

Gel'fand - Levitan - Marchenko equation

Maxim

May 2012

BOS

$$-\phi_{zz} + \phi_{xx} = u(x) \phi$$

$$\phi = \phi(x, z)$$

 $\uparrow$
 m^2

$$z \doteq t = \text{time}$$

(1)

$$\boxed{-\frac{1}{c^2} \phi_{tt} + \phi_{xx} = \frac{m^2 c^2}{\hbar^2} \phi}$$

Klein-Gordon

$$\boxed{c = \hbar = 1}$$

$$\phi(x, z) = \delta(x - z) + K(x, z)$$

$$K(x, z < x) = 0$$

$$K(x, z) = \tilde{K}(x, z) \Theta(z - x)$$

Crucial:

(1)

 $\Downarrow$

$$-K_{zz} + K_{xx} = u(x) \delta(x - z) + u(x) K$$

$$-\frac{\partial^2}{\partial z^2} (\tilde{K}(x, z) \Theta(z - x)) + \frac{\partial^2}{\partial x^2} (\tilde{K}(x, z) \Theta(z - x)) = u(x) \delta(x - z) +$$

 $\Downarrow$

$$+ u(x) \tilde{K}(x, z) \Theta(z - x)$$

$$\uparrow - 2 \left(\frac{\partial}{\partial z} \tilde{K}(x, z) \right) \delta(z - x) - \tilde{K}(x, z) \delta'(z - x) -$$

$$- 2 \left(\frac{\partial}{\partial x} \tilde{K}(x, z) \right) \delta(z - x) + \tilde{K}(x, z) \delta'(z - x) + \text{regular} = u(x) \delta(x - z) + \text{regular}$$

↓

②

$$-2 \left(\frac{\partial}{\partial x} \tilde{K}(x, z) + \frac{\partial}{\partial z} \tilde{K}(x, z) \right) \delta(x-z) + \text{regular} = u(x) \delta(x-z) + \text{regular}$$

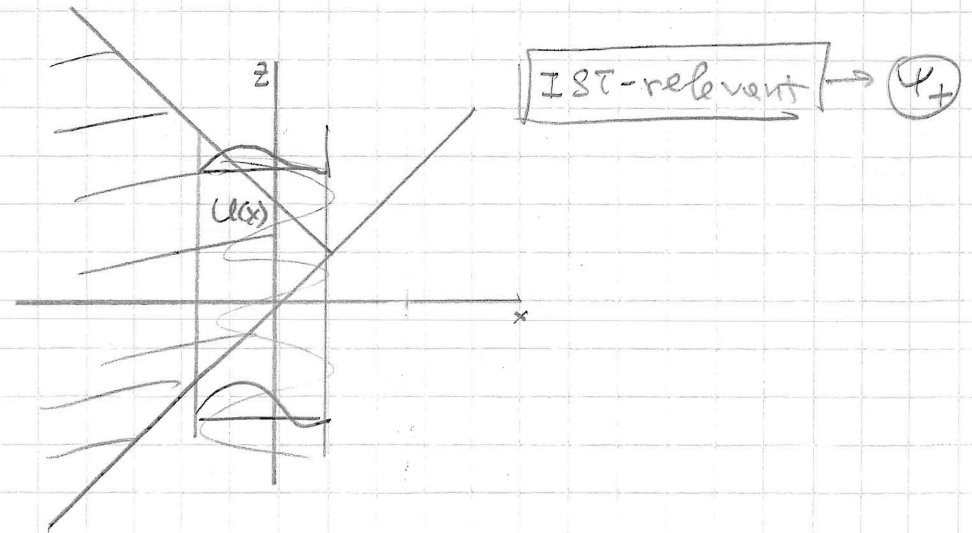
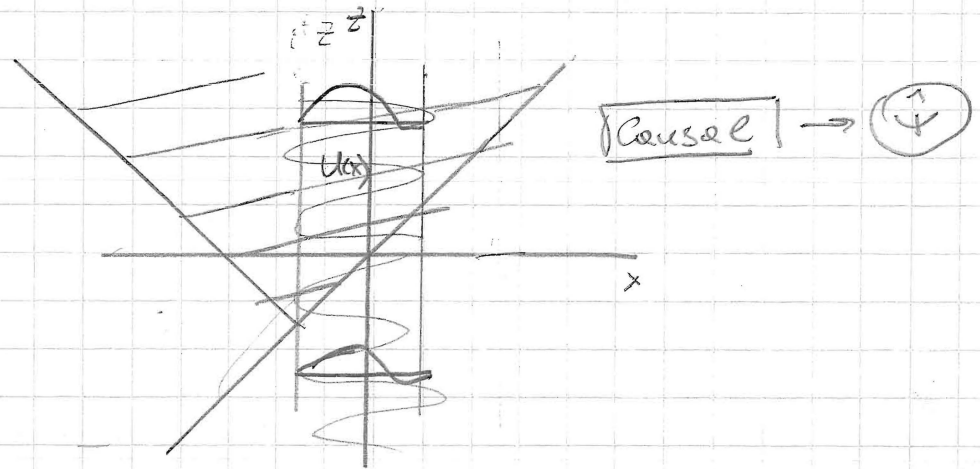
$$\frac{d}{dx} \tilde{K}(x, x) = -\frac{1}{2} u(x)$$

Causell solution:

$$\tilde{K}(x, x) = \frac{1}{2} \int_{-\infty}^x dx^p u(x^p)$$

IST-relevant solution:
IST

$$\tilde{K}(x, x) = \frac{1}{2} \int_{-\infty}^x dx^p u(x^p)$$



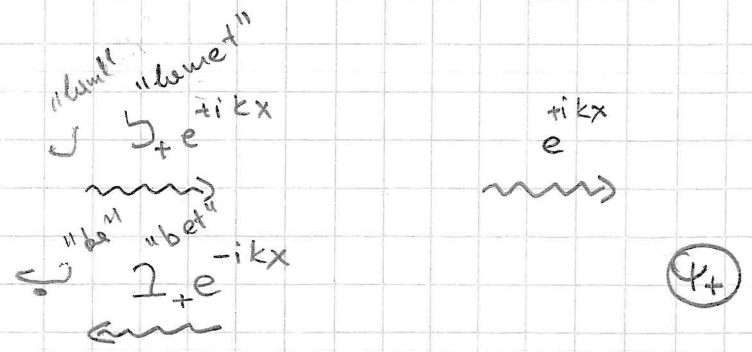
$$\int dz (1) e^{ikz} \phi(x, z) \xrightarrow{z \rightarrow \pm\infty} 0$$

$$k^2 \int dz \phi(x, z) e^{ikz} + \int dz \phi_{xx}(x, z) e^{ikz} = u(x) \int dz \phi(x, z)$$

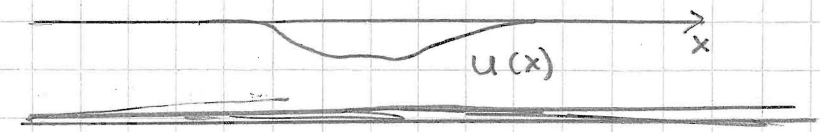
$$\psi_+(x, k) \equiv \int dz \phi(x, z) e^{ikz}$$

$$-(\psi_+)_{xx} + u(x) \psi_+ = k^2 \psi_+ \quad (2)$$

Schrödinger $-\frac{\hbar^2}{2m} \psi_{xx} + u(x) \psi(x) = \frac{\hbar^2 k^2}{2m} \psi$



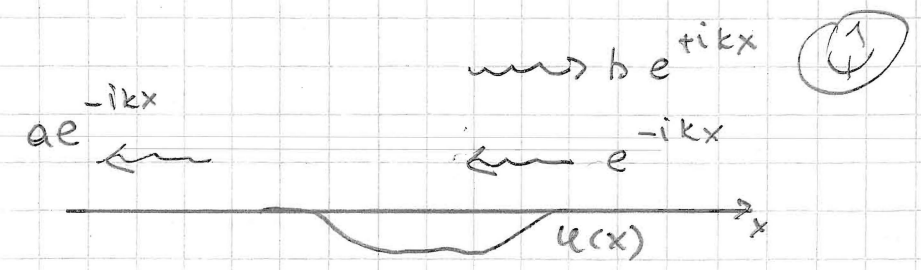
$$\psi_+ = e^{+ikx} + \int_x^\infty K(x, z) e^{ikz} dz \quad (3)$$



Introduce:

$$\hat{\psi}(x, k) : -(\hat{\psi})_{xx} + u(x) \hat{\psi} = k^2 \hat{\psi}$$

$$\hat{\psi} = \begin{cases} e^{-ikx} + b(k) e^{+ikx} & x \rightarrow +\infty \rightarrow +\infty \\ a(k) e^{-ikx} & x \rightarrow -\infty \end{cases}$$



$$\boxed{\hat{\psi} = \psi_+^* + b(k) \psi_+} \quad (4)$$

(4.)

$$\hat{\psi}(x, k) = \underbrace{e^{-ikx} + \int_{-\infty}^{\infty} K(x, z^p) e^{-ikz^p} dz^p}_{\psi_+^*} + \underbrace{b(k) e^{+ikx} + b(k) \int_{-\infty}^{+\infty} K(x, z^p) e^{ikz^p} dz^p}_{b(k) \psi_+}$$

$$\frac{1}{2\pi} \int dk e^{ikz} x \quad \downarrow$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\psi}(x, k) e^{ikz} dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} e^{+ikz} dk + K(x, z) +$$

$$\frac{1}{2\pi} \int dk b(k) e^{+ik(x+z)} dk +$$

$$+ \frac{1}{2\pi} \int dk b(k) \left(\int_{-\infty}^{+\infty} K(x, z^p) e^{ikz^p} dz^p \right) e^{ikz} dk$$

Introduce :

$$\tilde{F}(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} b(k) e^{ikz} dk$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} (\hat{\psi}(x, k) - e^{-ikx}) e^{ikz} dk = \quad (5)$$

$$= K(x, z) + \tilde{F}(x, z) + \int_{-\infty}^{+\infty} K(x, y) \tilde{F}(y, z) dy$$

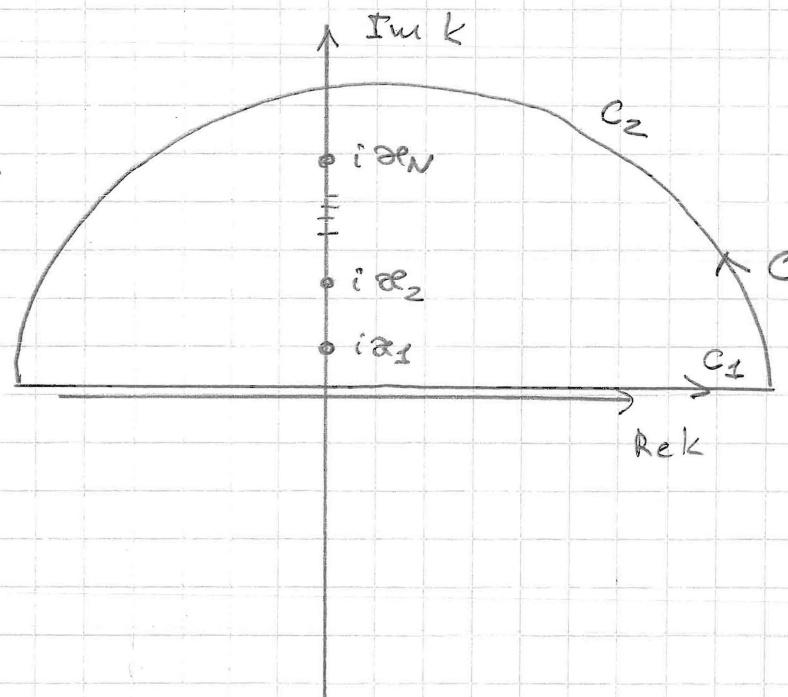
convolution

Explanation for the last term :

$$\begin{aligned} \int_{-\infty}^{+\infty} K(x, y) \tilde{F}(y, z) dy &= \int_{-\infty}^{+\infty} K(x, y) \frac{1}{2\pi} \int_{-\infty}^{+\infty} b(k) e^{ik(y+z)} dk dy = \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} b(k) \left(\int_{-\infty}^{+\infty} K(x, z') e^{ik(z'+z)} dz' \right) dk \end{aligned}$$

Righthand side :

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \underbrace{(\hat{\psi}(x, k) - e^{-ikx})}_{\substack{\downarrow k \rightarrow \infty \\ 0}} e^{+ikz} dk =$$



$$= \frac{1}{2\pi} \cdot 2\pi i \sum_{n=1}^N \text{Res}_{\hat{\psi} e^{ikz}}(i\alpha_n) =$$

$$= i \sum_{n=1}^N c_n^2 \psi_+(x, i\alpha_n) e^{-\alpha_n z} =$$

$$= - \sum_{n=1}^N c_n^2 \left\{ e^{-\alpha_n x} + \int_{-\infty}^{+\infty} k(x, y) e^{-\alpha_n y} dy \right\} e^{-\alpha_n z}$$

Result from the "residue set":

$$= - \sum_{n=1}^N c_n^2 e^{-\alpha_n(x+z)} -$$

$$- \int_{-\infty}^{+\infty} k(x, y) \left\{ \sum_{n=1}^N c_n^2 e^{-\alpha_n(y+z)} \right\} dy$$

$$\text{Res}_{\hat{\psi} e^{ikz}}(i\alpha_n) = i c_n^2 \psi_+(x, i\alpha_n) \cdot x e^{-\alpha_n z}$$

Define :

$$F(z) = \tilde{F}(z) + \sum_{n=1}^N c_n z e^{-\alpha_n z}$$

$$F(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} b(k) e^{ikz} dk + \sum_{n=1}^N c_n z e^{-\alpha_n z}$$

Then (5) becomes :

$$K(x, z) + F(x+z) + \int_x^z K(x, y) F(y+z) dy = 0$$
$$z > x > -\infty$$

Gel'fand - Levitan - Marchenko equation

$$u(x) = -2 \frac{d}{dx} K(x, x)$$