

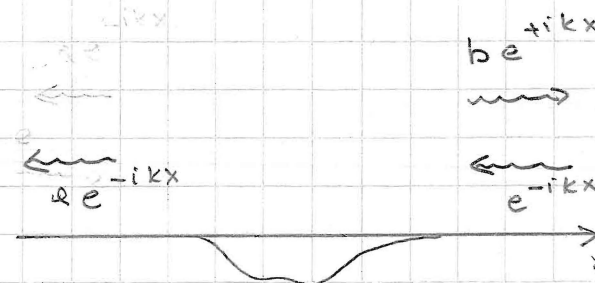
Residues of $\hat{\Psi} e^{ikz}$

Maxim
May 2012
BOS

$$-\Psi_{xx} + U(x)\Psi = k^2\Psi$$

1.

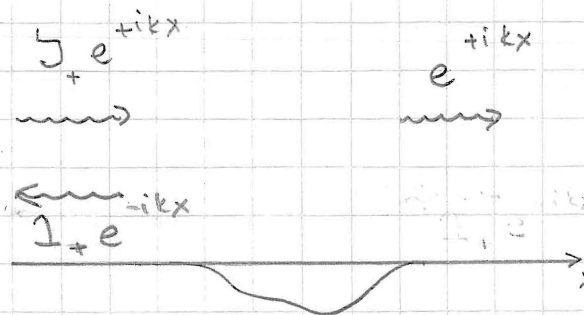
$$\hat{\Psi}(x; k) = \begin{cases} e^{-ikx} + b(k) e^{+ikx} & x \rightarrow +\infty \\ a(k) e^{-ikx} & x \rightarrow -\infty \end{cases}$$



$\hat{\Psi}$

$$\Psi_+(x; k) = e^{+ikx} + \int_x^\infty K(x, z) e^{ikz} dz$$

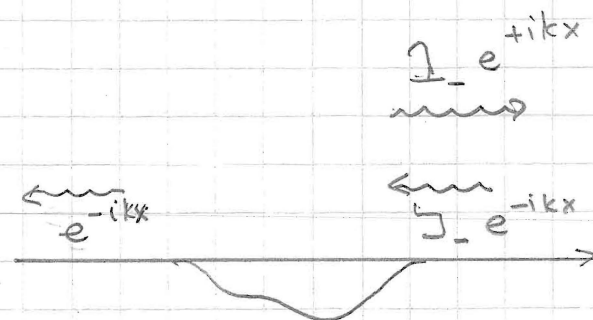
$$= \begin{cases} e^{+ikx} & x \rightarrow +\infty \\ \int_+(k) e^{+ikx} + \int_-(k) e^{-ikx} & x \rightarrow -\infty \end{cases}$$



Ψ_+

$$\Psi_-(x; k) = e^{-ikx} + \int_{-\infty}^x L(x, z) e^{-ikz} dz$$

$$= \begin{cases} \int_-(k) e^{-ikx} + \int_+(k) e^{+ikx} & x \rightarrow +\infty \\ e^{-ikx} & x \rightarrow -\infty \end{cases}$$



Ψ_-

$$\hat{\psi} = \psi_+^* + b \psi_+$$

$$= a \psi_-$$

$$\Rightarrow \begin{cases} \psi_+ = a^{-1} \\ \psi_- = b a^{-1} \end{cases}$$

$\Downarrow$

$$\psi_- = a^{-1} \psi_+^* + b a^{-1} \psi_+$$

$$\hat{\psi} = a \psi_-$$

(-1)

At $k = i\alpha_n \leftarrow$ bound states

$$\psi_+(i\alpha_n) = 0$$

$$\psi_-(i\alpha_n) = 0$$

$$\Rightarrow a(i\alpha_n) \rightarrow \infty$$

$$b(i\alpha_n) \rightarrow \infty$$

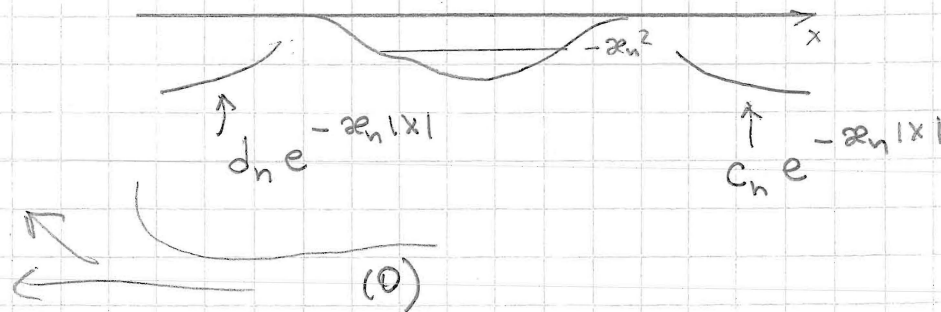
$$\psi_-(i\alpha_n) = \text{finite}$$

Normalized bound states:

$$\psi_n(x) : \int_{-\infty}^{+\infty} dx \psi_n^2(x) = 1$$

$$\psi_n(x) = c_n \psi_+(x; i\alpha_n)$$

$$= d_n \psi_-(x; i\alpha_n)$$



Wronskien

3.

$$\text{Wronskien: } W(\alpha, \beta) = \begin{vmatrix} \alpha(x) & \beta(x) \\ \alpha'(x) & \beta'(x) \end{vmatrix} = \alpha \beta' - \beta \alpha'$$

$$\beta(x) = \text{const} \times \alpha(x) \Rightarrow W(\alpha, \beta) = 0$$

$$\left. \begin{array}{l} -\alpha'' + u\alpha = k^2\alpha \\ -\beta'' + u\beta = k^2\beta \end{array} \right\} \Rightarrow \frac{d}{dx} W(\alpha, \beta) = 0 \Rightarrow W(\alpha, \beta) = \text{const}$$

Interpretation: ...

An intermediate goal: residues of $a(k)$

$$I. \quad W(\psi_-(k), \psi_+(k)) = \text{const} \quad (1)$$

$$\begin{aligned} W(\psi_-(k), \psi_+(k)) &= \overline{\psi_-} \psi_+ - \psi_- \overline{\psi_+} \\ &= (a^{-1} \psi_+^* + \cancel{b a^{-1} \psi_+}) \psi_+ - \psi_+ (a^{-1} (\psi_+^*)^* + \cancel{b a^{-1} \psi_+}) = \\ &= a^{-1} (\psi_+^* \psi_+ - \text{c.c.}) \xrightarrow{x \rightarrow \infty} 2ika \end{aligned} \quad (2)$$

$$(1) \& (2) \Rightarrow \boxed{W(\psi_-, \psi_+) = 2ika^{-1}} \quad (3)$$

II.

$$II. \quad \frac{\partial}{\partial k} \quad \left| -\psi_{xx} + u\psi = k^2 \psi \right. \quad (4) \quad (4)$$

$$-\psi_{xxk} + u\psi_k = 2k\psi + k^2\psi_k \quad (5) \quad (5)$$

$$-\psi_k \times (4) + \psi \times (5)$$

$\Downarrow$

$$+\psi_k \psi_{xx} - \psi \psi_{xxk} = +2k\psi^2$$

$$\frac{d}{dx} W(\psi_k, \psi) = 2k\psi^2$$

$$\boxed{W(\psi_k, \psi) = 2k \int^x \psi^2 dx} \quad (6)$$

III.

(book)

no need

$$W(\psi_{+k}, \psi_+) \Big|_{\substack{x \rightarrow +\infty \\ k = i\epsilon_n}} - W(\psi_{+k}, \psi_+) \Big|_{\substack{x \rightarrow -\infty \\ k = i\epsilon_n}} = 2i\epsilon_n \int_{-\infty}^{+\infty} \psi_+^2 dx = \frac{2i\epsilon_n}{c_n^2}$$

$\psi_n = c_n \psi_+$

(5)

$$\boxed{\begin{aligned} & \textcircled{x} \\ & W(\psi_{+k}, \psi_+) \Big|_{\substack{x \rightarrow +\infty \\ k = i\epsilon_n}} - W(\psi_{+k}, \psi_+) \Big|_{\substack{x \rightarrow -\infty \\ k = i\epsilon_n}} = \frac{2i\epsilon_n}{c_n^2} \end{aligned}} \quad (7)$$

IV.

V.

Result of I:

$$\frac{\partial}{\partial k} W(\beta, \psi_+) = 2i\epsilon_n$$

$$\begin{aligned} \Downarrow \quad \frac{d}{dk} (\psi_- \psi_+^* - \psi_-^* \psi_+) &= \psi_{-k} \psi_+^* - \psi_{-k}^* \psi_+ + \psi_- \psi_{+k}^* - \psi_-^* \psi_{+k} = \\ &= \boxed{W(\psi_{-k}, \psi_+) + W(\psi_-, \psi_{+k}) = 2i\epsilon_n - 2ik \frac{a^*}{a^2}} \end{aligned} \quad (8)$$

Rem: if k_0 = simple pole $a(k)$, then

$$\text{Res}_{a(k)} [k_0] = - \frac{a^*}{(a^* a^2)} \Big|_{k=k_0}$$

V.
(Maximum)

$$W(\psi_{-k}, \psi_{-}) \Big|_{\substack{x \rightarrow +\infty \\ k = i\alpha_n}} - W(\psi_{-k}, \psi_{-}) \Big|_{\substack{x \rightarrow -\infty \\ k = i\alpha_n}} = 2i\alpha_n \int_{-\infty}^{+\infty} \psi_{-}^2 dx = \frac{2i\alpha_n}{d_n^2}$$

(6.)

$$\boxed{W(\psi_{-k}, \psi_{-}) \Big|_{\substack{x \rightarrow +\infty \\ k = i\alpha_n}} - W(\psi_{-k}, \psi_{-}) \Big|_{\substack{x \rightarrow -\infty \\ k = i\alpha_n}} = \frac{2i\alpha_n}{d_n^2}} \quad (9)$$

~~~~~

VI.

$$(8) \Big|_{\substack{x \rightarrow +\infty \\ k = i\alpha_n}}$$

$\Downarrow$

$$\begin{aligned} & W(\psi_{-k}, \psi_{+}) \Big|_{\substack{x \rightarrow +\infty \\ k = i\alpha_n}} + W(\psi_{-}, \psi_{+k}) \Big|_{\substack{x \rightarrow +\infty \\ k = i\alpha_n}} \\ & \quad \frac{d_n}{c_n} \psi_{-} \quad \frac{c_n}{d_n} \psi_{+} \end{aligned} = 2i\alpha_n \left( + 2\alpha_n \frac{a^p}{a^2} \right) \Big|_{k = i\alpha_n}$$

$\Downarrow$

$$\left( \frac{d_n}{c_n} \right)^2 W(\psi_{-k}, \psi_{-}) \Big|_{\substack{x \rightarrow +\infty \\ k = i\alpha_n}} + \left( \frac{c_n}{d_n} \right)^2 W(\psi_{+}, \psi_{+k}) \Big|_{\substack{x \rightarrow +\infty \\ k = i\alpha_n}} = 2\alpha_n \frac{a^p}{a^2} \Big|_{k = i\alpha_n} \quad (*)$$

$\Downarrow (9)$



↓ (9)

$$\left( \frac{d_n}{c_n} \right) W(\psi_{-k}, \psi_{-}) \Big|_{\substack{x \rightarrow -\infty \\ k = i\alpha_n}} + 2i\alpha_n \frac{1}{c_n d_n} + \left( \frac{c_n}{d_n} \right) W(\psi_{+}, \psi_{+k}) \Big|_{\substack{x \rightarrow \infty \\ k = i\alpha_n}} = 2\alpha_n \frac{a^p}{a^2} \Big|_{k = i\alpha_n}$$

Diagram 1: A complex plane with a branch cut along the imaginary axis from  $i\alpha_n$  to  $i\infty$ . A contour is shown in the lower half-plane, consisting of a large semi-circle  $\Gamma_R$  and a small semi-circle  $\gamma_\epsilon$  around the pole at  $i\alpha_n$ . Arrows indicate the direction of integration. The integrand is  $e^{-ikx}$  for  $x \rightarrow -\infty$  and  $e^{-ikx}$  for  $x \rightarrow \infty$ .

Diagram 2: A complex plane with a branch cut along the imaginary axis from  $i\alpha_n$  to  $i\infty$ . A contour is shown in the upper half-plane, consisting of a large semi-circle  $\Gamma_R$  and a small semi-circle  $\gamma_\epsilon$  around the pole at  $i\alpha_n$ . Arrows indicate the direction of integration. The integrand is  $e^{+ikx}$  for  $x \rightarrow +\infty$  and  $e^{+ikx}$  for  $x \rightarrow -\infty$ .

$$\Downarrow - \frac{a^p}{a^2} \Big|_{k = i\alpha_n} = - \frac{i}{c_n d_n}$$

↓

$$\boxed{\left. \frac{d}{dk} \left( \frac{1}{a} \right) \right|_{k = i\alpha_n} = - \frac{i}{c_n d_n}}$$

(10)

$$\text{VII} \quad \left. \begin{array}{l} |c_n| < \infty \\ |d_n| < \infty \end{array} \right\} \Leftarrow \text{normalizability of } \psi_n(x)$$

↓

$$\left| \frac{d}{dk} \left( \frac{1}{a} \right) \right|_{k = i\alpha_n} > 0 \quad \Rightarrow \quad \begin{array}{l} i\alpha_n = \text{finite order pole} \\ i\alpha_n = \text{simple pole} \end{array} \Rightarrow \boxed{\text{Res}_a(i\alpha_n) = i c_n d_n}$$

(11)

$$\text{VIII. } \hat{\psi} e^{ikz} \stackrel{\text{Eq.}}{=} \hat{\psi}_0 e^{ikz}$$

(8.)

$$\begin{aligned} \text{Res}_{\hat{\psi}(x) \text{ack}} e^{ikz} (k=i\alpha_n) &= \psi_-(x, i\alpha_n) e^{-\alpha_n z} \text{Res}_{\text{ack}} (k=i\alpha) = \\ &= \underbrace{\psi_-(x, i\alpha_n)}_{\text{Eq. (0)}} e^{-\alpha_n z} i c_n d_n = i c_n^2 \psi_+(x, i\alpha_n) e^{-\alpha_n z} \end{aligned}$$

$$\frac{c_n}{d_n} \psi_+(x, i\alpha_n)$$

$$\boxed{\text{Res}_{\hat{\psi} e^{ikz}} (k=i\alpha_n) = i c_n^2 \psi_+(x, i\alpha_n) e^{-\alpha_n z}}$$

(12)



# IX. Discussion

$$\int dk \hat{\Psi}(x, k) e^{ikz} = -2\pi \sum_{n=1}^N c_n^2 \Psi_+(x, i\alpha_n) e^{-\alpha_n z}$$

$$\stackrel{2\pi i \sum_{n=1}^N \text{Res}(i\alpha_n)}{\Psi_n = c_n \Psi_+(i\alpha_n)} = -2\pi \sum_{n=1}^N \Psi_n(x) (c_n e^{-\alpha_n z}) \quad (13)$$

$$\Psi(x, k) \equiv \frac{1}{\sqrt{2\pi}} \hat{\Psi}(x, k) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-ikx} + \frac{1}{\sqrt{2\pi}} b(k) e^{+ikx} & x \rightarrow +\infty \\ \frac{1}{\sqrt{2\pi}} a(k) e^{-ikx} & x \rightarrow -\infty \end{cases}$$

Completeness:

$$\int dk \Psi(x, k) \Psi^*(z, k) + \sum_{n=1}^N \Psi_n(x) \Psi_n^*(z) = \delta(x-z) \quad (14)$$

Consider  $z \rightarrow \infty$ . There

$$\left. \begin{aligned} \Psi^*(z, k) &\rightarrow \frac{1}{\sqrt{2\pi}} e^{+ikz} \\ \Psi_n^*(z) &\rightarrow c_n e^{-\alpha_n z} \\ \delta(x-z) &\rightarrow 0 \end{aligned} \right\} \Rightarrow (14) \rightarrow (13) \text{ consistency } \text{😊}$$