficient condition for such a *nice* segment to exist is that the translation surface has no vertical connection, or no horizontal connection.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Proposition

The data for S' are obtained from the data for S by a succession of Rauzy-Veech elementary operations.