

**2358-20**

**Joint ICTP-IAEA Workshop on Nuclear Structure Decay Data: Theory and  
Evaluation**

*6 - 17 August 2012*

**Introduction to Nuclear Physics - 2**

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# Nuclear Structure (II) Collective models

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*NSDD Workshop, Trieste, August 2012*

# Some collective models

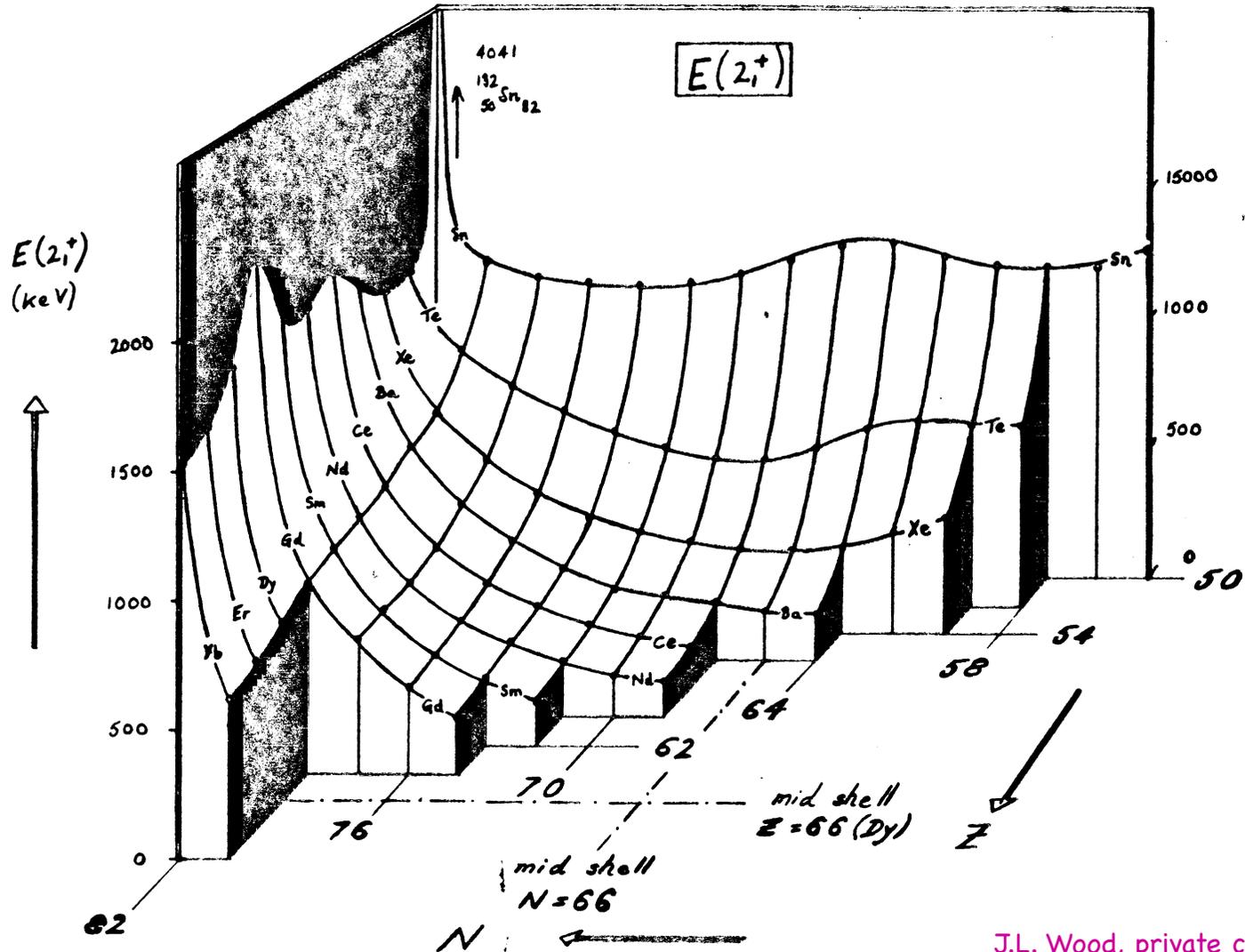
(Rigid) rotor model

(Harmonic quadrupole) vibrator model

Liquid-drop model of vibrations and rotations

Interacting boson model

# Evolution of $E_x(2^+)$



J.L. Wood, private communication

NSDD Workshop, Trieste, August 2012

# Symmetric top

Energy spectrum:

$$E_{\text{rot}}(I) = \frac{\hbar^2}{2\mathfrak{I}} I(I+1)$$

$$\equiv A I(I+1), \quad I = 0, 2, 4, \dots$$

$E(I) - E(I-2)$

Large deformation  $\Rightarrow$   
large  $\mathfrak{I} \Rightarrow$  low  $E_x(2^+)$ .

$R_{42}$  energy ratio:

$$E_{\text{rot}}(4^+) / E_{\text{rot}}(2^+) = 3.333\dots$$

$$6^+ \quad \underline{\quad\quad\quad 42A}$$

22A

$$4^+ \quad \underline{\quad\quad\quad 20A}$$

14A

$$2^+ \quad \underline{\quad\quad\quad 6A}$$

$$0^+ \quad \underline{\quad\quad\quad 0}$$

6A

# Rigid rotor model

Hamiltonian of quantum-mechanical rotor in terms of 'rotational' angular momentum  $R$ :

$$\hat{H}_{\text{rot}} = \frac{\hbar^2}{2} \left[ \frac{R_1^2}{\mathfrak{S}_1} + \frac{R_2^2}{\mathfrak{S}_2} + \frac{R_3^2}{\mathfrak{S}_3} \right] = \frac{\hbar^2}{2} \sum_{i=1}^3 \frac{R_i^2}{\mathfrak{S}_i}$$

Nuclei have an additional intrinsic part  $H_{\text{intr}}$  with 'intrinsic' angular momentum  $J$ .

The total angular momentum is  $I=R+J$ .

# Rigid axially symmetric rotor

For  $\mathfrak{I}_1 = \mathfrak{I}_2 = \mathfrak{I} \neq \mathfrak{I}_3$  the rotor hamiltonian is

$$\hat{H}_{\text{rot}} = \sum_{i=1}^3 \frac{\hbar^2}{2\mathfrak{I}_i} I_i^2 = \frac{\hbar^2}{2\mathfrak{I}} \sum_{i=1}^3 I_i^2 + \frac{\hbar^2}{2} \left( \frac{1}{\mathfrak{I}_3} - \frac{1}{\mathfrak{I}} \right) I_3^2$$

Eigenvalues of  $H_{\text{rot}}$ :

$$E_{KI} = \frac{\hbar^2}{2\mathfrak{I}} I(I+1) + \frac{\hbar^2}{2} \left( \frac{1}{\mathfrak{I}_3} - \frac{1}{\mathfrak{I}} \right) K^2$$

Eigenvectors  $|KIM\rangle$  of  $H_{\text{rot}}$  satisfy:

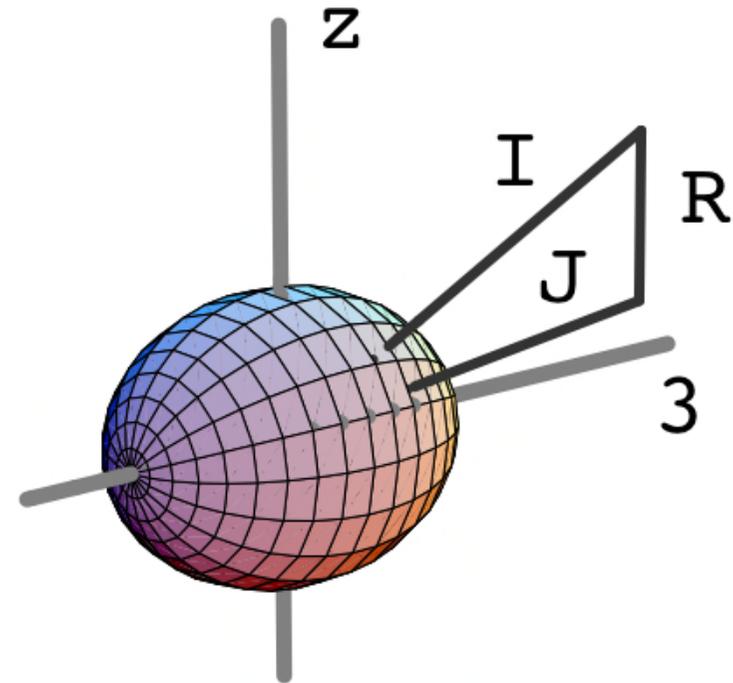
$$I^2 |KIM\rangle = I(I+1) |KIM\rangle,$$

$$I_z |KIM\rangle = M |KIM\rangle, \quad I_3 |KIM\rangle = K |KIM\rangle$$

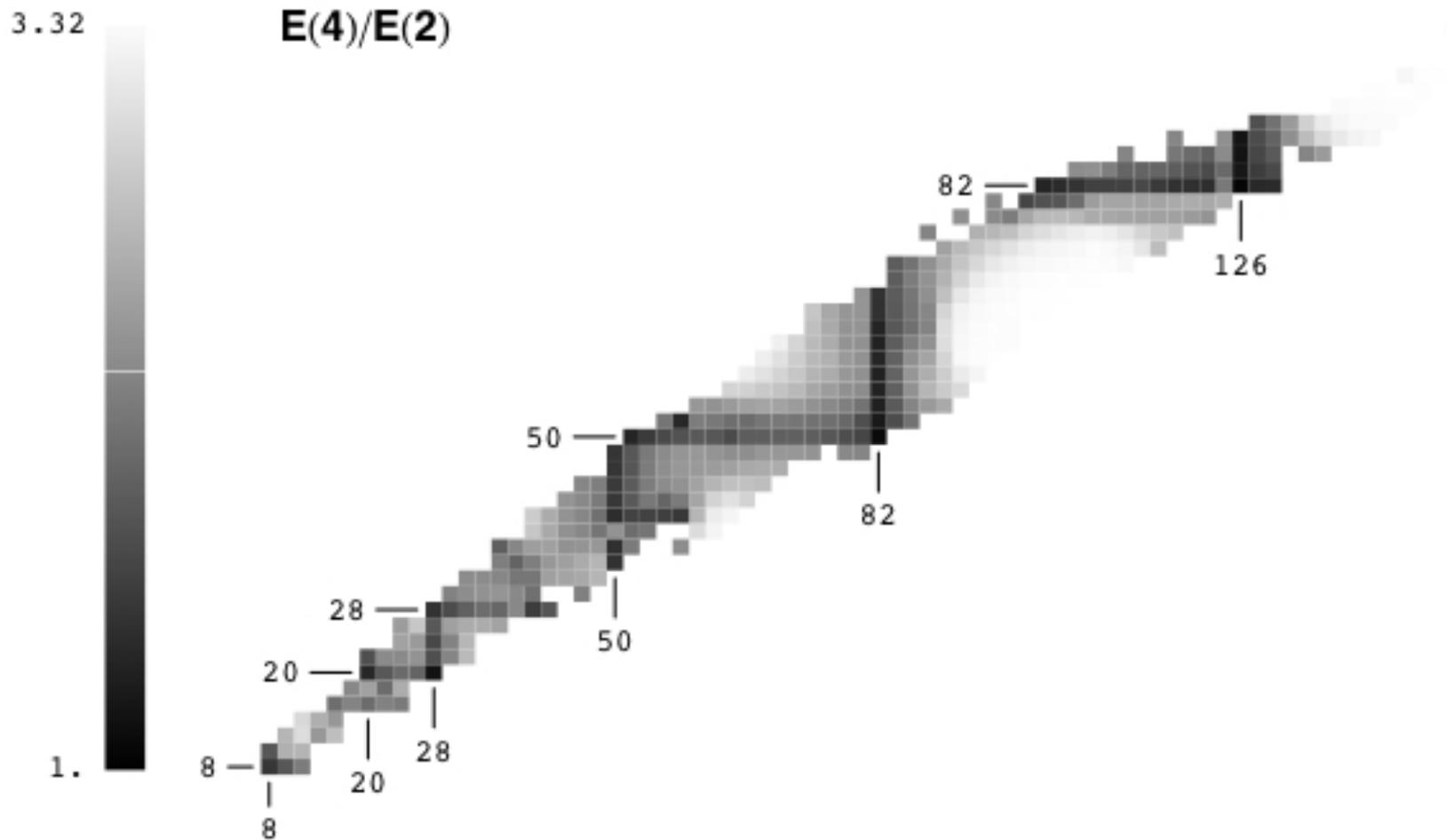
# Ground-state band of axial rotor

The ground-state spin of even-even nuclei is  $I=0$ . Hence  $K=0$  for ground-state band:

$$E_I = \frac{\hbar^2}{2\mathcal{I}} I(I+1)$$



# The ratio $R_{42}$



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# Electric (quadrupole) properties

Partial  $\gamma$ -ray half-life:

$$T_{1/2}^\gamma(\mathbf{E}\lambda) = \ln 2 \left\{ \frac{8\pi}{\hbar} \frac{\lambda + 1}{\lambda [(2\lambda + 1)!!]^2} \left( \frac{E_\gamma}{\hbar c} \right)^{2\lambda + 1} B(\mathbf{E}\lambda) \right\}^{-1}$$

Electric quadrupole transitions:

$$B(\mathbf{E}2; I_i \rightarrow I_f) = \frac{1}{2I_i + 1} \sum_{M_i} \sum_{M_f \mu} \left| \langle I_f M_f | \sum_{k=1}^A e_k r_k^2 Y_{2\mu}(\theta_k, \varphi_k) | I_i M_i \rangle \right|^2$$

Electric quadrupole moments:

$$eQ(I) = \langle IM = I | \sqrt{\frac{16\pi}{5}} \sum_{k=1}^A e_k r_k^2 Y_{20}(\theta_k, \varphi_k) | IM = I \rangle$$

# Magnetic (dipole) properties

Partial  $\gamma$ -ray half-life:

$$T_{1/2}^{\gamma}(\text{M}\lambda) = \ln 2 \left\{ \frac{8\pi}{\hbar} \frac{\lambda + 1}{\lambda [(2\lambda + 1)!!]^2} \left( \frac{E_{\gamma}}{\hbar c} \right)^{2\lambda + 1} B(\text{M}\lambda) \right\}^{-1}$$

Magnetic dipole transitions:

$$B(\text{M}1; I_i \rightarrow I_f) = \frac{1}{2I_i + 1} \sum_{M_i} \sum_{M_f \mu} \left| \langle I_f M_f | \sum_{k=1}^A (g_k^l l_{k,\mu} + g_k^s s_{k,\mu}) | I_i M_i \rangle \right|^2$$

Magnetic dipole moments:

$$\mu(I) = \langle IM = I | \sum_{k=1}^A (g_k^l l_{k,z} + g_k^s s_{k,z}) | IM = I \rangle$$

# E2 properties of rotational nuclei

*Intra-band E2 transitions:*

$$B(E2; KI_i \rightarrow KI_f) = \frac{5}{16\pi} \langle I_i K 20 | I_f K \rangle^2 e^2 Q_0(K)^2$$

*E2 moments:*

$$Q(KI) = \frac{3K^2 - I(I+1)}{(I+1)(2I+3)} Q_0(K)$$

$Q_0(K)$  is the 'intrinsic' quadrupole moment:

$$e\hat{Q}_0 \equiv \int \rho(\mathbf{r}') r'^2 (3\cos^2 \theta' - 1) d\mathbf{r}', \quad Q_0(K) = \langle K | \hat{Q}_0 | K \rangle$$

# E2 properties of gs bands

For the ground state ( $K=I$ ):

$$Q(K = I) = \frac{I(2I-1)}{(I+1)(2I+3)} Q_0(K)$$

For the gsb in even-even nuclei ( $K=0$ ):

$$B(E2; I \rightarrow I-2) = \frac{15}{32\pi} \frac{I(I-1)}{(2I-1)(2I+1)} e^2 Q_0^2$$

$$Q(I) = -\frac{I}{2I+3} Q_0$$

$$\Rightarrow |eQ(2_1^+)| = \frac{2}{7} \sqrt{16\pi \cdot B(E2; 2_1^+ \rightarrow 0_1^+)}$$

# Generalized intensity relations

Mixing of  $K$  arises from

*Dependence of  $Q_0$  on  $I$  (stretching)*

*Coriolis interaction*

*Triaxiality*

Generalized *intra-* and *inter-*band matrix elements  
(eg E2):

$$\frac{\sqrt{B(\text{E2}; K_i I_i \rightarrow K_f I_f)}}{\left| \langle I_i K_i \ 2K_f - K_i | I_f K_f \rangle \right|} = M_0 + M_1 \Delta + M_2 \Delta^2 + \dots$$

$$\text{with } \Delta = I_f(I_f + 1) - I_i(I_i + 1)$$



# Modes of nuclear vibration

Nucleus is considered as a droplet of nuclear matter with an equilibrium shape. Vibrations are modes of excitation around that shape.

Character of vibrations depends on symmetry of equilibrium shape. Two important cases in nuclei:

*Spherical equilibrium shape*

*Spheroidal equilibrium shape*

# Vibrations about a spherical shape

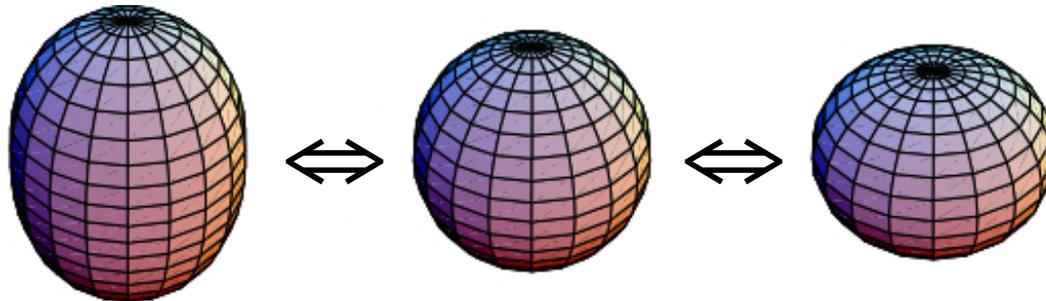
Vibrations are characterized by a multipole quantum number  $\lambda$  in surface parametrization:

$$R(\theta, \varphi) = R_0 \left( 1 + \sum_{\lambda} \sum_{\mu=-\lambda}^{+\lambda} \alpha_{\lambda\mu} Y_{\lambda\mu}^*(\theta, \varphi) \right)$$

$\lambda=0$ : *compression (high energy)*

$\lambda=1$ : *translation (not an intrinsic excitation)*

$\lambda=2$ : *quadrupole vibration*



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# Properties of spherical vibrations

Energy spectrum:

$$E_{\text{vib}}(n) = \left(n + \frac{5}{2}\right)\hbar\omega, n = 0, 1, \dots$$

$$\frac{3}{\text{---}} 6^+ 4^+ 3^+ 2^+ 0^+$$

$R_{42}$  energy ratio:

$$E_{\text{vib}}(4^+) / E_{\text{vib}}(2^+) = 2$$

$$\frac{2}{\text{---}} 4^+ 2^+ 0^+$$

E2 transitions:

$$B(E2; 2_1^+ \rightarrow 0_1^+) = \alpha^2$$

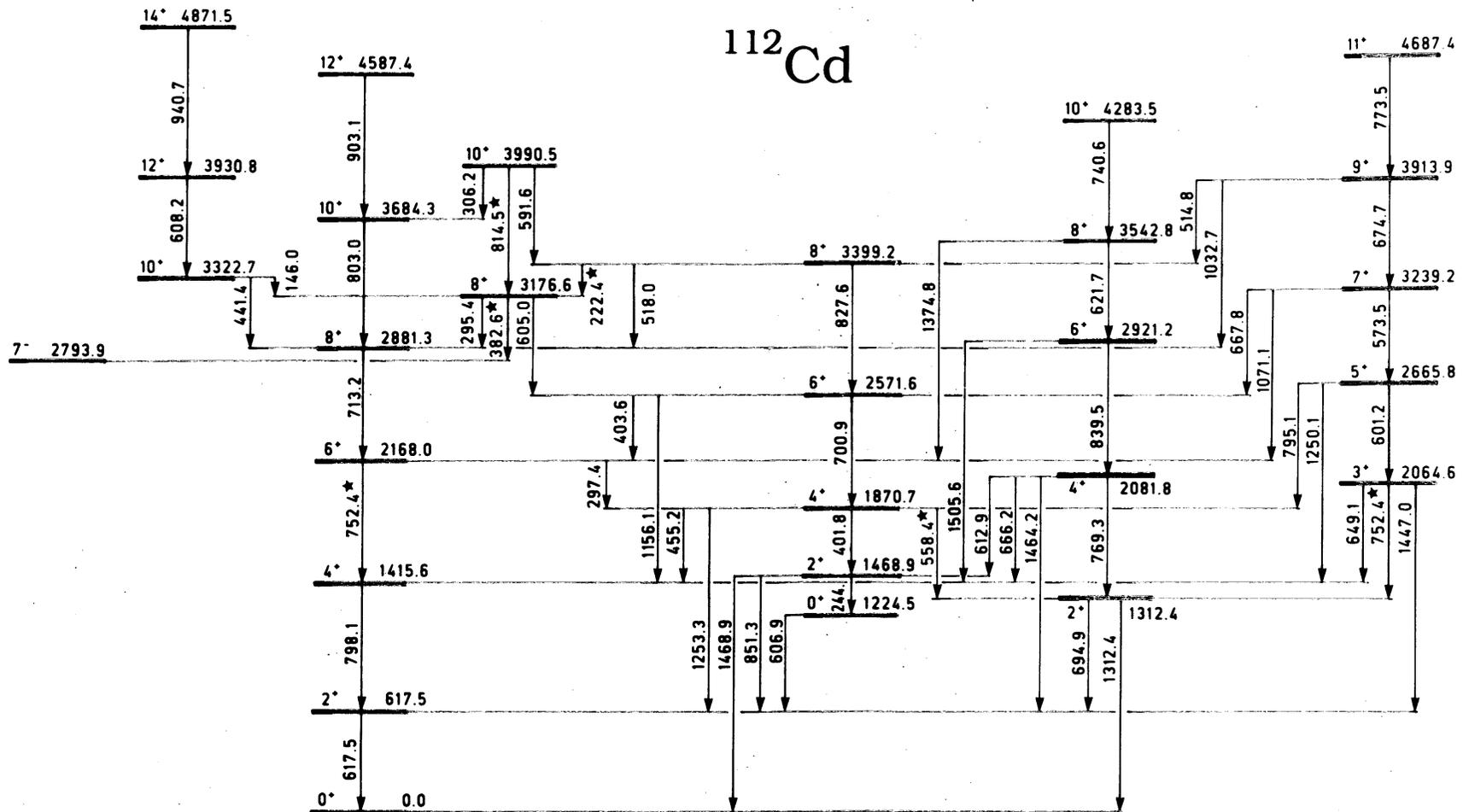
$$B(E2; 2_2^+ \rightarrow 0_1^+) = 0$$

$$\frac{1}{\text{---}} 2^+$$

$$B(E2; n = 2 \rightarrow n = 1) = 2\alpha^2$$

$$\frac{0}{\text{---}} 0^+$$

# Example of $^{112}\text{Cd}$



Y - BAND

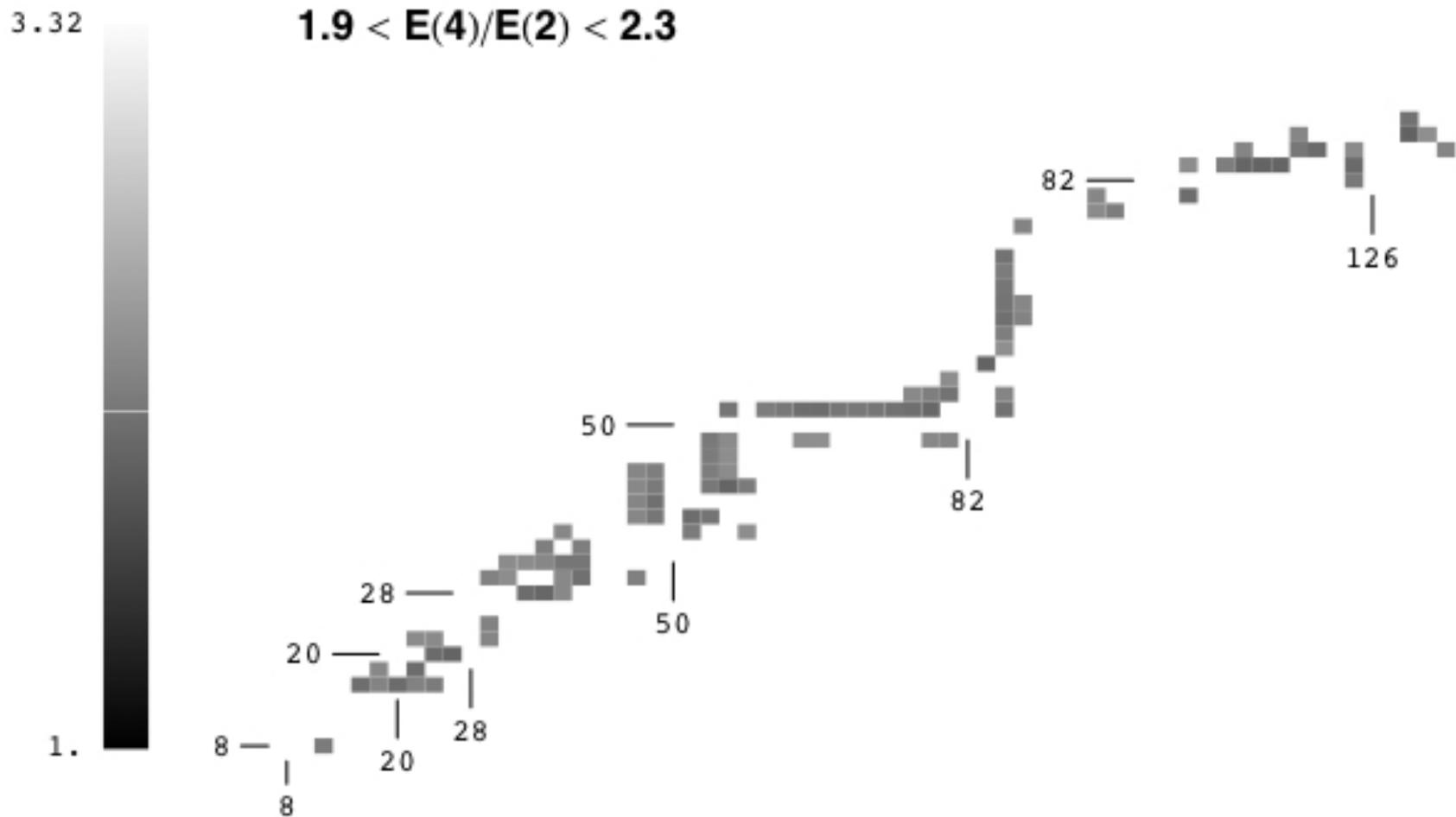
G - BAND

I - BAND

QUASI -  $\gamma$  BAND

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# Possible vibrational nuclei from $R_{42}$



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# Vibrations of a spheroidal shape

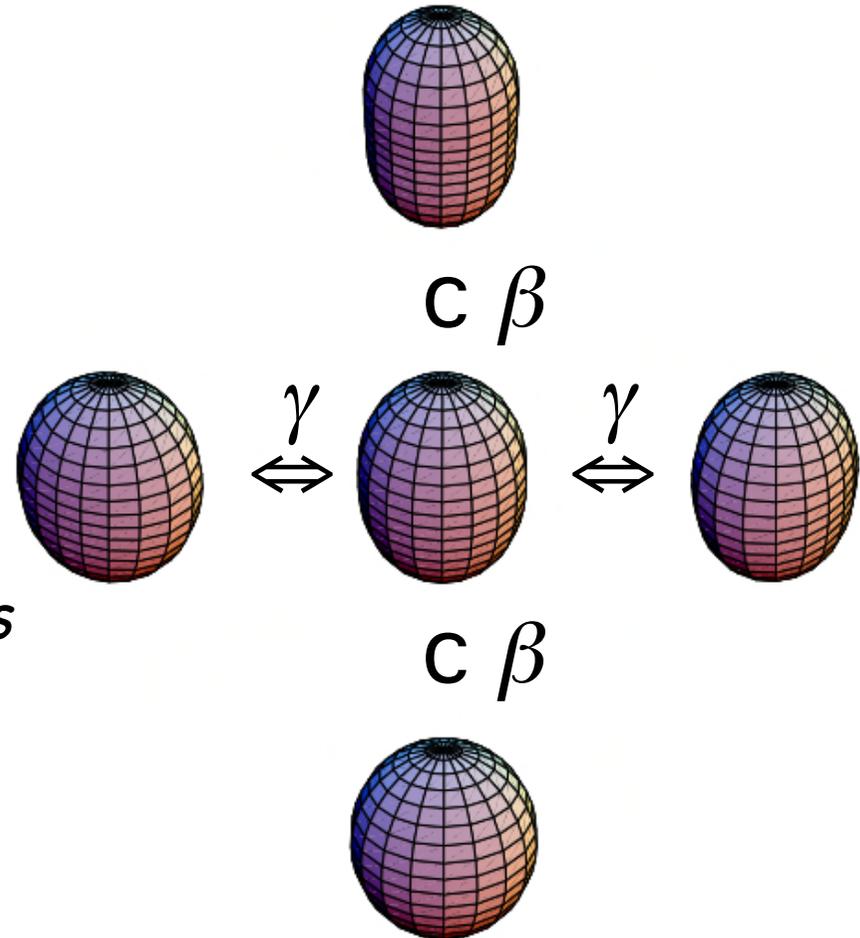
The vibration of a shape with axial symmetry is characterized by  $a_{\lambda\nu}$ .

Quadrupole oscillations:

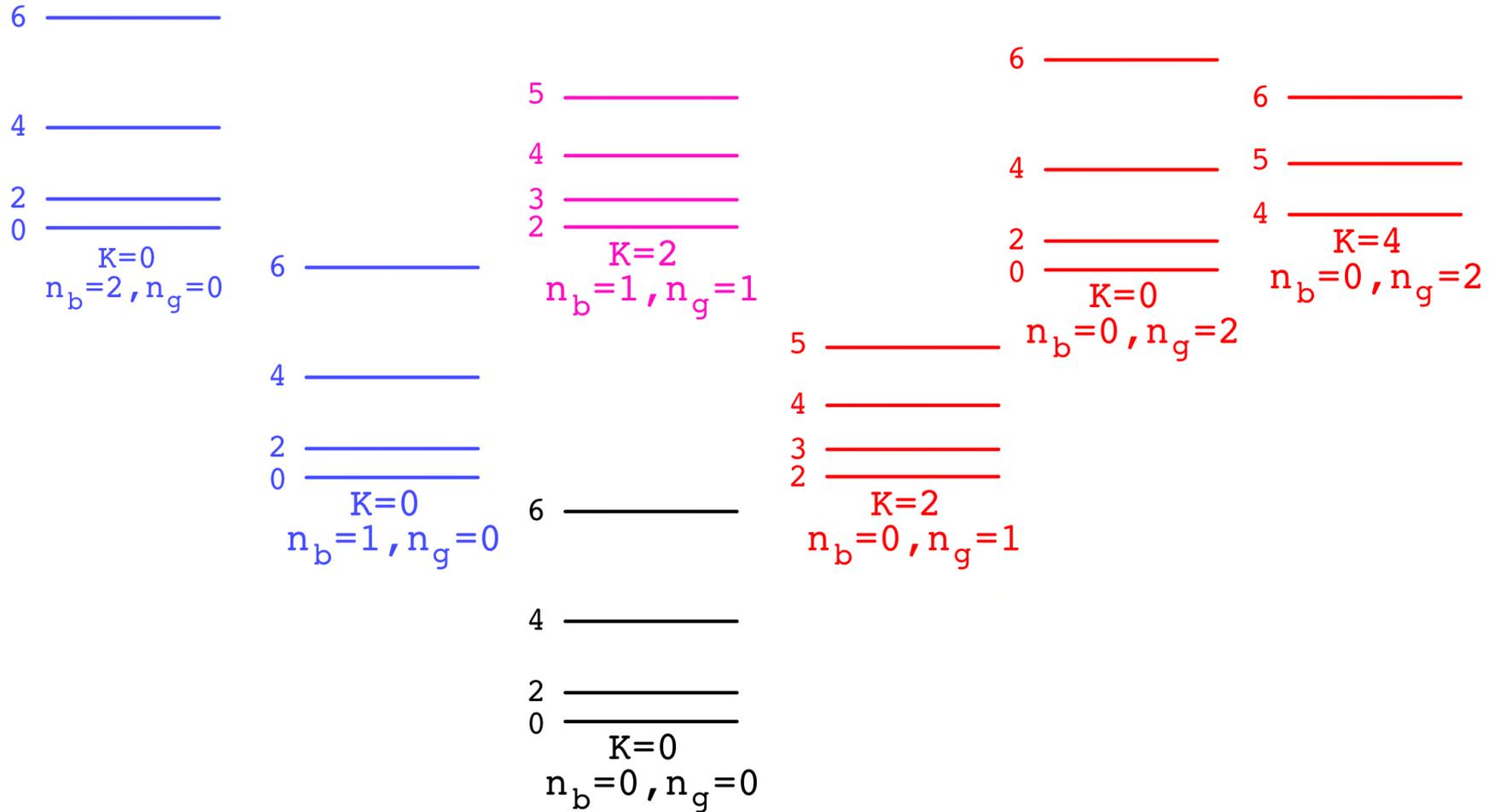
$\nu=0$ : along the axis of symmetry ( $\beta$ )

$\nu=\pm 1$ : spurious rotation

$\nu=\pm 2$ : perpendicular to axis of symmetry ( $\gamma$ )

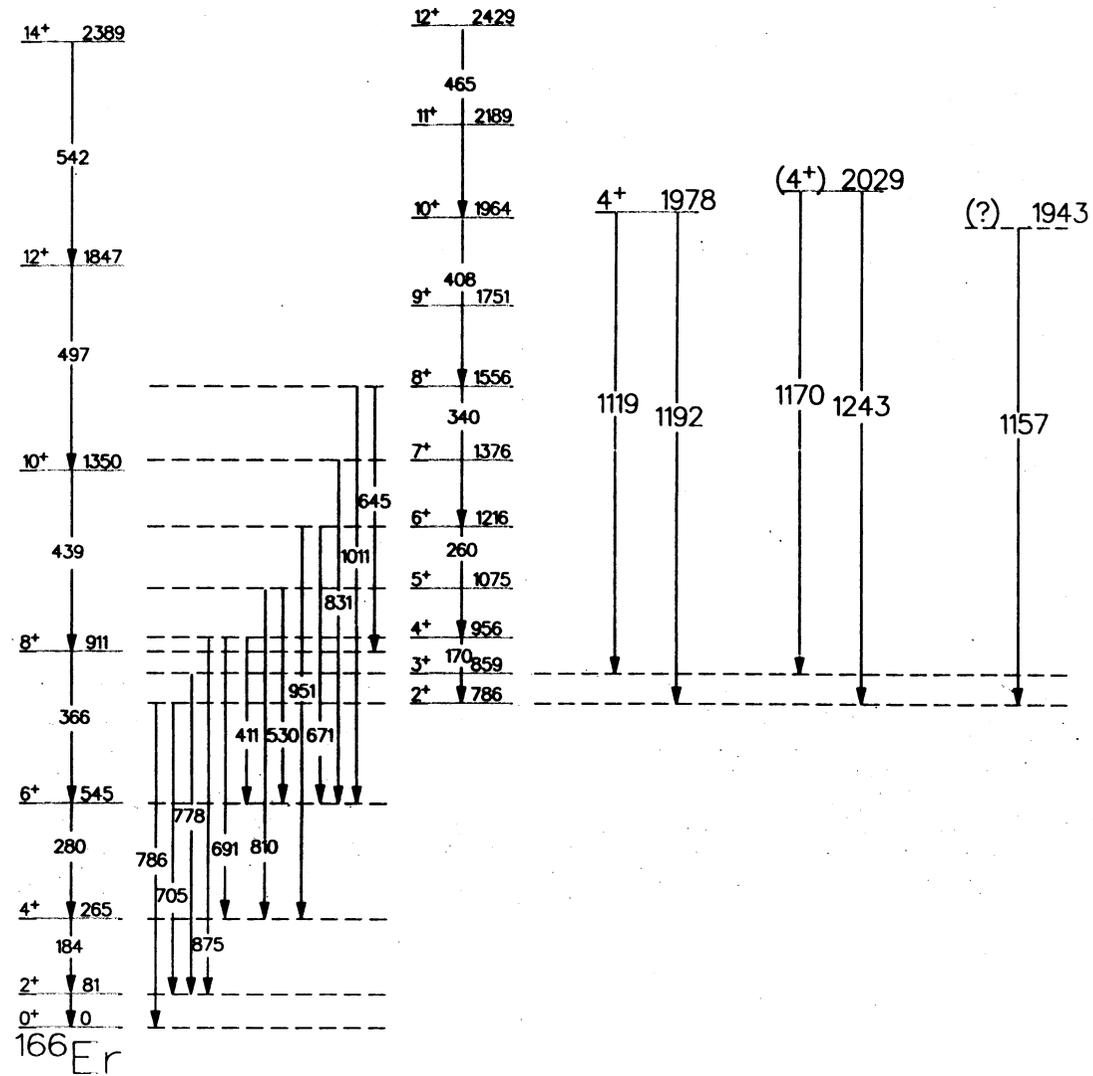


# Spectrum of spheroidal vibrations



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# Example of $^{166}\text{Er}$



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# Rigid triaxial rotor

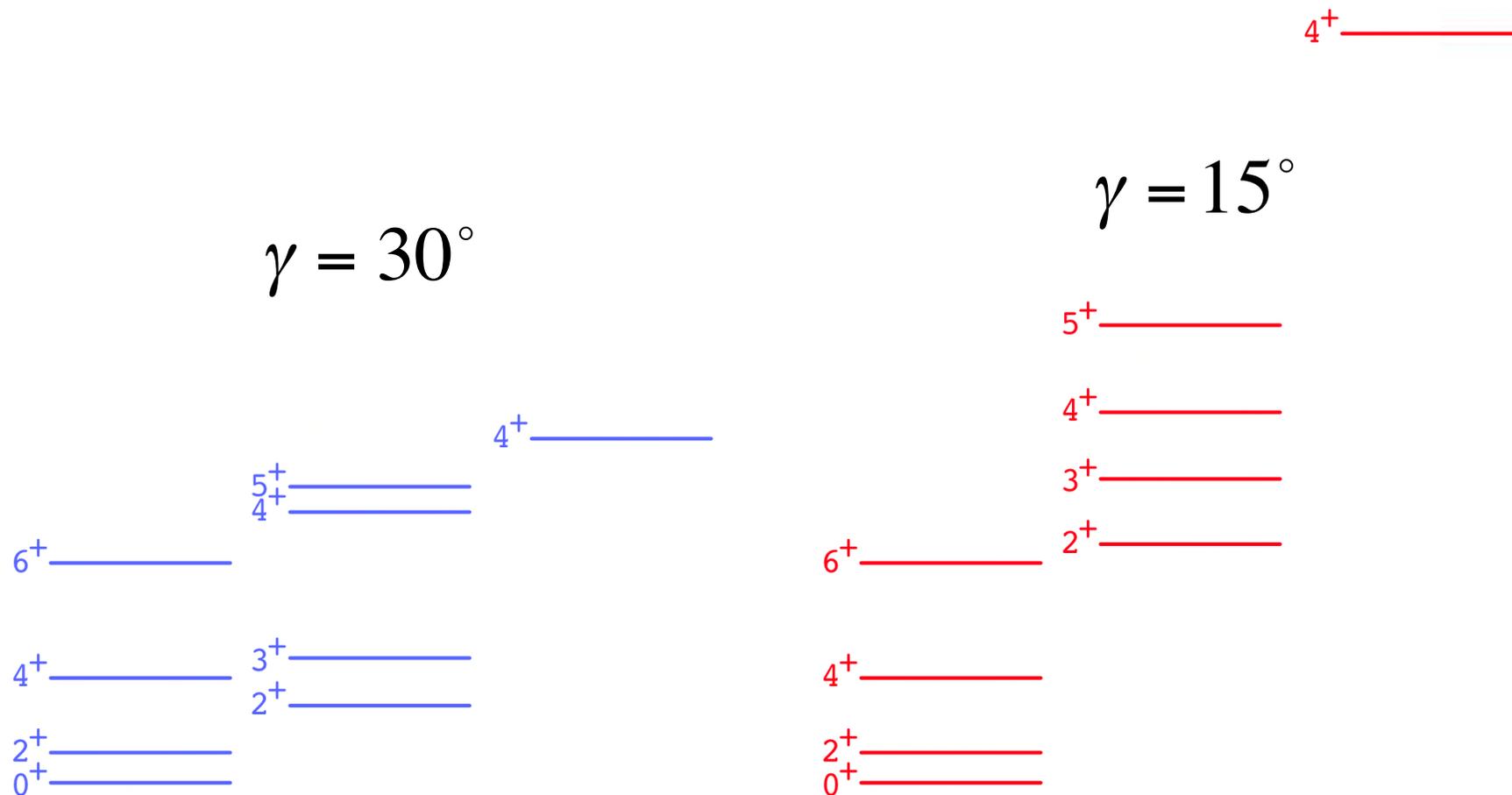
Triaxial rotor hamiltonian  $\mathfrak{I}_1 \neq \mathfrak{I}_2 \neq \mathfrak{I}_3$  :

$$\hat{H}'_{\text{rot}} = \sum_{i=1}^3 \frac{\hbar^2}{2\mathfrak{I}_i} I_i^2 = \underbrace{\frac{\hbar^2}{2\mathfrak{I}} I^2 + \frac{\hbar^2}{2\mathfrak{I}_f} I_3^2}_{\hat{H}'_{\text{axial}}} + \underbrace{\frac{\hbar^2}{2\mathfrak{I}_g} (I_+^2 + I_-^2)}_{\hat{H}'_{\text{mix}}}$$

$$\frac{1}{\mathfrak{I}} = \frac{1}{2} \left( \frac{1}{\mathfrak{I}_1} + \frac{1}{\mathfrak{I}_2} \right), \quad \frac{1}{\mathfrak{I}_f} = \frac{1}{\mathfrak{I}_3} - \frac{1}{\mathfrak{I}}, \quad \frac{1}{\mathfrak{I}_g} = \frac{1}{4} \left( \frac{1}{\mathfrak{I}_1} - \frac{1}{\mathfrak{I}_2} \right)$$

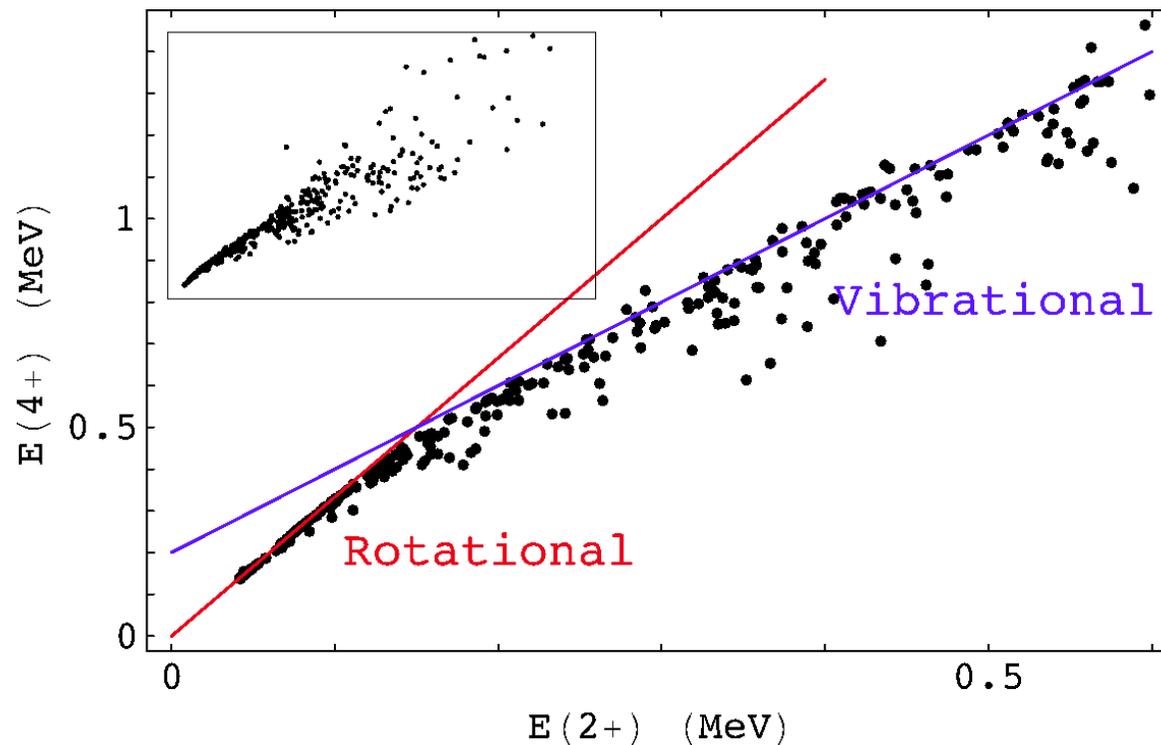
$H'_{\text{mix}}$  non-diagonal in axial basis  $|KIM\rangle \Rightarrow K$  is *not* a conserved quantum number

# Rigid triaxial rotor spectra



# Tri-partite classification of nuclei

Empirical evidence for seniority-type, vibrational- and rotational-like nuclei.



N.V. Zamfir *et al.*, Phys. Rev. Lett. 72 (1994) 3480

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# Interacting boson model

Describe the nucleus as a system of  $N$  interacting  $s$  and  $d$  bosons. Hamiltonian:

$$\hat{H}_{\text{IBM}} = \sum_{i=1}^6 \varepsilon_i \hat{b}_i^+ \hat{b}_i + \sum_{i_1 i_2 i_3 i_4 = 1}^6 v_{i_1 i_2 i_3 i_4} \hat{b}_{i_1}^+ \hat{b}_{i_2}^+ \hat{b}_{i_3} \hat{b}_{i_4}$$

Justification from

*Shell model:  $s$  and  $d$  bosons are associated with  $S$  and  $D$  fermion (Cooper) pairs.*

*Geometric model: for large boson number the IBM reduces to a liquid-drop hamiltonian.*

# Dimensions

Assume  $\Omega$  available 1-fermion states. Number of  $n$ -fermion states is  $\binom{\Omega}{n} = \frac{\Omega!}{n!(\Omega - n)!}$

Assume  $\Omega$  available 1-boson states. Number of  $n$ -boson states is  $\binom{\Omega + n - 1}{n} = \frac{(\Omega + n - 1)!}{n!(\Omega - 1)!}$

Example:  $^{162}\text{Dy}_{96}$  with 14 neutrons ( $\Omega=44$ ) and 16 protons ( $\Omega=32$ ) ( $^{132}\text{Sn}_{82}$  inert core).

*SM dimension:  $7 \cdot 10^{19}$*

*IBM dimension: 15504*

# Dynamical symmetries

Boson hamiltonian is of the form

$$\hat{H}_{\text{IBM}} = \sum_{i=1}^6 \varepsilon_i \hat{b}_i^+ \hat{b}_i + \sum_{i_1 i_2 i_3 i_4=1}^6 v_{i_1 i_2 i_3 i_4} \hat{b}_{i_1}^+ \hat{b}_{i_2}^+ \hat{b}_{i_3} \hat{b}_{i_4}$$

In general not solvable analytically.

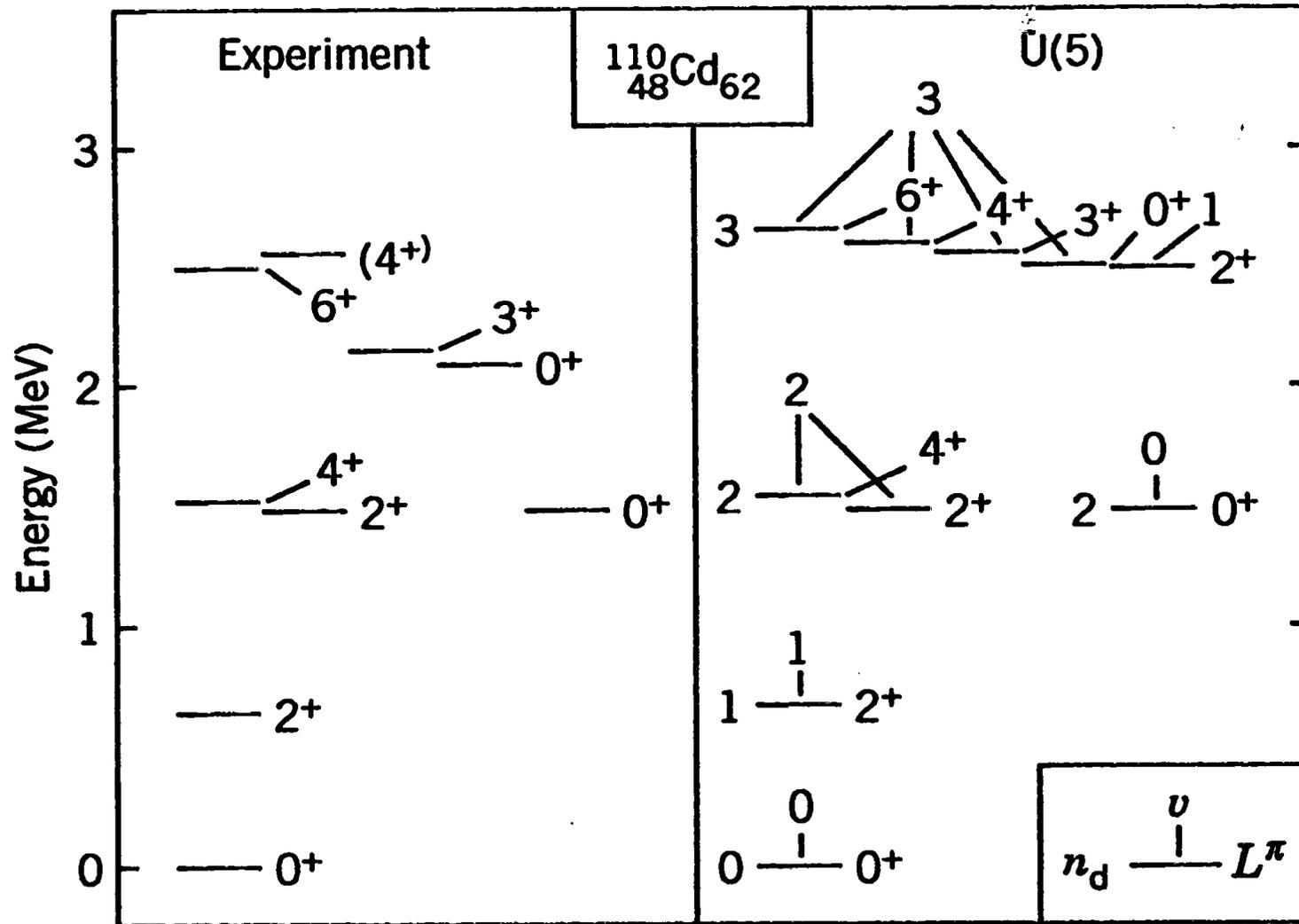
Three solvable cases with  $SO(3)$  symmetry:

$$U(6) \supset U(5) \supset SO(5) \supset SO(3)$$

$$U(6) \supset SU(3) \supset SO(3)$$

$$U(6) \supset SO(6) \supset SO(5) \supset SO(3)$$

# U(5) vibrational limit: $^{110}\text{Cd}_{62}$

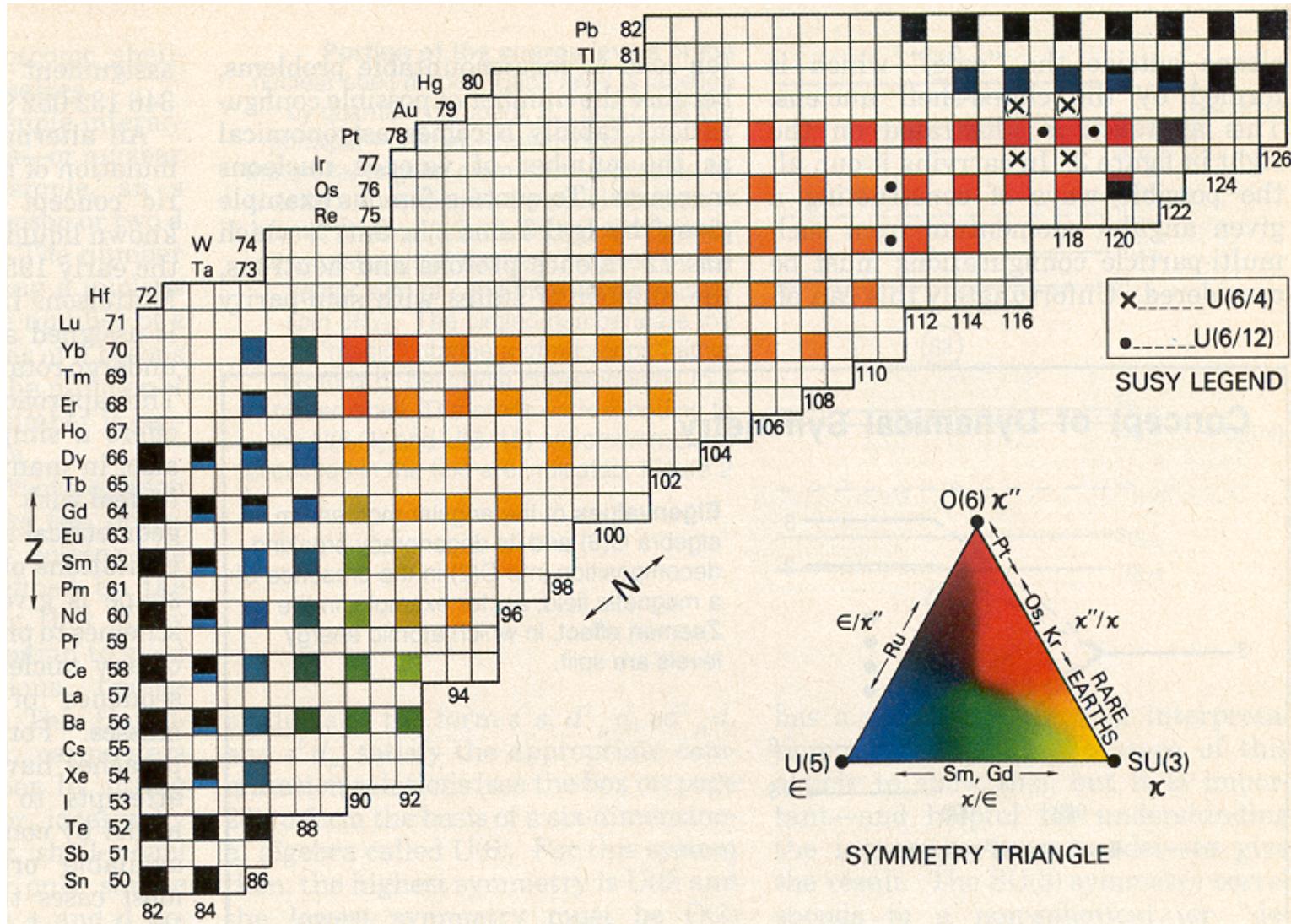


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# Applications of IBM



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# Classical limit of IBM

For large boson number  $N$ , a *coherent* (or *intrinsic*) state is an approximate eigenstate,

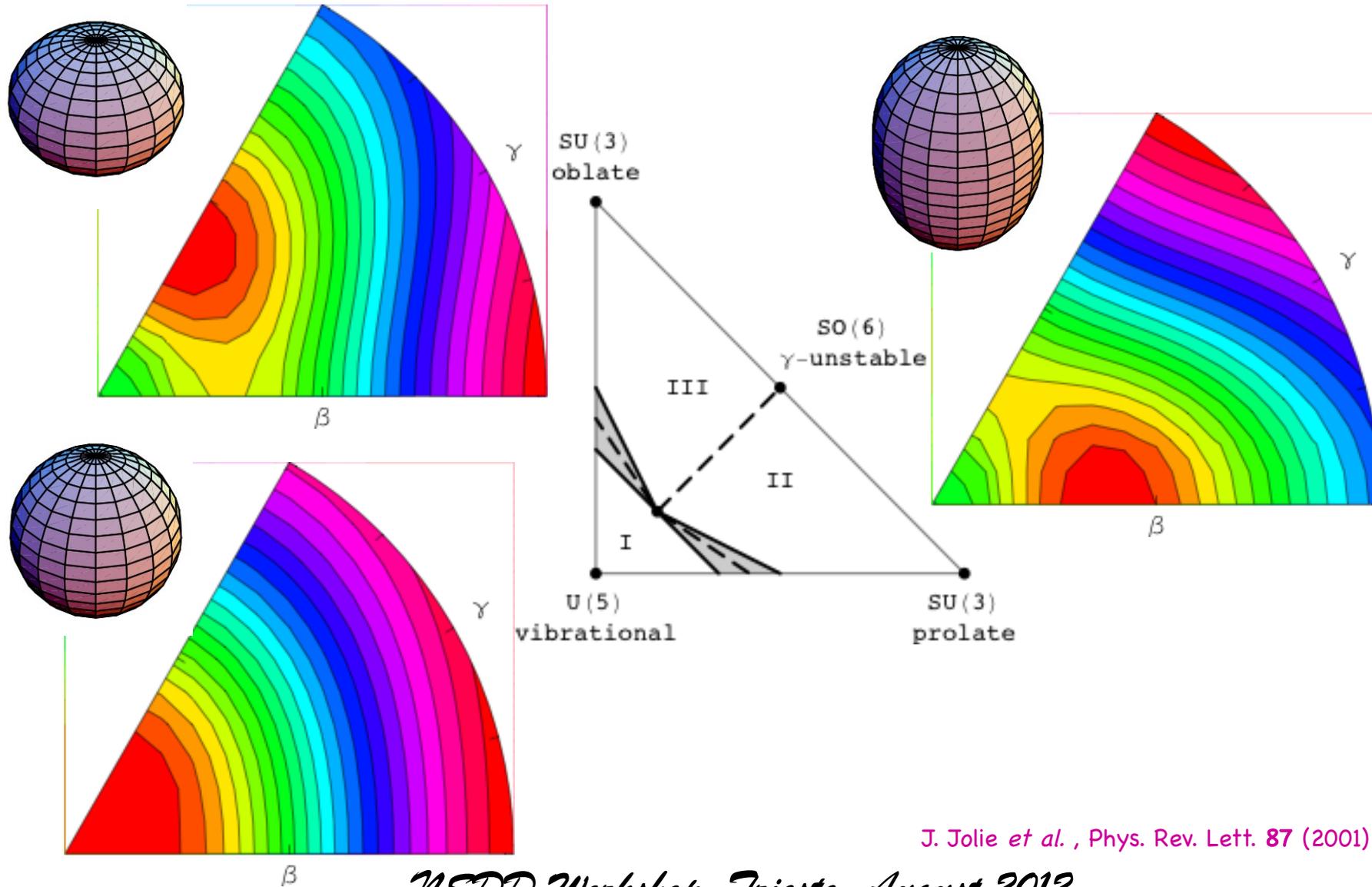
$$\hat{H}_{\text{IBM}}|N;\alpha_\mu\rangle \approx E|N;\alpha_\mu\rangle, \quad |N;\alpha_\mu\rangle \propto \left(s^+ + \sum_\mu \alpha_\mu d_\mu^+\right)^N |0\rangle$$

The real parameters  $\alpha_\mu$  are related to the three Euler angles and shape variables  $\beta$  and  $\gamma$ .

Any IBM hamiltonian yields energy surface:

$$\langle N;\alpha_\mu | \hat{H}_{\text{IBM}} | N;\alpha_\mu \rangle = \langle N;\beta\gamma | \hat{H}_{\text{IBM}} | N;\beta\gamma \rangle \equiv V(\beta,\gamma)$$

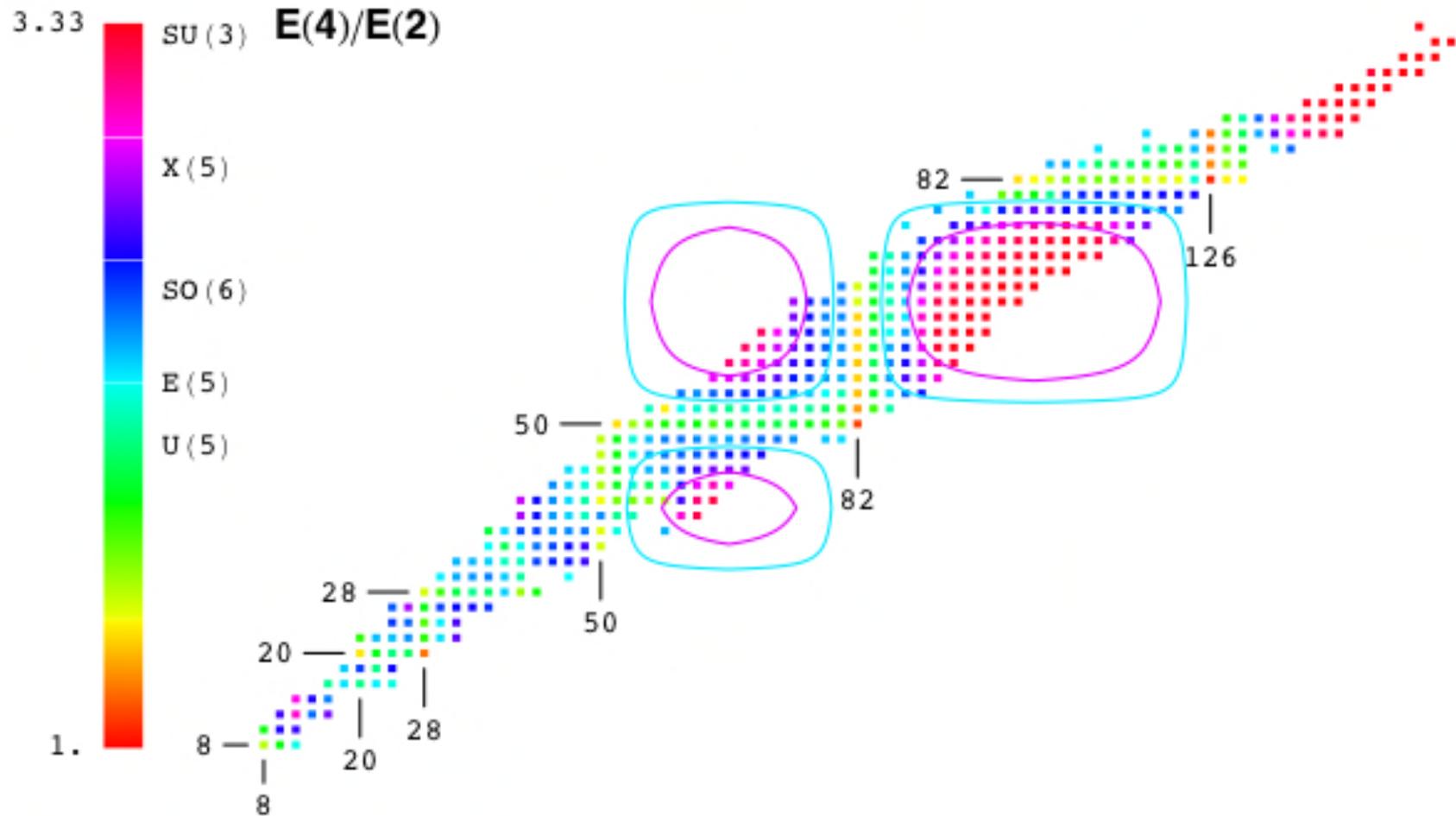
# Phase diagram of IBM



J. Jolie *et al.*, Phys. Rev. Lett. 87 (2001) 162501

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# The ratio $R_{42}$



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F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge UP, 1987).

# Extensions of IBM

Neutron and proton degrees freedom (IBM-2):

*F-spin multiplets ( $N_v + N_\pi = \text{constant}$ )*

*Scissors excitations*

Fermion degrees of freedom (IBFM):

*Odd-mass nuclei*

*Supersymmetry (doublets & quartets)*

Other boson degrees of freedom:

*Isospin  $T=0$  &  $T=1$  pairs (IBM-3 & IBM-4)*

*Higher multipole (g,...) pairs*

# Scissors mode

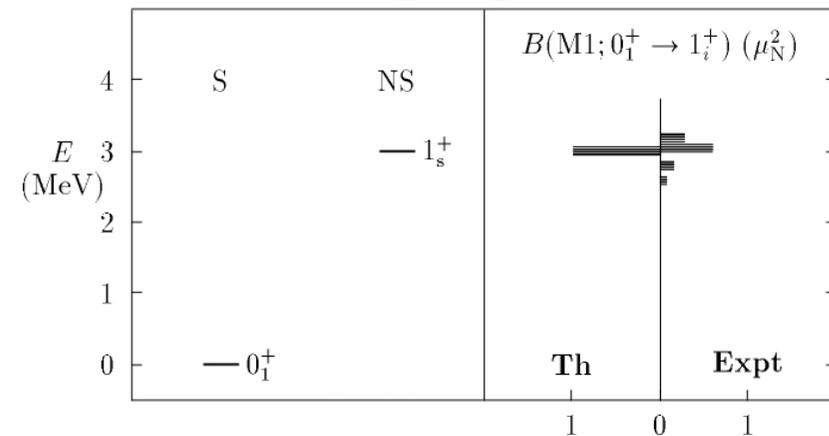
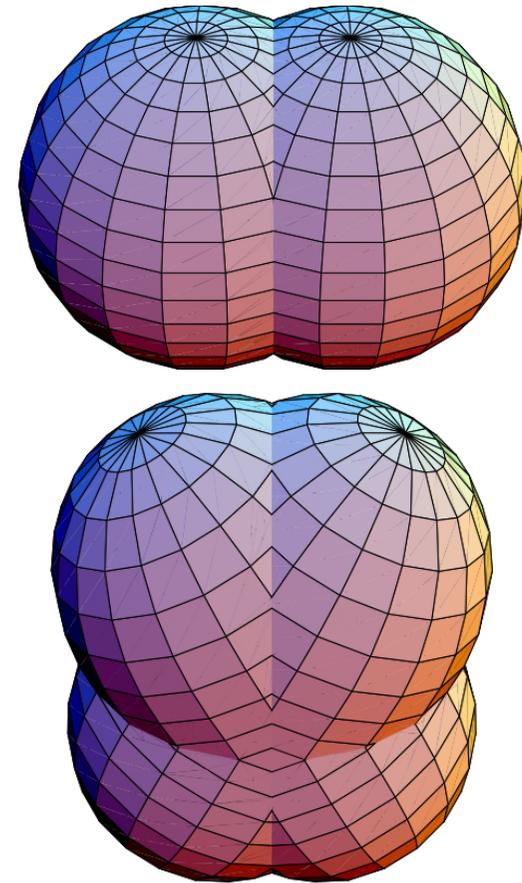
Collective displacement modes between neutrons and protons:

*Linear displacement (giant dipole resonance):*

$R_{\nu} - R_{\pi} \Rightarrow E1$  excitation.

*Angular displacement (scissors resonance):*

$L_{\nu} - L_{\pi} \Rightarrow M1$  excitation.

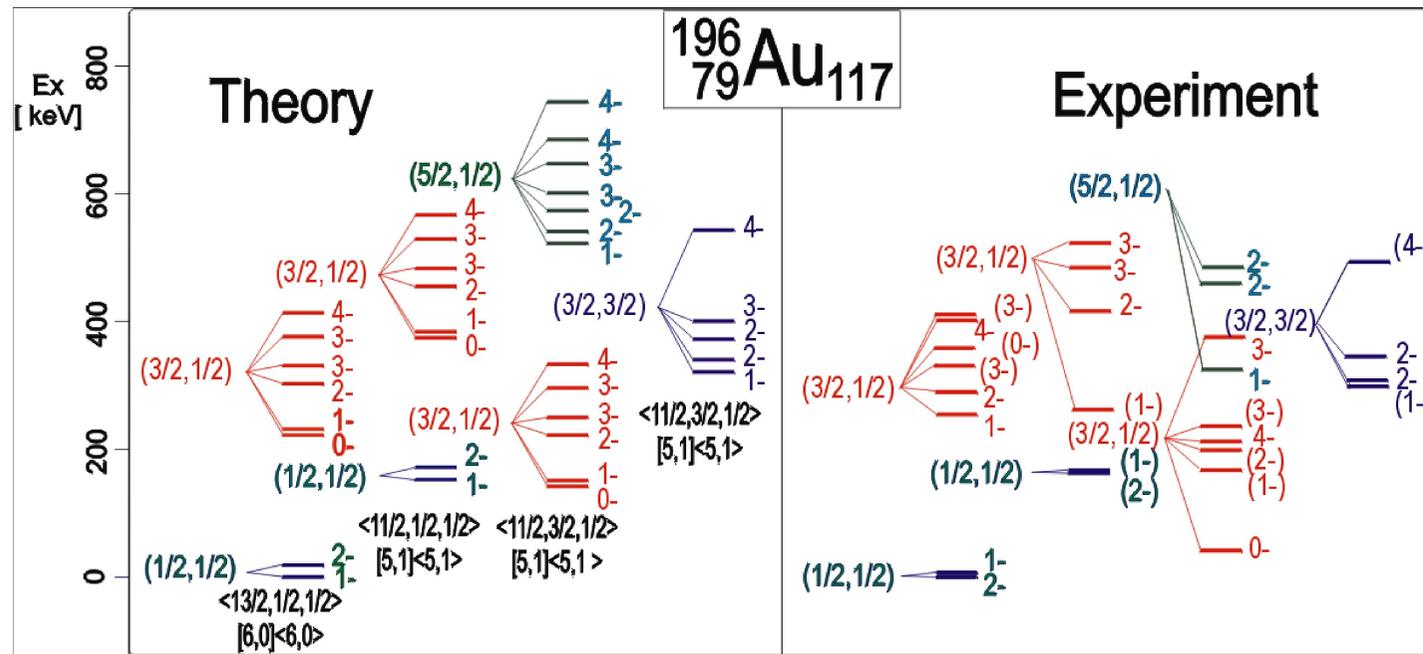


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# Supersymmetry

A simultaneous description of even- and odd-mass nuclei (doublets) or of even-even, even-odd, odd-even and odd-odd nuclei (quartets).

Example of  $^{194}\text{Pt}$ ,  $^{195}\text{Pt}$ ,  $^{195}\text{Au}$  &  $^{196}\text{Au}$ :



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# Bosons + fermions

Odd-mass nuclei are fermions.

Describe an odd-mass nucleus as  $N$  bosons + 1 fermion mutually interacting. Hamiltonian:

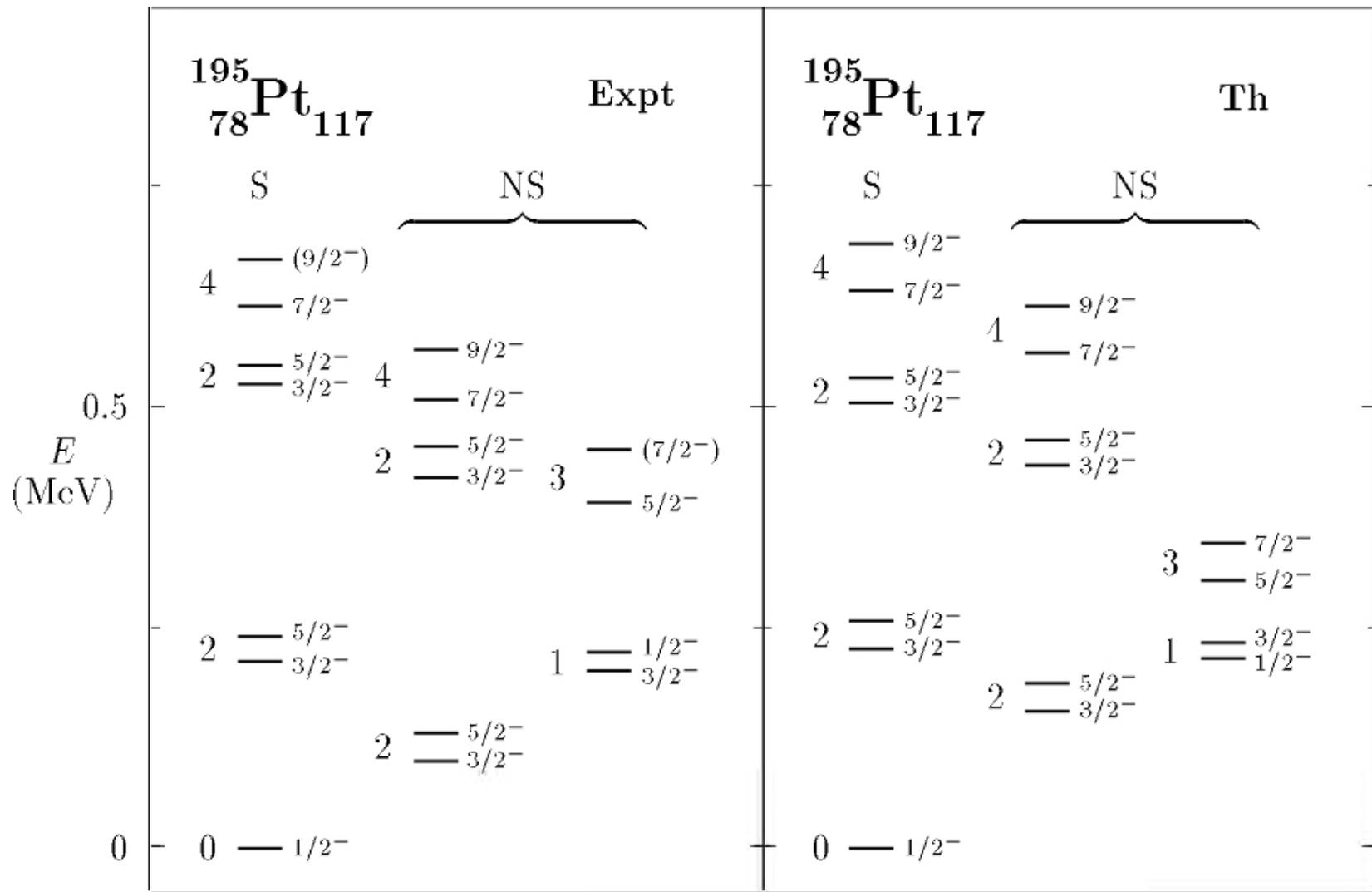
$$\hat{H}_{\text{IBFM}} = \hat{H}_{\text{IBM}} + \sum_{j=1}^{\Omega} \bar{\varepsilon}_j \hat{a}_j^+ \hat{a}_j + \sum_{i_1 i_2=1}^6 \sum_{j_1 j_2=1}^{\Omega} \bar{v}_{i_1 j_1 i_2 j_2} \hat{b}_{i_1}^+ \hat{a}_{j_1}^+ \hat{b}_{i_2} \hat{a}_{j_2}$$

Algebra:

$$\text{U}(6) \oplus \text{U}(\Omega) = \left\{ \begin{array}{l} \hat{b}_{i_1}^+ \hat{b}_{i_2} \\ \hat{a}_{j_1}^+ \hat{a}_{j_2} \end{array} \right\}$$

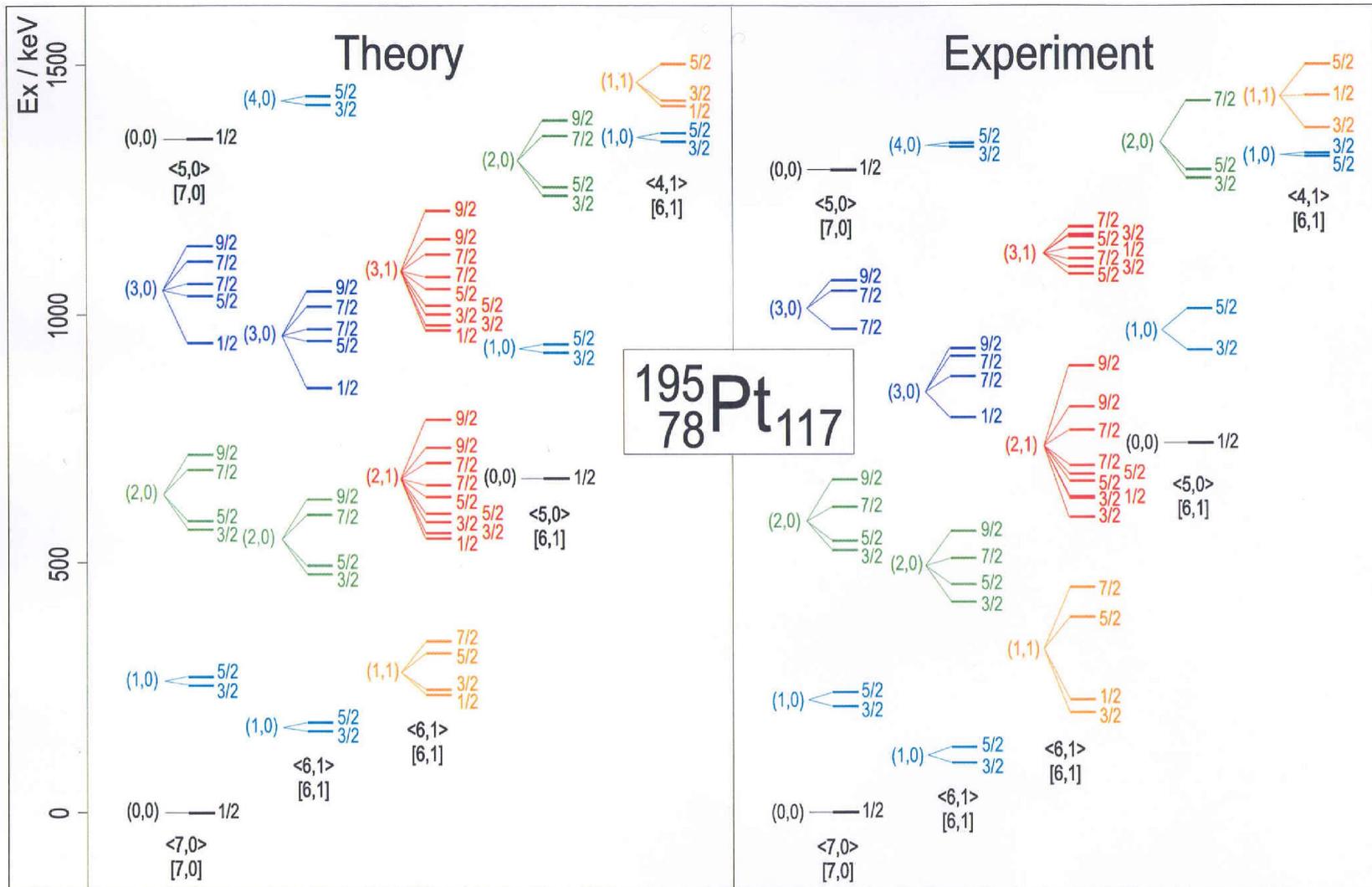
Many-body problem is solved analytically for certain energies  $\varepsilon$  and interactions  $v$ .

# Example: $^{195}\text{Pt}_{117}$



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# Example: $^{195}\text{Pt}_{117}$ (new data)



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# Nuclear supersymmetry

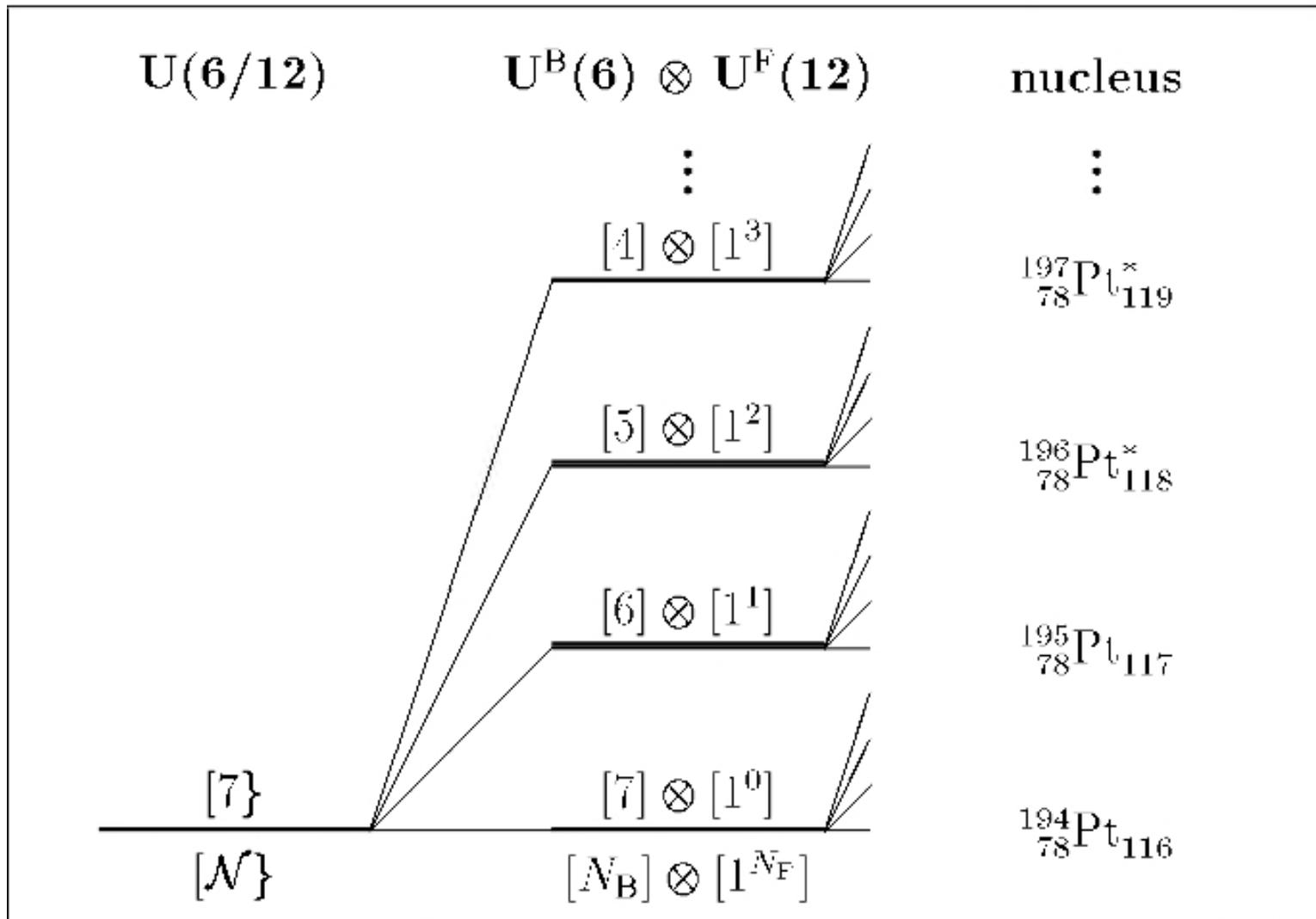
Up to now: separate description of even-even and odd-mass nuclei with the algebra

$$U(6) \oplus U(\Omega) = \left\{ \begin{array}{c} \hat{b}_{i_1}^+ \hat{b}_{i_2} \\ \hat{a}_{j_1}^+ \hat{a}_{j_2} \end{array} \right\}$$

Simultaneous description of even-even and odd-mass nuclei with the superalgebra

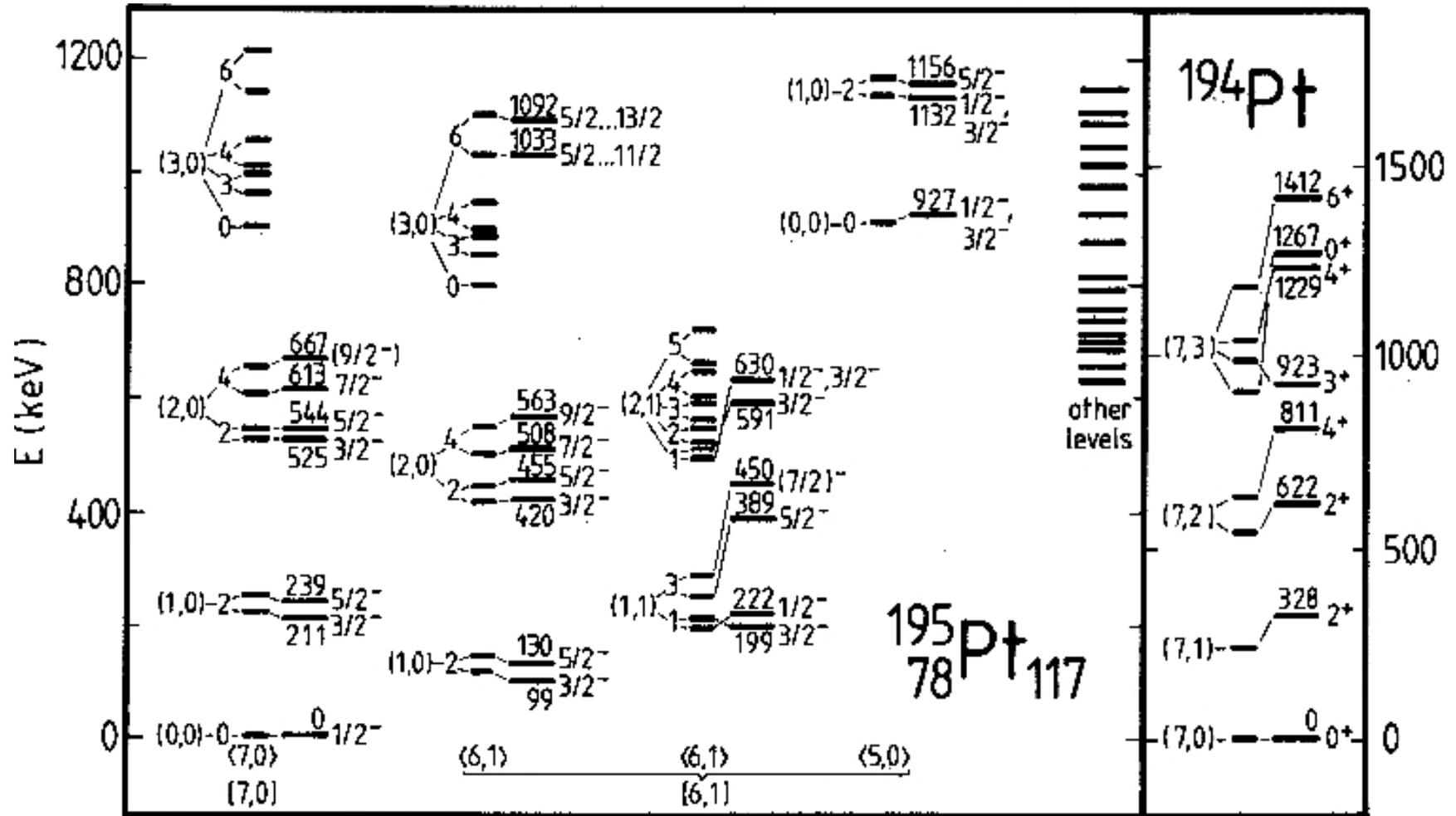
$$U(6/\Omega) = \left\{ \begin{array}{cc} \hat{b}_{i_1}^+ \hat{b}_{i_2} & \hat{b}_{i_1}^+ \hat{a}_{j_2} \\ \hat{a}_{j_1}^+ \hat{b}_{i_2} & \hat{a}_{j_1}^+ \hat{a}_{j_2} \end{array} \right\}$$

# U(6/12) supermultiplet



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# Example: $^{194}\text{Pt}_{116}$ & $^{195}\text{Pt}_{117}$



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