Statistical stability for Lorenz attractors

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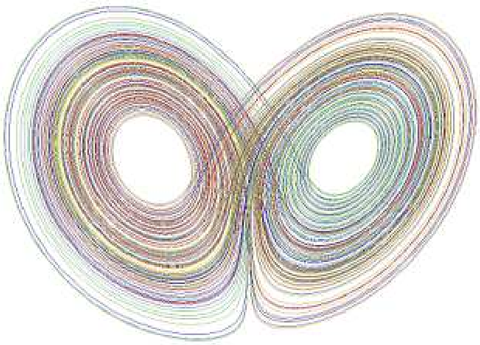
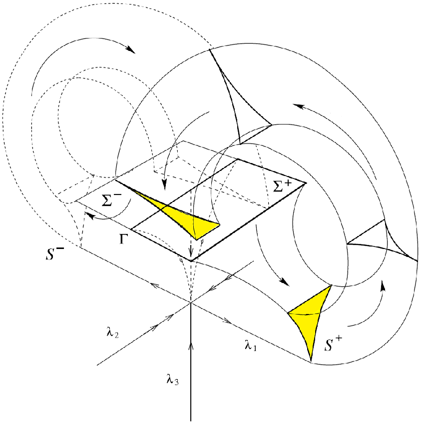
Lorenz system

Lorenz (1963) introduced a system of differential equations X in $\mathbb{R}^3$ having an attractor with sensitive dependence on the initial conditions.



Geometric model

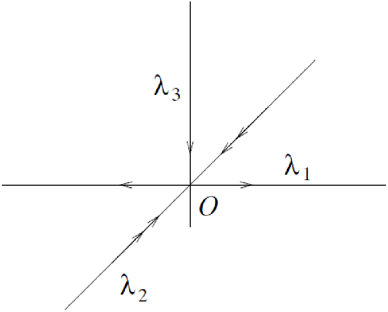
A geometric model for Lorenz equations was introduced in the seventies by Guckenheimer and Williams.



Singularity

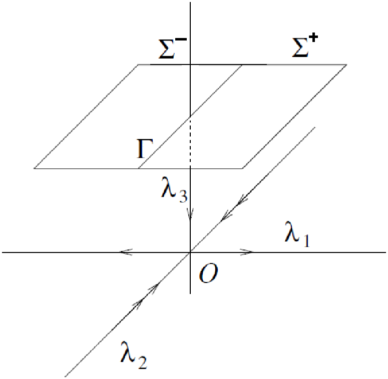
The vector field X is linear in a neighborhood of the singularity $(0, 0, 0)$ whose eigenvalues satisfy

$$0 < -\lambda_3 < \lambda_1 < -\lambda_2,$$

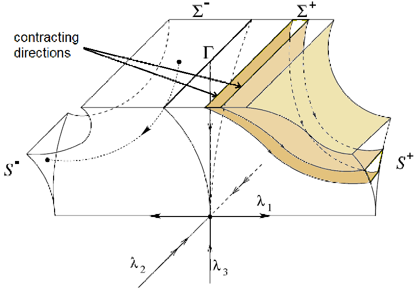


Cross-section

There is a cross-section Σ intersecting the stable manifold of the singularity along a curve Γ.

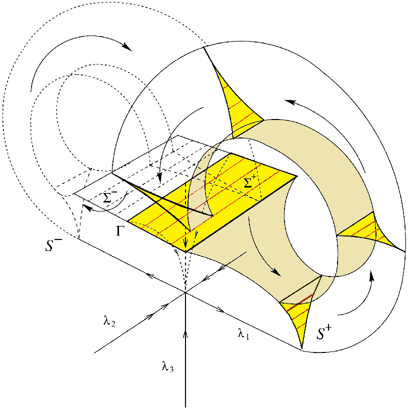


Local behaviour



$$\tau(x, y, 1) = -\frac{1}{\lambda_1} \log |x|$$

Return to Σ



$$\tau(x, y, 1) = -\frac{1}{\lambda_1} \log |x| + T_0,$$

The attractor

The geometric model admits a Lorenz-like attractor Λ:

- Λ is an invariant set under the flow;
- there is an open neighborhood U of Λ such that

$$\Lambda = \bigcap_{t>0} X_t(U);$$

- Λ contains a dense orbit;
- it has sensitive dependence on the initial conditions in U;
- Λ contains the singularity O.

Λ is a singular-hyperbolic attractor.

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Poincaré return map

The return map P admits a stable foliation $\mathcal{F}$ on Σ with the properties:

- invariant: the image by P of a leaf ξ in Σ distinct from Γ is contained in another stable leaf;
- contracting: the diameter of $P^n(\xi)$ goes to zero when $n \to \infty$, uniformly over all leaves;
- it induces a map f on the quotient space $\Sigma/\mathcal{F} \sim [-1,1] = I$.

The foliation $\mathcal{F}$ is C^1-Hölder when the vector field X is C^2.
Assuming the strong dissipative condition at the equilibrium

$$-\frac{\lambda_2}{\lambda_1} > -\frac{\lambda_3}{\lambda_1} + 2,$$

then $\mathcal{F}$ is C^2, and the one-dimensional quotient map f is C^2 smooth away from the singularity.

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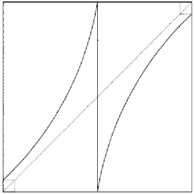

- f is discontinuous at $x = 0$ and
$$\lim_{x\to 0^+} f(x) = -1, \qquad \lim_{x\to 0^-} f(x) = 1;$$
- f is differentiable on $I \setminus \{0\}$ and
$$f'(x) > \sqrt{2}, \qquad \text{for all } x \in I \setminus \{0\};$$
- the derivative tends to infinity near 0
$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x) = +\infty.$$

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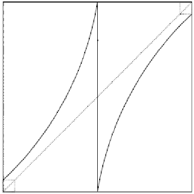

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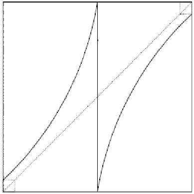

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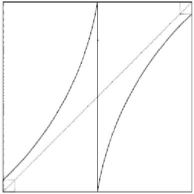

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Robustness

There is a C^2 neighborhood $\mathcal{U}$ of X such that for each $Y \in \mathcal{U}$

- U is a trapping region containing the cross-section Σ of Y;
- the maximal positively invariant subset $\Lambda_Y = \cap_{t\geq 0} Y^t(U)$ inside U is a Lorenz-like attractor;
- the first return Poincaré map P_Y on Σ admits a C^2 uniformly contracting foliation $\mathcal{F}_Y$.
- the induced one-dimensional quotient map $f_Y = P_Y/\mathcal{F}_Y$ is a C^2 Lorenz map;
- there exist (unique) SRB measures for the Lorenz map f_Y, the Poincaré map P_Y and the flow Y on U.

Theorem (Tucker)

For the classical parameter values, the Lorenz equations support a robust strange attractor.

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SRB measures

Theorem

Each Lorenz map f_Y has a unique ergodic acip $\bar{\mu}_Y$ whose density wrt Lebesgue has bounded variation.

$\bar{\mu}$ is an SRB measure: for Lebesgue almost every $x \in I$

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi \, d\bar{\mu},$$

for any continuous function $\varphi : I \to \mathbb{R}$.

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Statistical stability: continuous variation of the SRB measures with weak* topology as a function of the dynamical system.

Strong statistical stability: continuous variation of the densities of the SRB measures in the L^1-norm.

Theorem (Keller)

Lorenz maps are strongly statistically stable.

Theorem (A., Soufi)

Lorenz flows are statistically stable.

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SRB measures for the Poincaré return maps

Given a bounded function $\phi : \Sigma \to \mathbb{R}$, define

$$\phi_+(x) := \sup_{y \in \xi(x)} \phi(y) \qquad \text{and} \qquad \phi_-(x) := \inf_{y \in \xi(x)} \phi(y),$$

where $\xi(x)$ is the leaf in foliation $\mathcal{F}$ which contains x.

Lemma

Given any continuous function $\phi : \Sigma \to \mathbb{R}$ both limits

$$\lim_{n \to \infty} \int (\phi \circ P^n)_- d\bar{\mu} \qquad \text{and} \qquad \lim_{n \to \infty} \int (\phi \circ P^n)_+ d\bar{\mu}$$

exist and they coincide.

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Corollary

There is a (unique) probability P-invariant measure $\tilde{\mu}$ on Σ such that

$$\int \phi \, d\tilde{\mu} = \lim_{n\to\infty} \int (\phi \circ P^n)_- d\bar{\mu} = \lim_{n\to\infty} \int (\phi \circ P^n)_+ d\bar{\mu},$$

for every continuous function $\phi : \Sigma \to \mathbb{R}$.

Theorem

The Lorenz-like attractor supports a unique SRB measure μ defined for any continuous function $\varphi : \bar{U} \to \mathbb{R}$ as

$$\int \varphi \, d\mu = \frac{1}{\int \tau d\tilde{\mu}} \int \int_0^{\tau(x)} \varphi(X(x,t)) dt d\tilde{\mu}(x)$$

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There is a (unique) probability P-invariant measure $\tilde{\mu}$ *on* Σ *such that*

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Statistical stability for the Poincaré map

Proposition

If X_n is a sequence converging to X_0 in C^2 topology, then

$$\tilde{\mu}_n \longrightarrow \tilde{\mu}_0 \qquad \text{in weak}^* \text{ topology},$$

where $\tilde{\mu}_n = \tilde{\mu}_{X_n}$ for all $n \geq 0$.

We need to show that for any continuous $\varphi : \Sigma \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \int \varphi d\tilde{\mu}_n = \int \varphi d\tilde{\mu}_0.$$

By definition

$$\lim_{n\to\infty} \int \varphi d\tilde{\mu}_n = \lim_{n\to\infty} \lim_{m\to\infty} \int \inf(\varphi \circ P_n^m)\, d\bar{\mu}_n.$$

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We have

$$\left|\int \inf(\varphi \circ P_n^m)d\bar{\mu}_n - \int \inf(\varphi \circ P_0^m)d\bar{\mu}_0\right| \leq$$
$$\left|\int \inf(\varphi \circ P_n^m)d\bar{\mu}_n - \int \inf(\varphi \circ P_0^m)d\bar{\mu}_n\right|$$
$$+\left|\int \inf(\varphi \circ P_0^m)d\bar{\mu}_n - \int \inf(\varphi \circ P_0^m)d\bar{\mu}_0\right|.$$

The second term tends to zero because

$$\bar{\mu}_n \xrightarrow{\text{weak}^*} \bar{\mu}_0.$$

We are left to prove that the first term converges to zero when $n \to \infty$.

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\left|\int \inf(\varphi \circ P_n^m) d\bar{\mu}_n - \int \inf(\varphi \circ P_0^m) d\bar{\mu}_0\right| \leq \\
\left|\int \inf(\varphi \circ P_n^m) d\bar{\mu}_n - \int \inf(\varphi \circ P_0^m) d\bar{\mu}_n\right| \\
+ \left|\int \inf(\varphi \circ P_0^m) d\bar{\mu}_n - \int \inf(\varphi \circ P_0^m) d\bar{\mu}_0\right|.
\end{aligned}
$$

The second term tends to zero because

$$\bar{\mu}_n \xrightarrow{\text{weak}^*} \bar{\mu}_0.$$

We are left to prove that the first term converges to zero when $n \to \infty$.

Letting $\lambda =$ Lebesgue

$$
\begin{aligned}
&\left|\int \inf(\varphi \circ P_n^m) d\bar{\mu}_n - \int \inf(\varphi \circ P_0^m) d\bar{\mu}_n\right| \\
&\quad = \left|\int \inf(\varphi \circ P_n^m) \frac{d\bar{\mu}_n}{d\lambda} d\lambda - \int \inf(\varphi \circ P_0^m) \frac{d\bar{\mu}_n}{d\lambda} d\lambda\right| \\
&\quad \leq \int \left|\inf(\varphi \circ P_n^m) - \inf(\varphi \circ P_0^m)\right| \left|\frac{d\bar{\mu}_n}{d\lambda}\right| d\lambda \\
&\quad \leq C \int \left|\inf(\varphi \circ P_n^m) - \inf(\varphi \circ P_0^m)\right| d\lambda
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The rate of the contraction of the stable foliation on the cross-section can be taken the same for all vector fields. So, the last expression can be made uniformly small.

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Statistical stability for the flow

Theorem

Let X_n be any sequence converging to X_0 in C^2 topology. Then

$$\mu_n \longrightarrow \mu_0, \qquad \text{in the weak}^* \text{ topology.}$$

$\left|\int \varphi \, d\mu_n - \int \varphi \, d\mu_0\right|$ is bounded by the sum of the terms

$$\left|\frac{1}{\int \tau_n \, d\tilde{\mu}_n} - \frac{1}{\int \tau_0 d\tilde{\mu}_0}\right| \int \int_0^{\tau_0(x)} |\varphi(X_0(x,t))| dt d\tilde{\mu}_0(x),$$

and

$$\frac{1}{\int \tau_n d\tilde{\mu}_n} \left| \int \int_0^{\tau_n(x)} \varphi(X_n(x,t)) dt d\tilde{\mu}_n - \int \int_0^{\tau_0(x)} \varphi(X_0(x,t)) dt d\tilde{\mu}_0 \right|.$$

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The statistical stability of the Poincaré return map and the fact that

$$\tau_n(x, y, 1) \sim -\log |x - c_n|,$$

where the c_n is the discontinuity point of the map f_{X_n}, yield

Lemma

$$\lim_{n \to +\infty} \int \tau_n \, d\tilde{\mu}_n = \int \tau_0 \, d\tilde{\mu}_0$$

And defining

$$h_n(x) = \int_0^{\tau_n(x)} \varphi(X_n(x, t)) \, dt, \quad \text{for } n \geq 0$$

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Contracting Lorenz flow

Replace the usual expanding condition $\lambda_3 + \lambda_1 > 0$ in the original Lorenz vector field by the contracting condition

$$\lambda_3 + \lambda_1 < 0.$$

There is a trapping region U for X_0 on which $\Lambda = \cap_{t \geq 0} X_0^t(U)$ is a singular-hyperbolic attractor.

Λ is 2-dimensionally almost persistent in the C^3 topology: X_0 is a 2-dimensional density point of the set of vector fields Y for which $\Lambda_Y = \cap_{t \geq 0} Y^t(U)$ is an attractor.

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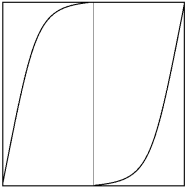
The quotient map $f_0 : I \setminus \{0\} \to I$ satisfies:

- $\lim_{x \to 0^{\pm}} f_0(x) = \mp 1$;
- ± 1 are pre-periodic and repelling;
- f_0 is of class C^3 on $I \setminus \{0\}$ with negative Schwarzian derivative;



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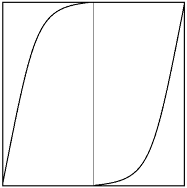

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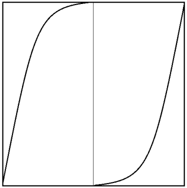

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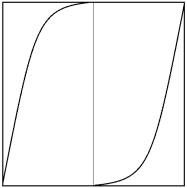

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Theorem (Rovella)

There is $E \subseteq [0,1]$ with full density at 0 such that:

1. *for all $a \in E$, f_a is of class C^3 on $x \in I \setminus \{0\}$ and satisfies*

$$K_2|x|^{s-1} \le f_a'(x) \le K_1|x|^{s-1};$$

2. *there exists $\lambda > 1$ such that for $a \in E$*

$$(f_a^n)'(\pm 1) > \lambda^n \quad \text{for all } n \ge 0;$$

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Assume f is nonuniformly expanding:

$$\exists c > 0 : \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(f'(f^i(x))) > c, \qquad \text{Lebesgue a.e. } x$$

with slow recurrence to the critical set:

$$\forall \epsilon > 0 \ \exists \delta > 0 : \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} -\log d_\delta(f^i(x), \mathcal{C}) \le \epsilon, \qquad \text{Lebesgue a.e. } x$$

where d_δ is the δ-truncated distance is defined as

$$d_\delta(x,y) = \begin{cases} |x-y| & \text{if} \quad |x-y| \le \delta, \\ 1 & \text{if} \quad |x-y| > \delta. \end{cases}$$

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This allows us to introduce the expansion time

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the recurrence time

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Assume there are $C > 1$ and $\gamma > 1$ such that $\left|\Gamma_n^f\right| \leq Cn^{-\gamma}$ for all $n \geq 1$ and $f \in \mathbf{F}$. Then, each $f \in \mathbf{F}$ is strongly statistically stable.

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Rovella maps are nonuniformly expanding with slow recurrence to the critical set. Moreover, there are $C, \tau > 0$ such that for all $n \in \mathbb{N}$ and $a \in E$,

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