$$\boxed{-\Psi_{xx} + U(x)\Psi = k^2\Psi}$$

Residues of $\hat{\Psi} e^{ikz}$

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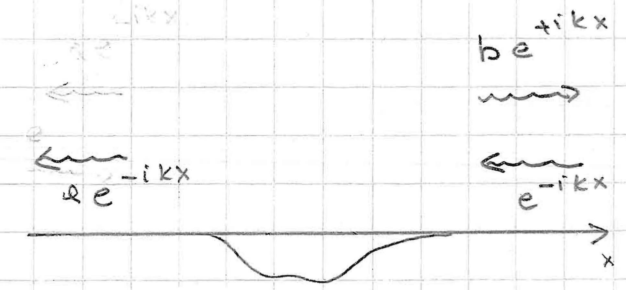
BOS

$\hat{\Psi}$

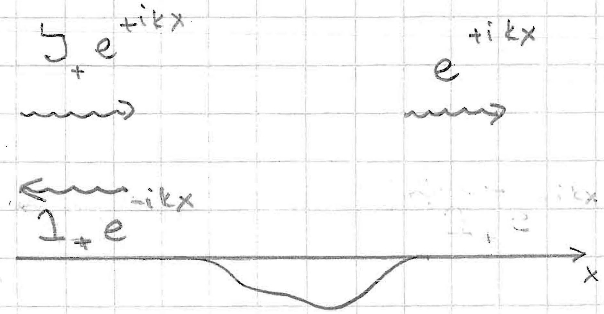
$$\hat{\Psi}(x;k) = \begin{cases} e^{-ikx} + b(k)\, e^{+ikx} & x \to +\infty \\ a(k)\, e^{-ikx} & x \to -\infty \end{cases}$$



Ψ_+

$$\Psi_+(x;k) = e^{+ikx} + \int_x^{\infty} K(x,z)\, e^{ikz}\, dz$$

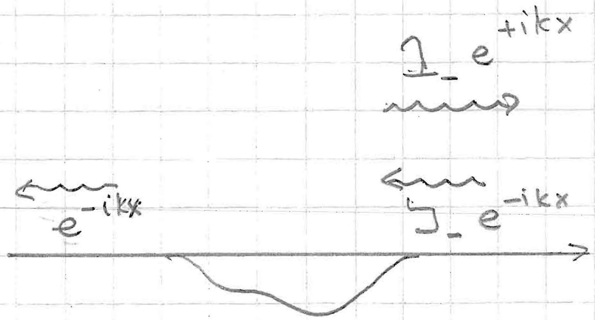
$$= \begin{cases} e^{+ikx} & x \to +\infty \\ \mathcal{J}_+(k)\, e^{+ikx} + \mathcal{I}_+(k)\, e^{-ikx} & x \to -\infty \end{cases}$$



Ψ_-

$$\Psi_-(x;k) = e^{-ikx} + \int_{-\infty}^{x} L(x,z)\, e^{-ikz}\, dz$$

$$= \begin{cases} \mathcal{J}_-(k)\, e^{-ikx} + \mathcal{I}_-(k)\, e^{+ikx} & x \to +\infty \\ e^{-ikx} & x \to -\infty \end{cases}$$



$$\hat{\Psi} = \Psi_+^* + b\,\Psi_+ \qquad\qquad \hat{\Psi} = a\,\Psi_-$$

$$= a\,\Psi_- \qquad \Rightarrow \qquad J_- = a^{-1}, \quad I_- = b\,a^{-1} \qquad\qquad (-1)$$

$$\Downarrow$$

$$\Psi_- = a^{-1}\,\Psi_+^* + b\,a^{-1}\,\Psi_+$$

At $k = i\varkappa_n$ $\leftarrow$ bound states

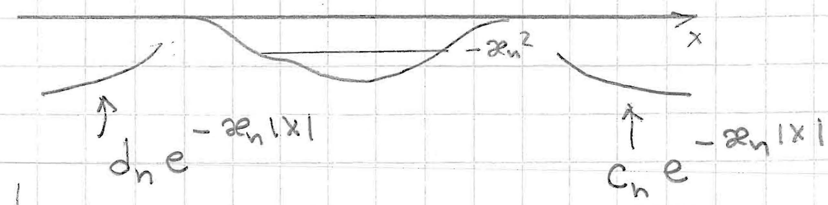
$$J_+(i\varkappa_n) = 0$$

$$J_-(i\varkappa_n) = 0 \qquad \Rightarrow \qquad a(i\varkappa_n) \to \infty, \quad b(i\varkappa_n) \to \infty$$

$I_-(i\varkappa_n) = \text{finite}$

Normalized bound states : Ψ_n

$$\Psi_n(x) : \int_{-\infty}^{+\infty} dx\, \Psi_n^2(x) = 1$$



$$\Psi_n(x) = c_n\, \Psi_+(x; i\varkappa_n)$$

$$= d_n\, \Psi_-(x; i\varkappa_n) \qquad\qquad (0)$$

Wronskians

Wronskian: $W(\alpha,\beta) \equiv \begin{vmatrix} \alpha(x) & \beta(x) \\ \alpha_x(x) & \beta_x(x) \end{vmatrix} = \alpha\beta_x - \beta\alpha_x$

$$\beta(x) = \text{const} \times \alpha(x) \Longrightarrow W(\alpha,\beta) = 0$$

$$\left.\begin{aligned} -\alpha_{xx} + u\alpha &= k^2\alpha \\ -\beta_{xx} + u\beta &= k^2\alpha \end{aligned}\right\} \Longrightarrow \frac{d}{dx} W(\alpha,\beta) = 0 \Rightarrow W(\alpha,\beta) = \text{const}$$

An intermediate goal: residues of $a(k)$

I. $W(\psi_-(k), \psi_+(k)) = \text{const}$ (1)

$$W(\psi_-(k), \psi_+(k)) \xlongequal{\psi_- = a^{-1}\psi_+^* + ba^{-1}\psi_+}$$

$$= (a^{-1}\psi_+^* + \cancel{ba^{-1}\psi_+})\psi_+' - \psi_+(a^{-1}(\psi_+^*)' + \cancel{ba^{-1}\psi_+'}) =$$

$$= a^{-1}(\psi_+^*\psi_+' - c.c.) \xrightarrow{x\to\infty} 2ika \quad (2)$$

$$(1)\&(2) \Longrightarrow \boxed{W(\psi_-, \psi_+) = 2ika^{-1}} \quad (3)$$

II. $\frac{\partial}{\partial k}$ $\Big|\ -\psi_{xx} + u\psi = k^2\psi$ (4)

$$-\psi_{xxk} + u\psi_k = 2k\psi + k^2\psi_k \quad (5)$$

$$-\psi_k \times (4) + \psi \times (5)$$

$\Downarrow$

$$+\psi_k\psi_{xx} - \psi\psi_{xxk} = +2k\psi^2$$

$$\frac{d}{dx} W(\psi_k, \psi) = 2k\psi^2$$

$$\boxed{W(\psi_k, \psi) = 2k\int^x \psi^2 dx} \quad (6)$$

III.

(book)

do not need

$$W(\Psi_{+k}, \Psi_+)\Big|_{\substack{x\to+\infty \\ k=i\varkappa_n}} - W(\Psi_{+k}, \Psi_+)\Big|_{\substack{x\to-\infty \\ k=i\varkappa_n}} = 2i\varkappa_n \int_{-\infty}^{+\infty} \Psi_+^2\, dx \underset{\Psi_n = C_n\Psi_+}{=\!=} \frac{2i\varkappa_n}{C_n^2}$$

IV.

$$\boxed{(*)\quad W(\Psi_{+k}, \Psi_+)\Big|_{\substack{x\to+\infty \\ k=i\varkappa_n}} - W(\Psi_{+k}, \Psi_+)\Big|_{\substack{x\to-\infty \\ k=i\varkappa_n}} = \frac{2i\varkappa_n}{C_n^2}} \qquad (7)$$

IV.

Result of I:

$$\frac{\partial}{\partial k}\Big|\ W(B)_{-}\ \Psi_+) = 2ika$$

$$\Downarrow$$

$$\frac{d}{dk}\left(\Psi_- \Psi_+' - \Psi_-'\Psi_+\right) = \Psi_{-k}\Psi_+' - \Psi_{-k}'\Psi_+ + \Psi_-\Psi_{+k}' - \Psi_-'\Psi_{+k} =$$

$$= \boxed{W(\Psi_{-k}, \Psi_+) + W(\Psi_-, \Psi_{+k}) = 2ia^{-1} - 2ik\,\frac{a'}{a^2}} \qquad (8)$$

Rem: if k_0 = simple pole $a(k)$, then

$$\operatorname*{Res}_{a(k)}[k_0] = -\frac{1}{(a'/a^2)}\Big|_{k=k_0}$$

V. (Maxim)

$$W(\Psi_{-k}, \Psi_-)\Big|_{\substack{x\to+\infty\\ k=i\varkappa_n}} - W(\Psi_{-k}, \Psi_-)\Big|_{\substack{x\to-\infty\\ k=i\varkappa_n}} = 2i\varkappa_n \int_{-\infty}^{+\infty} \Psi_-^2\, dx = \frac{2i\varkappa_n}{d_n^2}$$

$$\boxed{W(\Psi_{-k}, \Psi_-)\Big|_{\substack{x\to+\infty\\ k=i\varkappa_n}} - W(\Psi_{-k}, \Psi_-)\Big|_{\substack{x\to-\infty\\ k=i\varkappa_n}} = \frac{2i\varkappa_n}{d_n^2}} \quad (*) \qquad (9)$$

VI.

$$(8)\Big|_{\substack{x\to+\infty\\ k=i\varkappa_n}}$$

$$\Downarrow$$

$$W(\Psi_{-k}, \underbrace{\Psi_+}_{\frac{d_n}{c_n}\Psi_-})\Big|_{\substack{x\to+\infty\\ k=i\varkappa_n}} + W(\underbrace{\Psi_-}_{\frac{c_n}{d_n}\Psi_+}, \Psi_{+k})\Big|_{\substack{x\to+\infty\\ k=i\varkappa_n}} = \cancel{2ia}\Big|_{k=i\varkappa_n}^{\to 0} + 2\varkappa_n \frac{a^\circ}{a^2}\Big|_{k=i\varkappa_n}$$

$$\Downarrow$$

$$\left(\frac{d_n}{c_n}\right)^2 W(\Psi_{-k}, \Psi_-)\Big|_{\substack{x\to+\infty\\ k=i\varkappa_n}} + \left(\frac{c_n}{d_n}\right)^2 W(\Psi_+, \Psi_{+k})\Big|_{\substack{x\to+\infty\\ k=i\varkappa_n}} = 2\varkappa_n \frac{a^\circ}{a^2}\Big|_{k=i\varkappa_n} \quad (*)$$

$$\Downarrow (9)$$

$\Downarrow$ (9)

$$\left(\frac{d_n}{c_n}\right) \cancel{W(\underbrace{\psi_{-k}}_{\substack{x\to-\infty \\ -ike^{-ikx}}}, \underbrace{\psi_-}_{e^{-ikx}})}^{\to 0}\Big|_{\substack{x\to-\infty \\ k=i\varkappa_n}} + 2i\varkappa_n \frac{1}{c_n d_n} + \left(\frac{c_n}{d_n}\right) \cancel{W(\underbrace{\psi_+}_{\substack{x\to+\infty \\ e^{+ikx}}}, \underbrace{\psi_{+k}}_{ike^{+ikx}})}^{\to 0}\Big|_{\substack{x\to\infty \\ k=i\varkappa_n}} = 2\varkappa_n \frac{a^\cdot}{a^2}\Big|_{k=i\varkappa_n}$$

$$\Downarrow \quad -\frac{a^\cdot}{a^2}\Big|_{k=i\varkappa_n} = -\frac{i}{c_n d_n}$$

$\Downarrow$

$$\boxed{\frac{d}{dk}\left(\frac{1}{a}\right)\Big|_{k=i\varkappa_n} = -\frac{i}{c_n d_n}} \qquad (10)$$

VII $\quad \left.\begin{array}{l} |c_n| < \infty \\ |d_n| < \infty \end{array}\right\} \Leftarrow$ normalizability of $\psi_n(x)$

$\Downarrow$

$$\left|\frac{d}{dk}\left(\frac{1}{a}\right)_{k=i\varkappa_n}\right| > 0 \quad \xRightarrow{i\varkappa_n = \text{finite order pole}} \quad i\varkappa_n = \text{simple pole} \Rightarrow \boxed{\operatorname{Res}_a(i\varkappa_n) = i c_n d_n}$$

(11)

VIII. $\hat{\Psi} e^{ikz} \underset{\text{Eq. (1)}}{=} \Psi_- a\, e^{ikz}$

$$\mathrm{Res}_{\Psi_-(x,k)a(k)e^{ikz}}(k=i\varkappa_n) = \Psi_-(x,i\varkappa_n)\, e^{-\varkappa_n z}\, \mathrm{Res}_{a(k)}(k=i\varkappa) =$$

$$= \underbrace{\Psi_-(x,i\varkappa_n)}_{\text{Eq.(0)}}\, e^{-\varkappa_n z}\, i c_n d_n = i c_n^2\, \Psi_+(x,i\varkappa_n)\, e^{-\varkappa_n z}$$

$$\frac{c_n}{d_n}\Psi_+(x,i\varkappa_n)$$

$$\boxed{\mathrm{Res}_{\hat{\Psi}e^{ikz}}(k=i\varkappa_n) = i c_n^2\, \Psi_+(x,i\varkappa_n)\, e^{-\varkappa_n z}} \qquad (12)$$

IX. Discussion

$$\int dk\ \hat{\Psi}(x,k)\, e^{ikz} = -2\pi \sum_{n=1}^{N} C_n^2\, \Psi_+(x, i\varkappa_n)\, e^{-\varkappa_n z}$$

$$\overset{\parallel}{2\pi i \sum_{n=1}^{N} \mathrm{Res}(i\varkappa_n)} \qquad \overset{\Psi_n = C_n \Psi_+(i\varkappa_n)}{=} -2\pi \sum_{n=1}^{N} \Psi_n(x) \left(C_n e^{-\varkappa_n z} \right) \qquad (13)$$

$$\Psi(x,k) \equiv \frac{1}{\sqrt{2\pi}} \hat{\Psi}(x,k) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-ikx} + \frac{1}{\sqrt{2\pi}} b(k) e^{+ikz} & x \to +\infty \\ \frac{1}{\sqrt{2\pi}} a(k) e^{-ikx} & x \to -\infty \end{cases}$$

Completeness:

$$\int dk\ \Psi(x,k)\, \Psi^*(z,k) + \sum_{n=1}^{N} \Psi_n(x)\, \Psi_n^*(z) = \delta(x-z) \qquad (14)$$

Consider $z \to \infty$. There

$$\left.\begin{aligned} \Psi^*(z,k) &\to \frac{1}{\sqrt{2\pi}} e^{+ikz} \\ \Psi_n^*(z) &\to \underset{\uparrow \text{real}}{C_n}\, e^{-\varkappa_n z} \\ \delta(x-z) &\to 0 \end{aligned}\right\} \Longrightarrow (14) \to (13) \quad \text{consistency} \ \text{☺}$$