# Rigidity of certain solvable actions on the sphere

Masayuki ASAOKA *
Department of Mathematics, Kyoto University

For $k \geq 2$, *the solvable Baumslag-Solitar group* $BS(1,k)$ is defined by

$$BS(1,k) = \langle a, b \mid aba^{-1} = b^k \rangle.$$

This group admits a real-analytic and effective action $\rho_{std}$ on $S^1 = \mathbb{R}P^1 = \mathbb{R}\backslash\{0\}$ given by $\rho_{std}^a(x) = kx$, $\rho_{std}^b(x) = x + 1$ for $x \in \mathbb{R} = S^1\backslash\{\infty\}$ and $\rho_{std}^a(\infty) = \rho_{std}^b(\infty) = \infty$. In [2], Burslem and Wilkinson proved the following rigidity result on $BS(1,k)$-actions on $S^1$.

**Theorem 1** (Burslem-Wilkinson [2])**.** *For $k \geq 2$, any real-analytic and effective $BS(1,k)$-action on the circle $S^1$ is locally rigid.*

As a special case, $\rho_{std}$ is locally rigid.

In this talk, we discuss rigidity of higher dimensional analog of $\rho_{std}$. For $n \geq 1$ and $k \geq 2$, we define a solvable group $\Gamma_{n,k}$ by

$$\Gamma_{n,k} = \langle a, b_1, \ldots, b_n \mid ab_ia^{-1} = b_i^k, b_ib_j = b_jb_i \text{ for any } i,j = 1,\ldots,n \rangle.$$

Remark that $\Gamma_{1,k} \simeq \langle a, b_i \rangle \simeq BS(1,k)$. For a basis $(v_1, \ldots, v_n) \in \mathrm{GL}(n,\mathbb{R})$, we define a real-analytic $\Gamma_{n,k}$-action $\rho_B$ on the $n$-sphere $S^n = \mathbb{R}^n \cup \{\infty\}$ by $\rho_B^a(x) = kx$, $\rho_B^{b_i}(x) = x + v_i$ for $x \in \mathbb{R}^n = S^n\backslash\{\infty\}$ and $\rho_B^a(\infty) = \rho_B^{b_i}(\infty) = \infty$. It is not so hard to see that $\rho_B$ is not locally rigid if $n \geq 2$.

**Proposition 2** ([1])**.** *For $B, B' \in \mathrm{GL}(n,\mathbb{R})$, the actions $\rho_B$ and $\rho_{B'}$ are smoothly conjugate if and only if $B' = (cT)B$ for some $c > 0$ and $T \in \mathrm{O}(n)$.*

However, $\rho_B$ exhibits a form of rigidity, *i.e.*, it has a finite-dimansional deformation space in the following sense:

**Main Theorem** ([1])**.** *For any given $n, k \geq 2$ and $B \in \mathrm{GL}(n,\mathbb{R})$, any $C^\infty$ $\Gamma_{n,k}$-action $\rho$ sufficiently close to $\rho_B$ is smoothly conjugate to $\rho_{B'}$ for some $B' = B'(\rho)$. Moreover, $B'$ depends continuously on the action $\rho$.*

The proof is divided into three steps:

1. Show the persistence of the global fixed point $\infty$. It reduces the theorem to the rigidity problem of the local $\Gamma_{n,k}$-action associated with $\rho_B$.

*Partially supported by and Grant-in-Aid for Encouragement of Young Scientists (A)

2. Construct the (modified) non-linear deformation complex of the local $\Gamma_{n,k}$-action and show its finite-dimensionality.

3. Show the exactness of the (modified) linear deformation complex, and apply Weil's Implicit Function Theorem ([4])

It is natural to ask whether sub-actions of $\rho_B$ restricted to $\langle a, b_1 \rangle \simeq BS(1,k)$. For $n \geq 1$ and $v \in \mathbb{R}^n$, let $\rho_v$ be a $BS(1,k)$-action on $S^n$ given by $\rho_v^a(x) = kx$ and $\rho_v^b(x) = x + v$ for $x \in \mathbb{R}^n = S^n \backslash \{\infty\}$ and $\rho_v^a(\infty) = \rho_v^b(\infty) = \infty$. Remark that $\rho_v$ and $\rho_w$ are smoothly conjugate for any non-zero $v, w \in \mathbb{R}^n$.

**Question 3.** For $n \geq 2$, is $\rho_v$ locally rigid?

In the case $n = 2$, it can be shown that the local action at the global fixed point $\infty$ is *not* locally rigid and its deformation space as a local action is two-dimensional. However, is it unknown whether each non-trivial deformation can extend to a global action on $S^n$ or not. For $n \geq 3$, nothing is known.

## References


[1] M.Asaoka, Weak form of local rigidity of certain solvable actions on the sphere. Preprint, arXiv:1109.1902.

[2] L.Burslem and A.Wilkinson, Global rigidity of solvable group actions on $S^1$. *Geom. Topol.* **8** (2004), 877-924 (electronic).

[3] D.Fisher, Local Rigidity: Past, Present, Future. *Dynamics, Ergodic Theory and Geometry (Mathematical Sciences Research Institute Publications)*, 45–98, Cambridge University Press, 2007.

[4] A.Weil, Remarks on the cohomology of groups. *Ann. Math. (2)* **80** (1964), 149–157.