Lax pairing for NLS: AKNS canonical scheme vs. a “KdV-inspired” Heisenberd equation

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After:
Nonlinear Evolution Equations of Physical Significance, M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, Phys. Rev. Letters 31, 125-127 (1973).

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The Nonlinear Schrödinger Equation

$$\begin{aligned} i\,\partial_t q &= -\partial_{x,x} q + 2\,q^2\,r \\ i\,\partial_t r &= +\partial_{x,x} q - 2\,r^2\,q \end{aligned} \qquad (1)$$

$r = +\, q^*$
$i\partial_t q = -\partial_{x,x} q + 2\,|q|^2\, q$

$r = -\, q^*$
$i\partial_t q = -\partial_{x,x} q - 2\,|q|^2\, q$

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The canonical AKNS Lax Pair

$$\begin{pmatrix} \partial_x \psi = P\psi \quad (2) \\ \partial_t \psi = Q\psi \quad (3) \end{pmatrix} \Longrightarrow P_t + [P, Q] - Q_x = 0 \Longrightarrow (1)$$

$$P = \begin{pmatrix} -i\zeta & q[x,t] \\ r[x,t] & +i\zeta \end{pmatrix}$$

$$Q = \begin{pmatrix} -i\, r[x,t]\, q[x,t] - 2\, i\, \zeta^2 & +i\, \partial_x q[x,t] + 2\, q[x,t]\, \zeta \\ -i\, \partial_x r[x,t] + 2\, r[x,t]\, \zeta & +i\, r[x,t]\, q[x,t] + 2\, i\, \zeta^2 \end{pmatrix}$$

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The “KdV-inspired” Heisenberd equation

$\partial_t L + [L, M] = 0 \Longrightarrow (1)$

$$L = \begin{pmatrix} +\partial_x & -q[x,t] \\ +r[x,t] & -\partial_x \end{pmatrix}$$

$$M = i \begin{pmatrix} +2\,\partial_{x,x} - q[x,t]\,r[x,t] & -2\,q[x,t]\,\partial_x - q^{(1,0)}[x,t] \\ +2\,r[x,t]\,\partial_x + r^{(1,0)}[x,t] & -2\,\partial_{x,x} + q[x,t]\,r[x,t] \end{pmatrix}$$

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AKNS vs Heisenberg connection

$$P = S + \begin{pmatrix} -i\,\zeta & 0 \\ 0 & +i\,\zeta \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & q[x,t] \\ r[x,t] & 0 \end{pmatrix}$$

$$L = \sigma_3(-S + \partial_x) = \begin{pmatrix} +\partial_x & -q[x,t] \\ +r[x,t] & -\partial_x \end{pmatrix}$$

$$M \overset{\text{on solutions of } (2)}{=} Q$$

$$\zeta = i\,\lambda$$

(2) ==> $L\psi = \lambda\psi$

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