Lax pairing for the Korteweg – de Vries equation

equan

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BOS

KdV

$$u_t - 6uu_x + u_{xxx} = 0$$

John Scott Russell	1834
Lord Rayleigh	~1870
Joseph Boussinsq	~1870
Korteweg & DeVries	1895 ← Gel'fand-Levitan 1951
Zabusky & Kruskal	1965
Lax	private 1967 → 1968
Gardner-Greene-Kruskal-Miura	1967

• Lax equation:

$$\boxed{\dot{\hat{L}} = [\hat{M}, \hat{L}]} \equiv \hat{M}\hat{L} - \hat{L}\hat{M}$$

$$\hat{L} = \hat{L}(t) \qquad \hat{M} = \hat{M}(t)$$

$$\boxed{L(t) = U(t)\,\overset{=L_0}{L(0)}\,U^{-1}(t)}$$

$$\boxed{U(t) = \underbrace{T\exp}\Big[\int_0^t dt'\, M(t')\Big] \equiv \sum_{n=0}^{\infty} \int_0^t dt_n \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_2} dt_1\; M(t_n)\, M(t_{n-1}) \ldots M(t_1)}$$

$$\dot{U} = M\,U$$

$$\dot{(U^{-1})} = -U^{-1}\dot{U}\,U^{-1}$$

$$\dot{L} = \dot{(U L_0 U^{-1})} = \dot{U} L_0 U^{-1} + U L_0 \dot{(U^{-1})} = M U L_0 U^{-1} - U L_0 U^{-1} \dot{U} U^{-1} =$$

$$= M \underbrace{U L_0 U^{-1}}_{L} - \underbrace{U L_0 U^{-1}}_{L} M \underbrace{U U^{-1}}_{I} = ML - LM = [M, L] \qquad \text{Q.E.D.}$$

$$\boxed{\begin{aligned} L_0 |\Psi_0\rangle &= \lambda |\Psi_0\rangle \\ |\Psi(t)\rangle &= U |\Psi_0\rangle \end{aligned}}$$

$$L(t)\,|\Psi(t)\rangle = U L_0 \underbrace{U^{-1} U}_{I} |\Psi_0\rangle = U \underbrace{L_0 |\Psi_0\rangle}_{\lambda |\Psi_0\rangle} = \lambda\, |\Psi(t)\rangle$$

$$\boxed{L(t)\,|\Psi(t)\rangle = \lambda\, |\Psi(t)\rangle}$$

Idea:

$\hat{L} = \hat{L}(u(x,t))$ = solvable IS probem $= -\partial_x^2 + u(x,t)$

$\hat{M} = \hat{M}(u(x,t)) = ?$

<u>nontrivial</u> equation for $u(x,t)$

$$\boxed{M = M_3(x,t)\,\partial_x^3 + M_2(x,t)\,\partial_x^2 + M_1(x,t)\,\partial_x + M_0(x,t)} \leftarrow \text{ansatz}$$

$$\boxed{M_n(x,t) = M_n(u(x,t), u_x(x,t), u_{xx}(x,t), \ldots)}$$

$$u_t = (M_3\,\partial_x^3 + M_2\,\partial_x^2 + M_1\,\partial_x + M_0)(-\partial_x^2 + u) - (-\partial_x^2 + u)(M_3\,\partial_x^3 + M_2\,\partial_x^2 + M_1\,\partial_x + M_0)$$

$$= A_4(x)\,\partial_x^4 + A_3(x)\,\partial_x^3 + A_2(x)\,\partial_x^2 + A_1(x)\,\partial_x + A_0(x)$$

$u(x,t)$

$A_n(x) = 0 \qquad n = 1, 2, 3, 4$

$u_t = A_0(x)$! $\quad \Rightarrow \quad u_t = +M_1 u_x + (M_0)_{xx} + M_2 u_{xx} + M_3 u_{xxx}$

$u(x,t)$

equations = 4

☺

unknown functions = 4

A_n(x) should read A_n(x,t)

$A_4 = 0 \quad \Rightarrow \quad +2(M_3)_x = 0$

$A_3 = 0 \quad \Rightarrow \quad +2(M_2)_x + (M_3)_{xx} = 0$

$A_2 = 0 \quad \Rightarrow \quad +2(M_1)_{xx} + 3M_3 u_x + (M_2)_{xx} = 0$

$A_1 = 0 \quad \Rightarrow \quad +2(M_0)_x + 2M_2 u_x + (M_1)_{xx} + 3M_3 u_{xx} = 0$

want no explicit dependence on x or t

$$M_3(x,t) = m_3(t) = m_3 = \text{const}$$

$$M_2(x,t) = m_2(t) = m_2 = \text{const}$$

$$M_1(x,t) = -\frac{3}{2} m_3\, u(x,t) + \tilde{m}_1$$

$$M_0(x,t) = -\frac{3}{4} m_3\, u_x(x,t) - m_2\, u(x,t) + \tilde{m}_0$$

$$\Downarrow$$

$$u_t = \left(-\frac{3}{2} m_3 u + \tilde{m}_1\right) u_x + \frac{1}{4} m_3\, u_{xxx}$$

$$\Downarrow$$

$$\boxed{u_t + \left(\frac{3}{2} m_3 u - \tilde{m}_1\right) u_x - \frac{1}{4} m_3\, u_{xxx} = 0}$$

$$u(x,t) = \tilde{v}(x, \underbrace{t/(m_3/4)}_{t'})$$

$$-\frac{m_3}{4} \tilde{v}_{t'} + \left(\frac{3}{2} m_3 \tilde{v} - \tilde{m}_1\right) \tilde{v}_x - \frac{m_3}{4} v_{xxx} = 0$$

$$\tilde{v}_{t'} + \left(-6\tilde{v} + 4\frac{\tilde{m}_1}{m_3}\right) \tilde{v}_x + v_{xxx} = 0$$

$$\tilde{v}(x,\mathring{t}) = \tilde{\tilde{v}}(x,\mathring{t}) + \frac{2}{3}\frac{\tilde{m}_1}{m_3}$$

$$\boxed{\tilde{\tilde{v}}_{\mathring{t}} - 6\,\tilde{\tilde{v}}\,\tilde{\tilde{v}}_x + \tilde{\tilde{v}}_{xxx} = 0}$$

Without loss of generality:

$$\boxed{u_t - 6uu_x + u_{xxx} = 0}$$

Without loss of generality:

$$\boxed{\begin{array}{ll} m_3 = -4 & \tilde{m}_1 = 0 \\ m_2 = 0 & \tilde{m}_0 = 0 \end{array}}$$

$$\hat{M} = -4\partial_x^3 + 6u\partial_x + 3u_x$$

$$= -4\partial_x^3 + 3u\partial_x + 3\partial_x(u\,\cdot)$$

$$\boxed{\hat{M} = -4\partial_x^3 + 3\left(u\partial_x + \partial_x(u\,\cdot)\right)}$$

Solving KdV

1. $u(x,0)$

$\downarrow$

$$\psi(x,k|0) = \begin{cases} e^{-ikx} + b(k|0)e^{+ikx} & x \to +\infty \\ a(k|0)e^{-ikx} & x \to -\infty \end{cases} \longrightarrow b(k)$$

$$\underset{\int dx\, \psi^2 = 1}{\psi_n(x|0)} = \begin{cases} c_n(0)e^{-\varkappa_n x} & x \to +\infty \\ d_n(0)e^{+\varkappa_n x_n} & x \to -\infty \end{cases} \longrightarrow \{(\varkappa_n, c_n) \mid n = 1,2,\dots,N\}$$

2. $|\psi(t)\rangle = U|\psi(0)\rangle$

$$\psi(x|t) = T\exp\left[\int_0^t dt'\,\hat{M}(t')\right]\psi(x,0)$$

$$\hat{M}(t) \xrightarrow{x\to+\infty} -4\partial_x^3 \Rightarrow T\exp\left[\int_0^t dt'\,\hat{M}(t')\right] \xrightarrow{x\to+\infty} \exp[-4(\partial_x^3)t] \Rightarrow$$

$$\Rightarrow T\exp\left[\int_0^t dt'\,\hat{M}(t')\right] \begin{cases} \psi(x,k|0) \xrightarrow{x\to+\infty} \exp[-4ik^3t]\,e^{-ikx} + b(k|0) \times \exp[+4ik^3t]\,e^{+ikx} \\ \psi_n(x|0) \xrightarrow{x\to+\infty} c_n(0)\exp[+4\varkappa_n^3 t]\,e^{-\varkappa_n x} \end{cases}$$

3.1

$$\Psi(x|k,t) \xrightarrow{x\to+\infty} \exp[-i4ik^3t]\,e^{-ikx} + b(k|0)\exp[+i4k^3t]\,e^{+ikx} \propto$$

$$\propto e^{-ikx} + b(k|0)\exp[+i8k^3t]\,e^{ikx}$$

$$\boxed{b(k|t) = \exp[i8k^3t]\,b(k|0)}$$

3.2 Remark: $\frac{d}{dt}\int dx\, \psi^2 =$

$$= 2\int dx\, \psi(\hat{M}\psi)$$

skew-adjoint

$$\frac{d}{dt}\int dx\, \psi_n^2(x|t) = \int dx\, \psi_n(x|t)\left(\hat{M}\,\psi_n(x,t)\right) =$$

$$= -\int dx\, \left(\hat{M}\,\psi_n(x|t)\right)\psi_n(x,t) = 0$$

Thus: $\int dx\, \psi_n^2(x|0) = \text{const}$

$$\boxed{c_n(t) = \exp[+4\varkappa_n^3 t]\, c_n(0)}$$

Also

4.

$$F(\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \;\; b(k|t)\exp[ik\xi] + \sum_{n=1}^{N} c_n^2(t)\exp[-\varkappa_n \xi]$$

$$K(x, z|t) + F(x+z|t) + \int_x^{\infty} dy \;\; K(x, y|t)\, F(y+z|t) \qquad K(x, z<x) = 0$$

$$\boxed{u(x,t) = -2\frac{\partial}{\partial x} K(x, x+\epsilon)}$$