Birkhoff sums for interval exchange maps: the Kontsevich-Zorich cocycle (VII)

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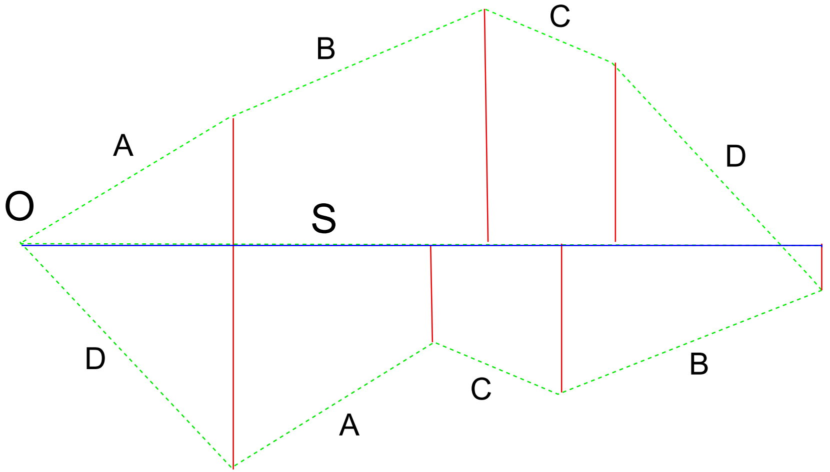
A sufficient condition for such a *nice* segment to exist is that the translation surface has no vertical connection, or no horizontal connection.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Proposition

The data for S' are obtained from the data for S by a succession of Rauzy-Veech elementary operations.

In order to obtain a fundamental domain for the action of the modular group on (marked) Teichmüller space, one asks that

$$|S| \geq 1 \geq |\widetilde{S}|,$$

where $\widetilde{S}$ is the initial segment obtained from S by **one step** of the Rauzy-Veech algorithm.

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For each combinatorial data π in a given Rauzy class $\mathcal{R}$, consider the *Veech box*

$$\mathcal{M}(\pi) := \{(\lambda, \tau, s), \lambda \in \mathbb{P}(\mathbb{R}_+^{\mathcal{A}}),\ \tau \in \mathbb{P}(\Theta_\pi), s \in [0, \log \frac{|S|}{|\widetilde{S}|}]\} \ .$$

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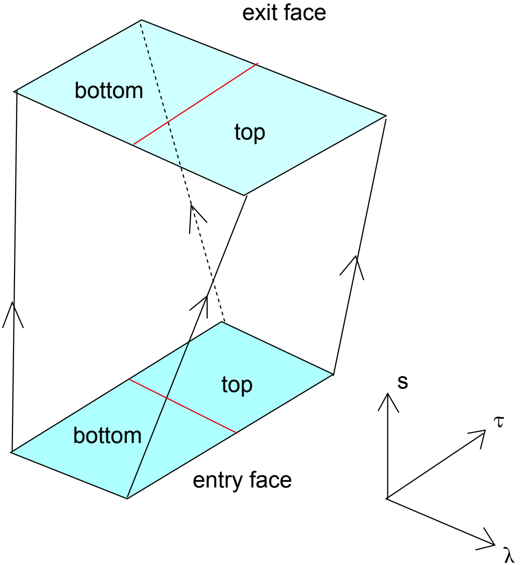
A Veech box , seen as a subset of $\mathcal{M}^1(M, \Sigma, \kappa)$,**is a flow box for the Teichmüller flow**.

A Veech box



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For every arrow $\pi \to \pi'$ of **top** type in the given Rauzy diagram $\mathcal{D}$, one glues the **top** half of the exit face of $\mathcal{M}(\pi)$ to the **top** half of the entry face of $\mathcal{M}(\pi')$,

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- The Teichmüller flow is $\mathcal{T}^t(\bar{\lambda}, \bar{\tau}, s) = (\bar{\lambda}, \bar{\tau}, s + t)$.

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- The Lyapunov exponents for the Teichmüller flow with respect to the Masur-Veech measure are obtained from those of the Kontsevich-Zorich cocycle.

The Kontsevich-Zorich cocycle (I)

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Recall the matrix $B_\gamma = \mathbf{1} + E_{\beta\alpha} \in SL(\mathbb{Z}^{\mathcal{A}})$ (with α, β the winner and loser of γ respectively). One has $B_\gamma(\mathrm{Im}(\Omega_\pi)) = \mathrm{Im}(\Omega_{\pi'})$.

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The restricted KZ-cocycle is the restriction in the fibers to $\mathrm{Im}(\Omega_\pi)$.

The Kontsevich-Zorich cocycle (I)

Let $\mathcal{D}$ be a Rauzy diagram supported by a Rauzy class $\mathcal{R}$, and let $\mathcal{V}$ be the associated Rauzy-Veech dynamics on $\mathcal{R} \times \mathbb{P}(\mathbb{R}_+^{\mathcal{A}})$.
To each iteration $\mathcal{V}(\pi, \lambda) = (\pi', \lambda')$ is associated an arrow $\gamma : \pi \to \pi'$ of $\mathcal{D}$.
Recall the matrix $B_\gamma = \mathbf{1} + E_{\beta\alpha} \in SL(\mathbb{Z}^{\mathcal{A}})$ (with α, β the winner and loser of γ respectively). One has $B_\gamma(\mathrm{Im}(\Omega_\pi)) = \mathrm{Im}(\Omega_{\pi'})$.
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(the extended cocycle acts trivially on the quotient $\mathbb{R}^{\mathcal{A}}/\mathrm{Im}(\Omega_\pi)$.

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Exercise

Use the description of the Teichmüller flow through Veech boxes to show that the discrete-time and continuous-time versions of the KZ-cocycle are indeed related!