# *Quantisation and Mabuchi energy*

Joel Fine

Université Libre de Bruxelles

ICTP Trieste,
28th June, 2012

# *Brief review of Mabuchi energy*

$L \to X$ ample line bundle over compact complex manifold.

Want to find "canonical" Kähler metric in $c_1(L)$.

Calabi's suggestion: minimise $L^2$-norm of curvature. Critical points are "extremal metrics", $\bar{\partial}(\nabla \operatorname{Scal}) = 0$.

When there are no holomorphic vector fields, these are just constant scalar curvature Kähler metrics. In this talk, mainly interested in this case.

Mabuchi phrased this problem in terms of the geometry of $\mathcal{H}$, the set of Kähler metrics in $c_1(L)$ and a special function, now called Mabuchi energy.

# *Brief review of Mabuchi energy*

$\mathcal{H}$ carries natural metric (Donaldson–Mabuchi–Semmes):

$$T_\omega \mathcal{H} = C_0^\infty(X, \mathbb{R}) \quad \text{and} \quad (\phi, \psi)_\omega = \int_X \phi\psi \, \frac{\omega^n}{n!}.$$

- This metric is non-positively curved.
- It is symmetric in the sense that $\nabla \operatorname{Rm} = 0$.

Mabuchi energy is function $E \colon \mathcal{H} \to \mathbb{R}$ defined up to a constant via the formula for its differential:

$$\mathrm{d}E_\omega(\phi) = \int_X \phi \, \operatorname{Scal}(\omega) \, \frac{\omega^n}{n!}.$$

## *Brief review of Mabuchi energy*

- Critical points of $E$ are constant scalar curvature Kähler metrics.
- $E$ is geodesically convex.
- Downward gradient flow of $E$ is Calabi flow:

$$\frac{\partial \omega}{\partial t} = 2i\bar{\partial}\partial \operatorname{Scal}(\omega).$$

Can understand geodesic convexity of $E$ in terms of its Hessian:

$$\operatorname{Hess} E = \mathcal{D}^* \mathcal{D},$$

where $\mathcal{D}(f) = \bar{\partial}(\nabla f)$.

Fourth order elliptic operator which is essential part of linearisation of scalar curvature operator and Calabi flow.

# *Brief review of Mabuchi energy*

What one might dream is true:

- There is a cscK metric iff the derivative of $E$ in every direction at infinity is positive. I.e., iff $E$ is "proper" in some sense.
- Running to infinity in $\mathcal{H}$ is a metric degeneration. Pulling back by diffeos can see it as a complex degeneration. Then derivative of $E$ in this direction should be calculable from this degeneration, the "Futaki invariant".
- So there is a cscK metric iff all Futaki invariants are positive, i.e., $(X, L)$ is K-stable. The famous DTY conjecture.
- If $(X, L)$ is K-stable then Calabi flow should find the cscK metric. If not it should metrically carry out the "worst" degeneration.

# *Projective metrics*

Given projective embedding $X \hookrightarrow \mathbb{CP}^N$ via holomorphic sections of $L$, restrict Fubini–Study metric to get $\omega \in \mathcal{H}$.

Can do this for higher and higher powers of $L$ to get more and more embeddings: a basis of $H^0(X, L^k)$ gives embedding $\iota\colon X \hookrightarrow \mathbb{CP}^{N_k}$ and hence a projective metric $\frac{1}{k}\iota^*\omega_{\mathsf{FS}} \in \mathcal{H}$.

Write $\mathcal{B}_k \subset \mathcal{H}$ for those projective metrics obtained via $H^0(X, L^k)$.

Have map in other direction $\mathcal{H} \to \mathcal{B}_k$, given as follows.

Given $\omega \in \mathcal{H}$, choose Hermitian metric $h$ in $L$ with curvature $\omega$. Now use $h^k$ and $\omega$ to define $L^2$-innerproduct on sections of $L^k$ and choose orthonormal basis of $H^0(X, L^k)$ to embed. Write $\omega_k \in \mathcal{H}$ for resulting projective metric.

# *Density of projective metrics*

**Theorem (Tian)**

*$\bigcup \mathcal{B}_k$ is dense in $\mathcal{H}$. More precisely, $\omega_k \to \omega$ as $k \to \infty$.*

$\omega_k = \omega + \frac{i}{k}\bar{\partial}\partial \log \rho_k$ where

$$\rho_k(x) = \sum |s_j(x)|^2$$

for orthonormal basis $s_j$ of holomorphic sections of $L^k$.

So Tian's theorem says $\rho_k$ is asymptotically constant. In fact,

$$\rho_k(x) = k^n + O(k^{n-1}).$$

(Here $n = \dim X$.)

To leading order in $k$, the sections $s_j$ are "evenly spread out".

## *Peaked sections*

To see why, consider trivial bundle $L \to \mathbb{C}^n$ with $\omega = 2i\mathrm{d}z \wedge \mathrm{d}\bar{z}$.

Hermitian metric in $L$ is $h = e^{-|z|^2}$, so metric in $L^k$ is $e^{-k|z|^2}$.

"Constant" section $s$ of $L^k = \mathbb{C} \times \mathbb{C}^n$ with unit $L^2$-norm has

$$|s|^2 = k^n e^{-k|z|^2}.$$

Now go back to $(L^k, X, k\omega)$. For large $k$, over a fixed ball in $X$ the metric $k\omega$ is almost flat.

"Glue in" above picture at $x \in X$ to get holomorphic section $s$ of $L^k$, localised at $x$, with $|s(x)|^2 \sim k^n$.

Any section $L^2$-orthogonal to $s$ must be very small at $x$, proving Tian's theorem.

## *Quantisation*

Tian's theorem says that projective metrics "at level $k$" approximate all Kähler metrics as $k \to \infty$.

In quantisation, aim to associate to each object defined on $\mathcal{H}$ a counterpart defined on $\mathcal{B}_k$. These counterparts should converge in some sense to original object as $k \to \infty$.

Reason for name "quantisation" is that $H^0(X, L^k)$ is supposed to be "state-space" of wave functions. As $k \to \infty$, can produce sections which are more and more peaked at points.

So think of $1/k$ as Planck's constant and $k \to \infty$ as the classical limit.

# *Balanced embeddings*

Donaldson worked out the "correct" quantisation of cscK metrics and Mabuchi energy.

$\mathbb{CP}^N \subset \mathrm{Herm}_0(N+1)$, a Euclidean vector space via $\langle A, B\rangle = \mathrm{tr}(AB)$. So can talk about centre of mass of $X \subset \mathbb{CP}^N$.

A projective submanifold is called *balanced* if it has centre of mass zero.

*Theorem (Luo, Zhang)*

*A complex submanifold $X \subset \mathbb{CP}^N$ can be moved via $\mathrm{GL}(N+1, \mathbb{C})$ to a balanced submanifold if and only if it is Chow stable.*

# *Balancing energy*

Balanced embeddings are critical points of function called balancing energy.

Given $X \subset \mathbb{CP}^N$, write $\mathcal{B}$ for all projectively equivalent embeddings mod unitary equivalence.

Balancing energy is function $F \colon \mathcal{B} \to \mathbb{R}$.

- Critical points of $F$ are balanced embeddings.
- $F$ is geodesically convex wrt natural negatively curved symmetric metric on $\mathcal{B} \cong \mathrm{SL}(N+1)/\,\mathrm{SU}(N+1)$.
- Balanced embedding exists iff derivatives of $F$ in all directions at infinity in $\mathcal{B}$ are positive.
- Directions at infinity are one-parameter subgroups of $\mathrm{SL}(N+1)$. Derivative at infinity in a direction is exactly the corresponding Chow weight, hence Luo–Zhang's theorem.

# *Quantisation of cscK metrics*

Donaldson's key observation:

- Balanced embeddings are the quantisation of cscK metrics.
- Balancing energy on $\mathcal{B}_k$ is the quantisation of Mabuchi energy.
- Luo–Zhang theorem is the quantisation of the DTY conjecture.

More precisely:

**Theorem (Donaldson)**

*Assume that $\mathrm{Aut}(X, L)/\mathbb{C}^*$ is discrete. Suppose also that $c_1(L)$ contains a cscK metric $\omega_{csc}$. Then for all large $k$, $\mathcal{B}_k$ contains a balanced embedding $\theta_k \in \mathcal{B}_k$ which is unique. Moreover, $\theta_k \to \omega_{csc}$ as $k \to \infty$.*

## *From cscK to balanced*

Important fact: $\omega \in \mathcal{B}_k$ is balanced iff $\omega = \omega_k$.

I.e., $(L, X) \subset (\mathcal{O}(1), \mathbb{CP}^N)$ is balanced iff when we pull back Fubini–Study, then use this to define $L^2$-innerproduct on $H^0(X, L)$, a second embedding of $X$ by an $L^2$-orthonormal basis gives us the *same* metric.

Recall $\omega_k = \omega + \frac{i}{k}\bar{\partial}\partial \log \rho_k(\omega)$ where

$$\rho_k(\omega) = k^n + O(k^{n-1})$$

In particular, $\rho_k(\omega_k) = k^n + O(k^{-1})$ and so given any $\omega$, the $k^{\text{th}}$ approximation $\omega_k$ is *nearly* balanced.

*Theorem (Catlin, Lu, Tian, Zelditch)*

$$\rho_k(\omega) = k^n + \mathrm{Scal}(\omega)k^{n-1} + \cdots$$

So if $\omega$ is cscK, $\omega_k$ is *really nearly* balanced!

# *From cscK to balanced*

Scalar curvature appears because as we try to push peaked sections together their essential supports take up volume.

Scalar curvature measures the difference in volume of small balls from Euclidean case.

To complete proof, want to flow $\omega_k$ down gradient of balancing energy $F_k \colon \mathcal{B}_k \to \mathbb{R}$ to reach a minimum.

To do this need uniform control of (amongst other things) the first eigenvalue of the Hessian of $F_k$ along the flow.

# *Convergence of the flows*

Downward gradient flow of Mabuchi energy is Calabi flow,

$$\frac{\partial \omega}{\partial t} = 2i\bar{\partial}\partial\,\mathrm{Scal}(\omega)$$

Downward gradient flow of balancing energy is called "balancing flow".

Given $\iota \colon X \to \mathbb{CP}^N$, have centre of mass $\bar{\mu}(\iota) \in \mathrm{Herm}_0$. Defines holomorphic vector field $V_{\bar{\mu}(\iota)}$ on $\mathbb{CP}^N$.

Balancing flow is

$$\frac{\mathrm{d}\iota}{\mathrm{d}t} = -\iota \circ V_{\bar{\mu}(\iota)}$$

*Theorem (F.)*

*Let $\omega \in \mathcal{H}$ and $\omega(t)$ solve Calabi flow starting at $\omega$. Write $\omega_k \in \mathcal{B}_k$ for Tian's sequence of approximations and let $\omega_k(t)$ solve balancing flow starting at $\omega_k$. Then $\omega_k(t) \to \omega(t)$ as $k \to \infty$.*

## *Applications*

First main application of Donaldson's result is uniqueness of $\omega_{csc} \in c_1(L)$. (Result since improved by Chen–Tian, Mabuchi.)

Second application: possible to numerically approximate $\omega_{csc}$. since finding balanced metric $\theta_k$ is finite dimensional problem.

Similarly, could use balancing flows to numerically approximate Calabi flow.

Might also hope to use balancing flows to understand long-time behaviour of Calabi flow.

E.g., in situation of Donaldson's theorem, know balancing flows $\omega_k(t)$ converge for small time to Calabi flow. Also know $\omega_k(\infty) = \theta_k$ converge to $\omega_{csc}$. What happens for large $t$?

Do the final directions of the flows even converge?

# *The Hessians*

Gradient flows arrive along eigendirections of the Hessian at the critical point.

So want to understand convergence of the Hessians of balancing energy to that of Mabuchi energy.

Recall $\mathrm{Hess}(E)(f) = \mathcal{D}^*\mathcal{D}f$ where $\mathcal{D}(f) = \bar{\partial}(\nabla f)$.

Given $\theta \in \mathcal{B}_k$, consider corresponding embedding $X \to \mathbb{CP}^N$ with normal bundle $E \to X$.

$T_\theta\mathcal{B}_k \cong \mathrm{Herm}_0(N+1)$. Given $A \in \mathrm{Herm}_0$, write $V_A$ for corresponding holomorphic vector field on $\mathbb{CP}^N$.

Define $P_k \colon T_p\mathcal{B}_k \to \Gamma(E)$ by $P_k(A) = \pi(V_A)$. Then,

$$\mathrm{Hess}\, F_k = P_k^* P_k$$

(Use Fubini–Study to define $L^2$-innerproduct on $\Gamma(E)$.)

# *Convergence of the eigenvalues*

Heuristically, $k^2 \operatorname{Hess} F_k \to \operatorname{Hess} E$ as $k \to \infty$.

Write $\lambda_m$ for $m^{\text{th}}$ eigenvalue of $\operatorname{Hess} E$, at $\omega$, counted with multiplicities, $\lambda_1 \leq \lambda_2 \leq \cdots$

Write $\nu_{k,m}$ for $m^{\text{th}}$ eigenvalue of $\operatorname{Hess} F_k$, at $\omega_k$, counted with multiplicities, $\nu_{k,1} \leq \nu_{k,2} \leq \cdots$

**Theorem (F.)**

*Assume that $\operatorname{Aut}(X, L)/\mathbb{C}^*$ is discrete. Then*

$$\nu_{k,m} = k^{-2}\lambda_m + O(k^{-3}).$$

*The same result is true if $\omega$ has constant scalar curvature and $\operatorname{Hess} F_k$ is taken at the balanced embeddings $\theta_k$.*

# *Convergence of the eigendirections*

For simplicity, assume that spectrum of $\operatorname{Hess} E$ is simple

$$\lambda_1 < \lambda_2 < \cdots$$

Previous result says that for large $k$, same is true for $\operatorname{Hess} F_k$.

Write $W_p \subset C^\infty(X, \mathbb{R})$ for the $\lambda_p$-eigenspace of $\mathcal{D}^* \mathcal{D}$.

Write $U_{k,p} \subset T_{\omega_k} \mathcal{B}_k$ for the $\nu_{k,p}$-eigenspace of $P_k^* P_k$.

Then the image of $V_{k,p}$ under the derivative of the inclusion $\mathcal{B}_k \to \mathcal{H}$ converges at $O(k^{-1})$ to $W_p$ as $k \to \infty$.

This derivative sends $A$ to the Kähler potential corresponding to the change in induced metric given by moving $X \subset \mathbb{CP}^N$ in the direction $V_A$.

## *Convergence of the eigendirections*

If $\lambda_p$ is multiple eigenvalue, let $q \geq p$ be such that

$$\lambda_{p-1} < \lambda_p = \lambda_{p+1} = \cdots = \lambda_q < \lambda_{q+1}$$

Write $U_{k,p,q}$ for span of the $\nu_{k,m}$-eigenvectors, $p \leq m \leq q$.

*Theorem (F.)*

*Assume that $\mathrm{Aut}(X,L)/\mathbb{C}^*$ is discrete. Then*

1. *The derivative of the inclusion $\mathcal{B}_k \to \mathcal{H}$, taken at $\omega_k$, is $O(k^{-1})$ from an isometric embedding when restricted to $U_{k,p,q}$.*
2. *The image of $U_{k,p,q}$ is $O(k^{-1})$ from $W_p$.*

*Same result holds if $\omega$ has constant scalar curvature and we consider eigendirections of $\mathrm{Hess}\, F_k$ taken at the balanced embeddings $\theta_k$.*