Quantization of an A_2 chamber

Maxim
BOS
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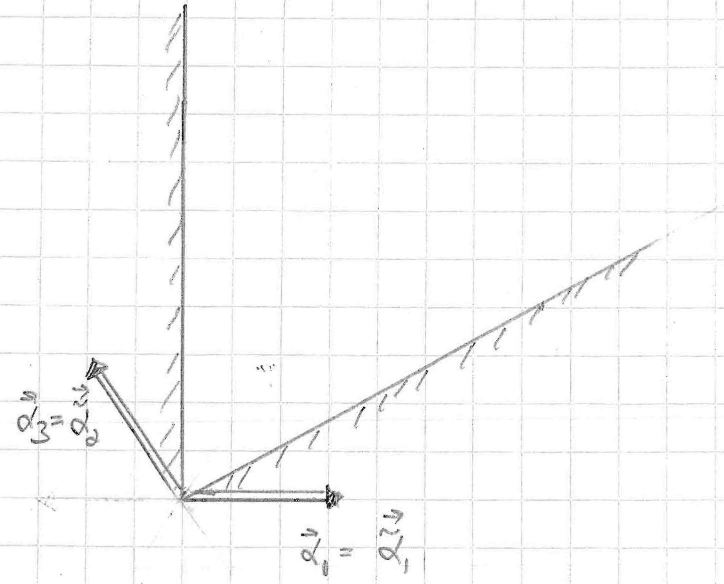
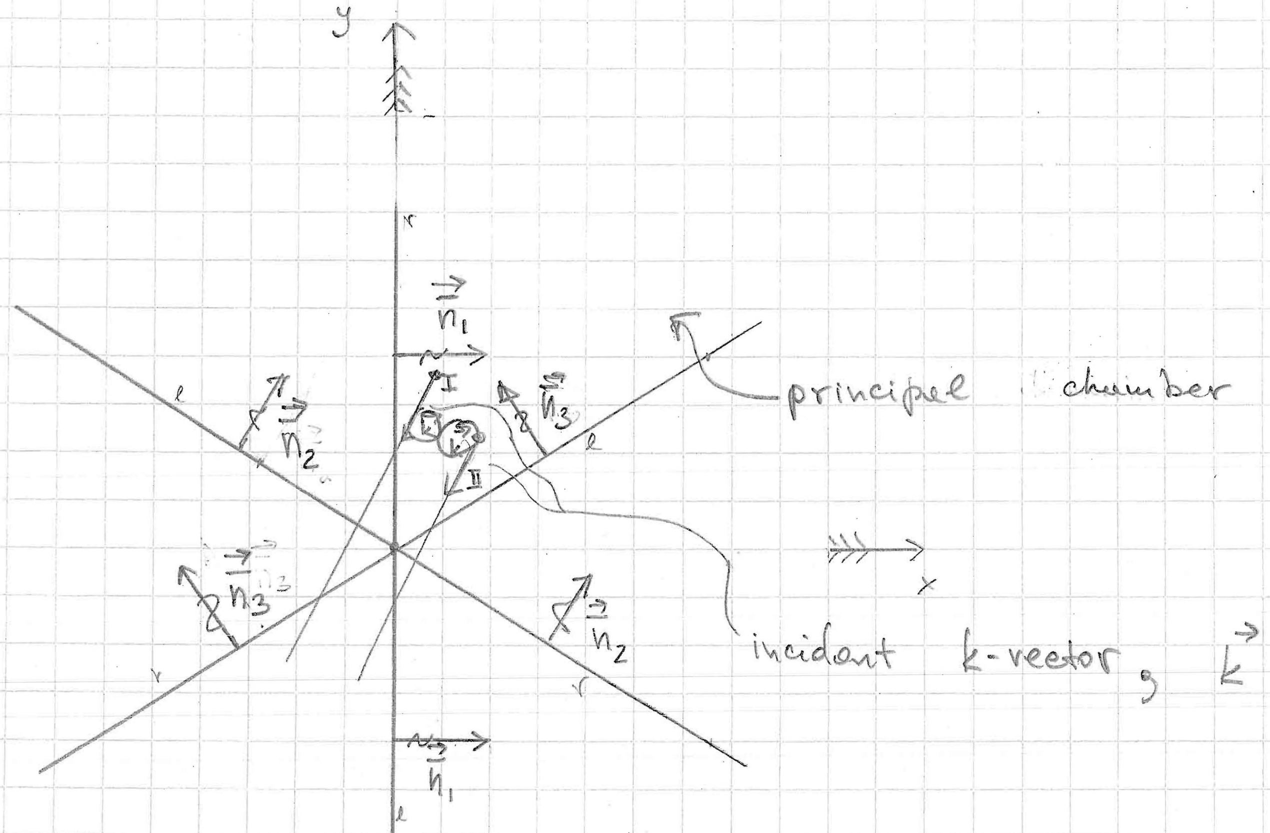
related to:
3 1D particles,
relative motion



$\Longrightarrow$ simple roots, $\tilde{\vec{\alpha}}_1, \tilde{\vec{\alpha}}_2, \dots, \tilde{\vec{\alpha}}_D$ [$D=2$, for A_2]



$$\vec{n}_j = \frac{\vec{\alpha}_j}{|\vec{\alpha}_j|}$$

$\vec{\alpha}_j$ = positive roots $\quad j = 1, 2, \ldots, N$

For A_2 : $N = 3$

$\begin{array}{c} A_2 \\ \parallel\!| \\ I_2(3) \end{array}$

$$\hat{h} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \tilde{g}\sum_{i=1}^{N} \delta(\vec{n}_i \cdot \vec{r})$$

$$N \overset{A_2}{=} 3$$

$$\hbar = m = 1 \qquad \tilde{a} \equiv -\frac{\hbar^2}{m\tilde{g}} \overset{\hbar=m=1}{=} -\frac{1}{\tilde{g}}$$

$$\hat{h} = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \tilde{g}\sum_{i=1}^{N} \delta(\vec{n}_i \cdot \vec{r})$$

Assume full mirror symmetry:

$$\Psi(\omega(\vec{r})) = \Psi(\vec{r})$$

any of the $N=3$ mirror reflections

Boundary condition:

$$\Psi\big|_{0+} = \Psi\big|_{0-}$$

$$\vec{n}\cdot\vec{\nabla}\Psi\Big|_{0+} - \vec{n}\cdot\vec{\nabla}\Psi\Big|_{0-} = -\frac{2}{\tilde{a}}\,\Psi\Big|_0 = 2\tilde{g}\,\Psi\Big|_0$$

Method of images

In principle domain: $\psi(x,y) = \eta \left\{ \exp[i \vec{k} \vec{r}\,] + \sum_{\hat{\omega}} A(\hat{\omega}) \exp[i \hat{\omega} \vec{k} \vec{r}\,] \right\}$

normalization

over all reflection group, as many elements as chambers

Boundary conditions:

$$A(\hat{R}_{\vec{n}_{j'}} \hat{\omega}) = (-1) \frac{1 + i \vec{n}_{j'} \vec{k} \tilde{a}}{1 - i \vec{n}_{j'} \vec{k} \tilde{a}} A_{\omega} \qquad (*)$$

reflection w.r.t. a mirror whose normal is $\vec{n}_j$

Impact parameter I:

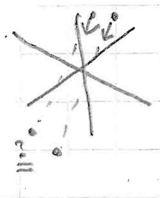
$$A(\hat{R}_{\vec{n}_j} \hat{R}_{\vec{n}_{j-1}} \cdots \hat{R}_{\vec{n}_1}) = \left[\prod_{j'=1}^{j} (-1) \frac{1 - i \vec{n}_{j'} \vec{k} \tilde{a}}{1 + i \vec{n}_{j'} \vec{k} \tilde{a}} \right] \eta$$

Impact parameter II:

$$A(\hat{R}_{\vec{n}_{N+1-j}} \hat{R}_{\vec{n}_{N-j}} \cdots \hat{R}_{N}) = \left[\prod_{j'=N}^{N+1-j} (-1) \frac{1 - i \vec{n}_{j'} \vec{k} \hat{a}}{1 + i \vec{n}_{j'} \vec{k} \hat{a}} \right] \eta$$

Yang-Baxter phenomenon:

$$A(\hat{R}_{\vec{n}_N}\,\hat{R}_{\vec{n}_{N-1}}\cdots\hat{R}_{\vec{n}_1}) = A(\hat{R}_{\vec{n}_1}\,\hat{R}_{\vec{n}_2}\cdots\hat{R}_{\vec{n}_N})$$



Conventional expression:

$$A(\hat{\omega}) = \prod_{j=1}^{N} \frac{1+i\,\vec{n}_{j}(\hat{\omega}\vec{k})\tilde{a}}{\vec{n}_{j}(\hat{\omega}\vec{k})\hat{a}}$$

Needs to be verified