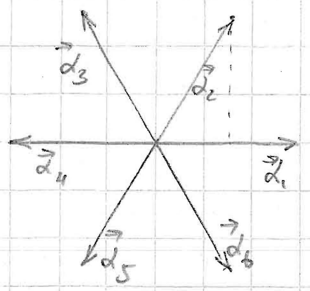
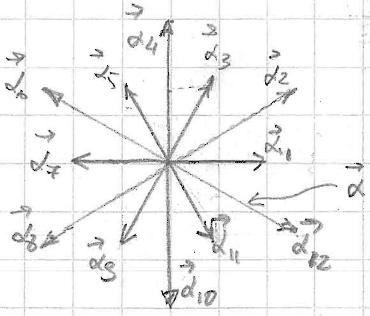
Root systems. Explicit examples of A_2, G_2, and $I_2(5)$

Maxim
BOS
May 20, 2012

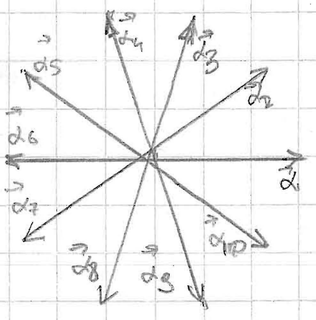
A_2



G_2



$I_2(5)$ a.k.a H_2



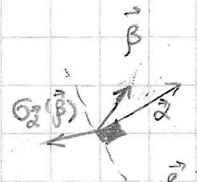
$\boxed{\text{root system, } \Phi}$

Euclideaan space, E

1. $\mathrm{Span}[\Phi] = E$

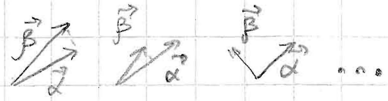
2. $\forall \vec{\alpha} \in \Phi : \; c\vec{\alpha} \in \Phi \Rightarrow c = \pm 1$ (*)

3. $\forall \vec{\alpha}, \vec{\beta} \in \Phi \quad \sigma_{\alpha}(\vec{\beta}) = \vec{\beta} - 2 \dfrac{(\vec{\alpha} \cdot \vec{\beta})}{\alpha^2} \vec{\alpha} \in \Phi$



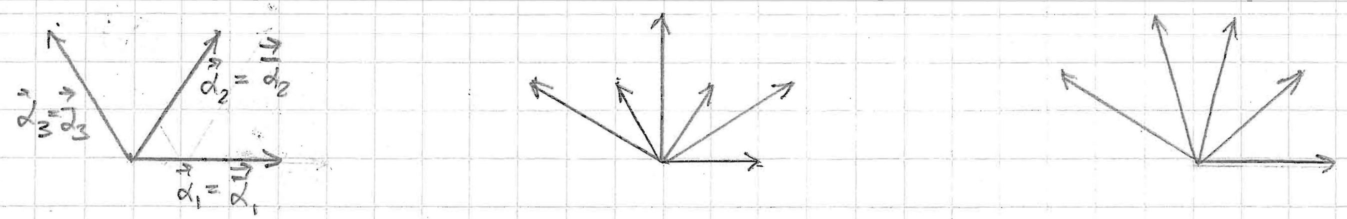
crystallographic

4. $\forall \vec{\alpha}, \vec{\beta} \in \Phi : \dfrac{(\vec{\alpha} \cdot \vec{\beta})}{\alpha^2} \vec{\alpha} = \dfrac{m}{2} \vec{\alpha}, \quad m \in \mathbb{Z}$



(*) Omitted for Calogero-Moser-Sutherland systems

A_2 G_2 $I_2(5)$

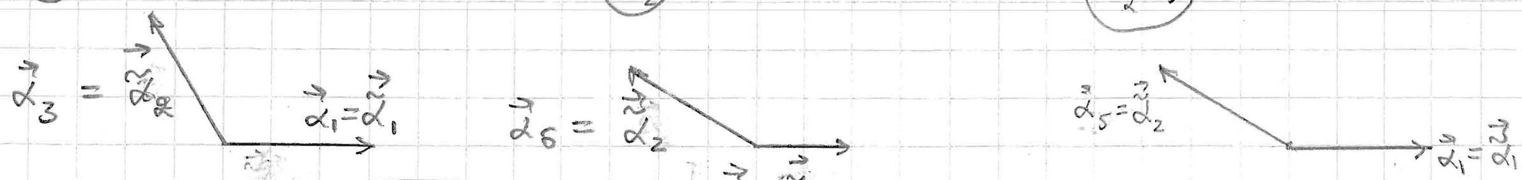


$\boxed{\text{positive roots, } \Phi_+}$

1. $\forall \vec\alpha \in \Phi : (\vec\alpha \in \Phi_+ \text{ and } -\vec\alpha \notin \Phi_+) \text{ or } (\vec\alpha \notin \Phi_+ \text{ and } -\vec\alpha \in \Phi_+)$

2. $(\forall \vec\alpha, \vec\beta \in \Phi_+ : \vec\alpha + \vec\beta \in \Phi) : \vec\alpha + \vec\beta \in \Phi_+$

A_2 G_2 $I_2(5)$



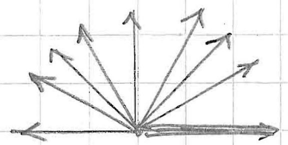
$\boxed{\text{simple roots, } \Delta}$ $\vec\alpha_1 = \vec{\tilde\alpha}_1$ <u>not</u> a sum of two positive roots

1. $(\nexists \vec\beta_1, \vec\beta_2 : (\vec\alpha, \vec\beta_1, \vec\beta_2 \in \Phi_+) \text{ and } \vec\beta_1 + \vec\beta_2 = \vec\alpha) \Longrightarrow \vec\alpha = \text{simple root}$

Property: $\forall \vec\alpha \in \Phi : (\underbrace{\forall n : c_n \geq 0}_{\Phi_+}) \text{ or } (\underbrace{\forall n : c_n \leq 0}_{\Phi_-})$, where $\vec\alpha = \sum_n c_n \vec{\tilde\alpha}_n$, $\Delta = \{\vec{\tilde\alpha}_n \mid n = 1, \dots, D\}$

Further properties:

1. $(\forall \Phi, \forall \vec{\alpha}_1, \vec{\alpha}_2 \in \Phi)$: $\widehat{\vec{\alpha}_1 \vec{\alpha}_2} = 0, \pm\frac{\pi}{6}, \pm\frac{\pi}{4}, \pm\frac{\pi}{3}, \pm\frac{\pi}{2}, \pm\frac{2\pi}{3}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{6}, \pi$



2. Irreducible root systems: $\nexists (E_1, \Phi_1), (E_2, \Phi_2)$: $E = E_1 \oplus E_2$ and $\Phi = \Phi_1 \cup \Phi_2$

For irreducible root systems:

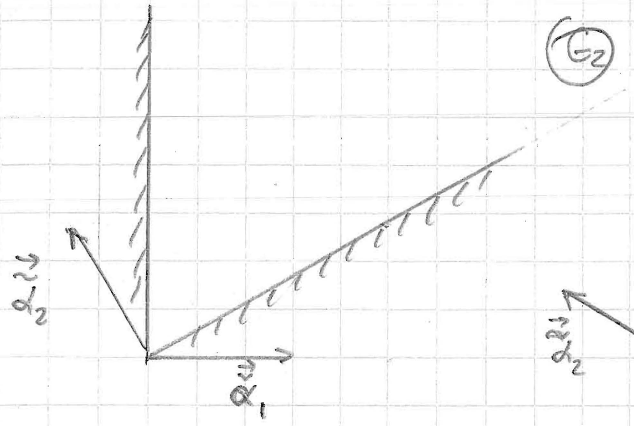
either simply-laced root system: $\forall \vec{\alpha} \in \Phi$: $|\vec{\alpha}| = \bar{\alpha}$ = the same for all

or non-simply-laced root system: $\forall \vec{\alpha} \in \Phi$: $|\vec{\alpha}|$ = either $\bar{\alpha}_{short}$ or $\bar{\alpha}_{long}$

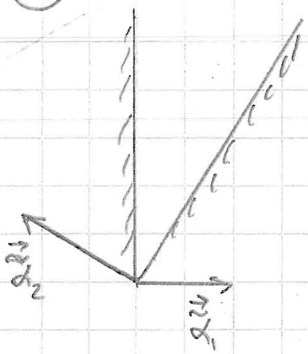
Simply laced = A, D, E

Non-simply laced = B, C, G, F

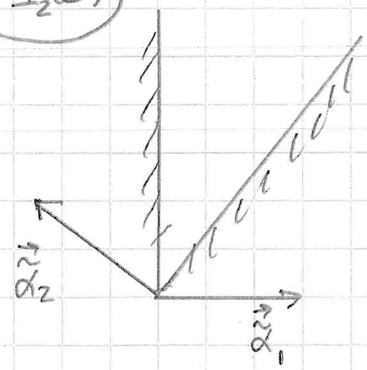
A_2



G_2



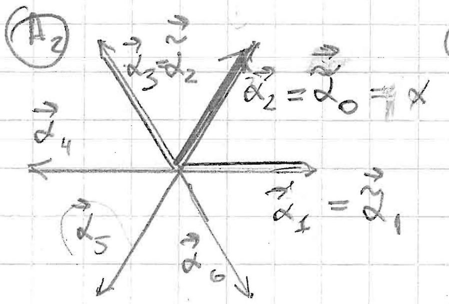
$I_2(5)$



Weyl chamber

$$\{\vec{r} \mid (\vec{\tilde{\alpha}}_n \cdot \vec{r}) > 0\}$$

A_2



$\vec\alpha_2 > \vec\alpha_1 \Leftarrow \vec\alpha_2 - \vec\alpha_1 = \vec\alpha_3$

$\vec\alpha_2 > \vec\alpha_3 \Leftarrow \vec\alpha_2 - \vec\alpha_3 = \vec\alpha_1$

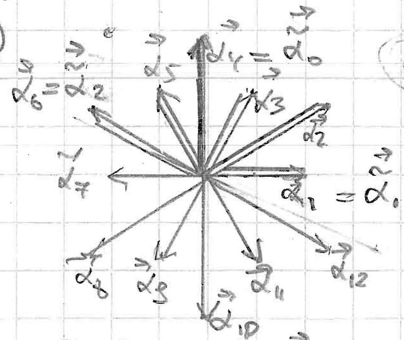
$\vec\alpha_2 > \vec\alpha_6 \Leftarrow \vec\alpha_2 - \vec\alpha_6 = \vec\alpha_2 + \vec\alpha_3$

$\vec\alpha_2 > \vec\alpha_5 \Leftarrow \vec\alpha_2 - \vec\alpha_5 = \vec\alpha_2 + \vec\alpha_1 + \vec\alpha_3$

$\vec\alpha_2 > \vec\alpha_4 \Leftarrow \vec\alpha_2 - \vec\alpha_4 = \vec\alpha_2 + \vec\alpha_1$

$\vec\alpha_3 ? \vec\alpha_1 ; \vec\alpha_1 > \vec\alpha_6 ; \vec\alpha_6 > \vec\alpha_5 ; \vec\alpha_4 > \vec\alpha_5 ; \vec\alpha_4 ? \vec\alpha_6 , \ldots$

G_2



$\vec\alpha_4 > \vec\alpha_3 \Leftarrow \vec\alpha_4 - \vec\alpha_3 = \vec\alpha_5$

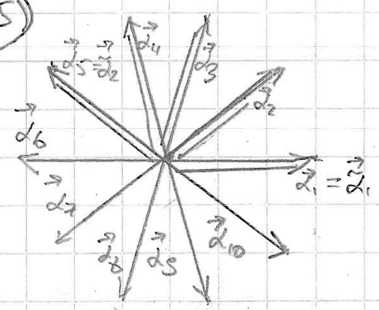
$\vec\alpha_4 > \vec\alpha_2 \Leftarrow \vec\alpha_4 - \vec\alpha_2 = \vec\alpha_6$

$\vec\alpha_4 > \vec\alpha_1 \Leftarrow \vec\alpha_4 - \vec\alpha_1 = \vec\alpha_6 + \vec\alpha_3$

...

$\boxed{\text{maximal root, } \tilde{\vec\alpha}_0}$

$I_2(5)$



No maximal root!

$\vec\alpha_3 > \vec\alpha_6 \Leftarrow \vec\alpha_3 - \vec\alpha_6 = \vec\alpha_1 + \vec\alpha_3$

$\vec\alpha_3 > \vec\alpha_7 \Leftarrow \vec\alpha_3 - \vec\alpha_7 = \vec\alpha_2 + \vec\alpha_3$

$\vec\alpha_3 > \vec\alpha_9 \ldots$

$\vec\alpha_3 > \vec\alpha_{10} \ldots$

$\vec\alpha_3 ? \vec\alpha_1 ; \vec\alpha_3 ? \vec\alpha_2 ; \vec\alpha_3 ? \vec\alpha_4 ; \vec\alpha_3 ? \vec\alpha_5 ;$

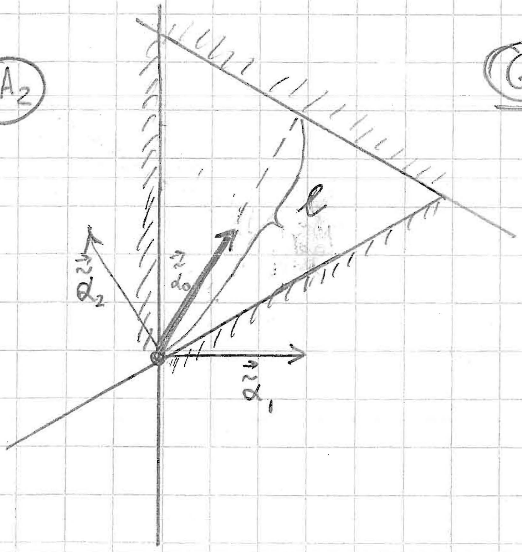
$\vec\alpha_3 ? \vec\alpha_8$

Partial order: Let $\vec\alpha, \vec\beta \in \Phi$. Then

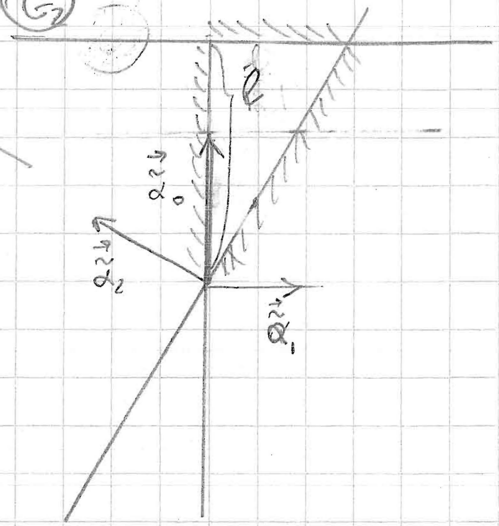
- $\vec\alpha > 0 \iff \vec\alpha \in \Phi_+$
- $\vec\alpha < 0 \iff \vec\alpha \in \Phi_-$
- $\vec\alpha > \vec\beta \iff \vec\alpha - \vec\beta = \sum_{\vec\gamma \in \Phi_+} \vec\gamma$

Maximal root: $\tilde{\vec\alpha}_0 : \forall \vec\alpha \neq \tilde{\vec\alpha}_0,\ \vec\alpha \in \Phi : \tilde{\vec\alpha}_0 > \vec\alpha$

A_2



G_2



Weyl alcove

$$\{\vec{r} \mid 0 < (\vec{\alpha} \cdot \vec{r}) < |\vec{\alpha}_0|\ell, \ \forall \vec{\alpha} \in \Phi_+ \}$$

or

$$\{\vec{r} \mid (\vec{\alpha}_n \cdot \vec{r}) > 0 \ \text{and} \ (\vec{\alpha}_0 \cdot \vec{r}) < |\vec{\alpha}_0|\ell \}$$