Birkhoff sums for interval exchange maps: the Kontsevich-Zorich cocycle (VIII)

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ICTP, May 30, 2012

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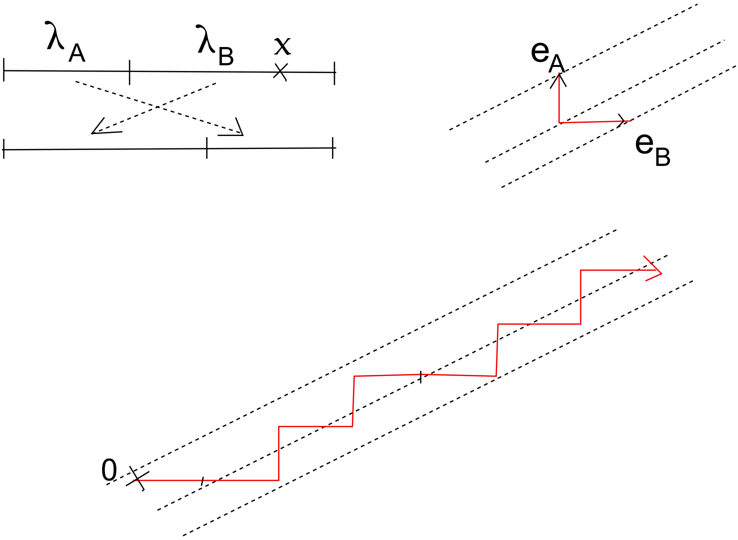
The sequence $(S_j(T,x))_{j\geq 0}$ defines a broken line in $\mathbb{R}^{\mathcal{A}}$ with segments parallel to the axes.

The case $d = 2$



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- Moreover, in this last case, there is a 2-dimensional subspace $D_2(\lambda) \supset D_1(\lambda)$ (independent of x) such that
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- This goes on till defining $\theta_g > 0$ and a g-dimensional subspace $D_g(\lambda)$ such that

$$\limsup_j \frac{\log(\mathrm{dist}(S_j(T, x), D_g(\lambda)))}{\log j} = 0.$$

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- The relation between $S_j(T,x)$ and the Lyapunov exponents of the KZ-cocycle (Zorich).
- The properties of these Lyapunov exponents (Forni,Avila-Viana).

Oseledets theorem for the KZ-cocycle

After the Zorich time reparametrization, the Rauzy-Veech dynamics $\mathcal{V}_Z$ on $\mathcal{R} \times \mathbb{P}(\mathbb{R}_+^{\mathcal{A}})$ is ergodic for the unique invariant probability which is a.c (in fact, equivalent) w.r.t. Lebesgue.

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The *multiplicity* of ω_i is $\dim E_{i-1} - \dim E_i$ (independent of (π, λ)).

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Forni had proved that 0 is not an exponent of the restricted KZ-cocycle.

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We explain thus (assuming the simplicity of the Lyapunov spectrum) a special case of the experimental results.

See Blackboard

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$$\begin{aligned} 2 = 1 + \theta_1 > \ldots &> 1 + \theta_g > 1 - \theta_g > 1 - \theta_1 = 0 = -1 + \theta_1 \\ > \ldots &> -1 + \theta_g > -1 - \theta_g > \ldots > -1 - \theta_1 = -2 \end{aligned}$$

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Theorem (Zorich)

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$$\begin{aligned} 2 = 1 + \theta_1 > \dots \quad > \quad & 1 + \theta_g > 1 - \theta_g > 1 - \theta_1 = 0 = -1 + \theta_1 \\ > \dots \quad > \quad & -1 + \theta_g > -1 - \theta_g > \dots > -1 - \theta_1 = -2 \end{aligned}$$

which are simple, and ± 1 with multiplicity $s - 1$.

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The second involves a deep combinatorial study of Rauzy diagrams. It is based on an induction on d: starting with a Rauzy diagram $\mathcal{D}$, Avila-Viana show how to relate $\mathcal{D}$ to simpler diagrams obtained by erasing one or several letters of the alphabet $\mathcal{A}$.

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- For $\underline{\ell} \in \Lambda^n$,
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- μ_- is the projection of $\widehat{\mu}$ on Σ_-.

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  Other Lie groups ($GL(d, \mathbb{R})$, $U(p, q)$, ...) may be considered with minor adaptations.
- For $1 \leqslant k \leqslant \frac{d}{2}$, $G(k)$ is the Grasmannian space of **isotropic** k-planes, $G(d-k)$ is the Grasmannian space of **coisotropic** $d-k$-planes.

Pinching

The cocycle is *pinching* if there exists $\underline{\ell}^* \in \Omega$ such that the matrix $A^{\underline{\ell}^*}$ has simple spectrum.

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Twisting

The pinching cocycle is *twisting* if , for some $\underline{\ell}^* \in \Omega$ as above, and any $1 \leqslant k \leqslant \frac{d}{2}$, there exists $\underline{\ell}(k) \in \Omega$ s.t.

$$A^{\underline{\ell}(k)} F \bigcap F' = \{0\},$$

for any $F \in G(k), F' \in G(d-k)$ which are **invariant under** $A^{\underline{\ell}^*}$**.**

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Theorem Avila-Viana

Under these conditions, the cocycle defined by $(A_\ell)_{\ell \in \Lambda}$ has simple Lyapunov spectrum w.r.t. μ.